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On diffusion dynamics for continuous systems with singular superstable interaction

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We consider the time evolution of states for continuous infinite particle systems which corresponds to nonequilibrium diffusion dynamics. For initial states μ_0 which are perturbations of the equilibrium we obtain a bound for finite volume nonequilibrium correlation functions and their continuity in time uniformly in volume for any finite time interval. This gives the possibility to construct the time evolution of correlation functions and corresponding states in the thermodynamic limit. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1690489]

I. INTRODUCTION

A diffusion of an interacting infinite particle system can be described by an infinite system of stochastic differential equations of the so-called gradient type:

$$dx_i(t) = -\sum_{j, i\neq j} \nabla\phi(x_i(t)-x_j(t))dt + \sqrt{\frac{2}{\beta}}\; dw_i(t). \tag{1.1}$$

Here $\phi:\mathbb{R}^d\backslash\{0\}\to\mathbb{R}$ $(\phi(x)=\phi(-x))$ is an interaction potential, $w_i(t)$ are independent standard Wiener processes in $\mathbb{R}^d$ and the parameter $\beta>0$ is the inverse temperature of the system. The physical background and motivation can be found in the article by Spohn[1] and references therein. The set of positions $\{x_i\}_{i\in\mathbb{N}}$ of identical particles is a locally finite subset in $\mathbb{R}^d$ and the set of all such subsets is the *configuration space* Γ:

$$\Gamma := \{\gamma\subset\mathbb{R}^d \quad | \quad |\gamma\cap K|<\infty \quad \textit{for any compact} \quad K\subset\mathbb{R}^d\},$$

where $|A|$ is the cardinality of A. Heuristically, any Gibbs measure μ on Γ corresponding to the interaction ϕ and the inverse temperature β is a stationary measure of the Markov process defined by (1.1). The corresponding Markov generator can be calculated by Ito's formula and defined in $L^2(\Gamma,\mu)$ on some domain of smooth cylinder functions F by the following expression:

$$(HF)(\gamma) = \sum_{x\in\gamma}\left(-\frac{1}{\beta}\Delta_x + \nabla_x U_\phi(\gamma)\cdot\nabla_x\right)F(\gamma), \tag{1.2}$$

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where

$$U_\phi(\gamma)=\sum_{\{x,y\}\subset\gamma}\phi(x-y),\quad \nabla_x U_\phi(\gamma)=\sum_{y\in\gamma\setminus x}\nabla\phi(x-y),\quad x\in\gamma. \tag{1.3}$$

Under some natural restrictions on the class of interaction potentials ϕ the generator H has a self-adjoint extension in $L^2(\Gamma,\mu)$ (see Ref. 2).

A rigorous study of (1.1) has been initiated by Lang[3] who has proved the existence of the so-called equilibrium stochastic dynamics which corresponds to (1.1) for a superstable, three times continuously differentiable potential with finite range. In more recent works by Osada,[4] Yoshida[5] and Albeverio *et al.*[2] the equilibrium stochastic dynamics was constructed by Dirichlet form methods for a wide class of potentials ϕ. The existence of the nonequilibrium dynamics was proved by Rost[6] and Lippner[7] in the one-dimensional case and by Fritz[8] for smooth superstable finite range potentials in the case $d\leqslant 4$.

To construct the nonequilibrium dynamics one can consider the corresponding semigroup $T_t=e^{-t\tilde{H}}$ on some class $\mathcal{F}(\Gamma)$ of observables $F:\Gamma\to\mathbb{R}$ defined by the Kolmogorov equation

$$\frac{\partial F_t}{\partial t}=-\tilde{H}F_t,\quad F_0\in\mathcal{F}(\Gamma), \tag{1.4}$$

where $\tilde{H}$ is the Friedrichs extension of H on $L^2(\Gamma,\mu)$ for some fixed Gibbs measure μ. On the other hand, instead of the evolution of observables one can consider the evolution of states, i.e., the evolution of probability measures on Γ. Such evolution is defined by the adjoint semigroup via the following equation:

$$\frac{d}{dt}\mu_t=-H^*\mu_t. \tag{1.5}$$

In the case of a finite particle system this equation can be rewritten in terms of the densities $\mathcal{D}(t,\gamma)$ w.r.t. Lebesgue measure $d\gamma=dx_1\cdots dx_N$ $(|\gamma|=N<\infty)$. Then (1.5) is, sometimes, called the generalized Smoluchowski equation (see, e.g., Ref. 9).

For infinite particle systems initial states μ_0 are not absolutely continuous w.r.t. any standard measure and the time evolution of densities has no rigorous sense. Below we consider an alternative approach in terms of correlation functions which correspond to the states of the system. To define these correlation functions we introduce the space of *finite configurations* Γ_0:

$$\Gamma_0:=\bigcup_{n\in\mathbb{N}_0}\Gamma^{(n)},\quad \Gamma^{(n)}:=\{\gamma\in\Gamma \quad|\quad |\gamma|=n\},\quad \mathbb{N}_0=\mathbb{N}\cup\{0\}. \tag{1.6}$$

Γ_0 is naturally equipped with the Borel σ-algebra $\mathfrak{B}(\Gamma_0)$ given by the disjoint union of the measurable spaces $(\Gamma^{(n)},\mathfrak{B}(\Gamma^{(n)}))$. For any bounded $Y\in\mathfrak{B}(\mathbb{R}^d)$ the topology of

$$\Gamma_Y^{(n)}:=\{\gamma\in\Gamma \quad|\quad \gamma\cap(\mathbb{R}^d\setminus Y)=\emptyset,\quad |\gamma|=n\}$$

is induced by the bijection between $\Gamma_Y^{(n)}$ and the symmetrization $\tilde{Y}^n/S_n$ of $\tilde{Y}^n$ (see Ref. 10 for details), where S_n is the permutation group over $\{1,\ldots,n\}$,

$$\tilde{Y}^n:=\{(x_1,\ldots,x_n) \quad|\quad x_i\in Y,\quad x_i\neq x_j, i\neq j\},$$

and we denote by $\Gamma_Y=\cup_{n=0}^\infty\Gamma_Y^{(n)}$ the set of configurations in Y.

Starting with an intensity measure $\sigma=z\,dx$ $(z>0)$ on $\mathfrak{B}(\mathbb{R}^d)$ we introduce the product-measure $\sigma^{\otimes n}$ on $(\mathbb{R}^d,\mathfrak{B}(\mathbb{R}^{dn}))$ and denote $\sigma^{(n)}:=\sigma^{\otimes n}\circ(s_n)^{-1}$, where s_n is the map $s_n:\mathbb{R}^{dn}\ni(x_1,\ldots,x_n)\mapsto\{x_1,\ldots,x_n\}\in\Gamma^{(n)}$. The Lebesgue–Poisson measure λ_σ on $\mathfrak{B}(\Gamma_0)$ is defined by the formula

$$\lambda_\sigma := \sum_{n\geq 0} \frac{1}{n!} \sigma^{(n)}. \tag{1.7}$$

Definition 1.1: Let $G:\Gamma_0 \to \mathbb{R}$ *be a measurable function with local support [i.e., there exists a bounded* $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$ *such that* $G \upharpoonright (\Gamma_0 \backslash \Gamma_\Lambda) = 0$*]. Define the function* KG: $\Gamma \to \mathbb{R}$ *as*

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta). \tag{1.8}$$

The summation in (1.8) is taken over all finite subconfigurations $\eta \subset \gamma$.

Remark 1.1: Functions on Γ can be considered as *observables* of our infinite particle system, and functions on Γ_0 can be interpreted as *quasi-observables*. The mapping (1.8) was introduced by Lenard[11] in order to give an abstract definition of the correlation functions in classical statistical mechanics. For a detailed study of properties of the K-transform in the framework of harmonic analysis on configuration spaces we refer to Refs. 12–14.

For a given probability measure μ on $\mathfrak{B}(\Gamma)$ one can define the correlation measure ρ_μ on $\mathfrak{B}(\Gamma_0)$ by

$$\rho_\mu(A) := \int_\Gamma (K\mathbb{1}_A)(\gamma)\mu(d\gamma), \tag{1.9}$$

where $\mathbb{1}_A$ is the indicator function of a set $A \in \mathfrak{B}(\Gamma_0)$. Assuming that ρ_μ is absolutely continuous w.r.t. λ_σ we can define the correlation functional

$$k(\eta) = k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_\sigma}(\eta). \tag{1.10}$$

In statistical physics it is useful to work with the corresponding family of correlation functions

$$k^{(n)} := k \upharpoonright \Gamma^{(n)}, \quad n \geq 0. \tag{1.11}$$

Under certain general conditions on the interaction potential the correlation functions $k^{(n)} = k^{(n)}(x_1, \ldots, x_n) := k^{(n)}(x)_n$ are bounded measurable functions on some Banach space (for example, E_ξ in Ref. 15). For any $G \in L^1(\Gamma_0, \rho_\mu)$ the following formula is true (see Ref. 12 for details):

$$\int_\Gamma (KG)(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\rho_\mu(d\eta) = \int_{\Gamma_0} G(\eta)k(\eta)\lambda_\sigma(d\eta). \tag{1.12}$$

To construct the dynamics for correlation functions, let us consider the K-transform of the generator H which is defined by

$$\hat{H} = K^{-1}HK \tag{1.13}$$

on a proper set $\mathcal{F}_0(\Gamma_0)$ of functions on Γ_0 (quasi-observables). The corresponding evolution of quasi-observables is given then by the following equation:

$$\frac{\partial G_t}{\partial t} = -\hat{H}G_t, \quad G_0 \in \mathcal{F}_0(\Gamma_0). \tag{1.14}$$

We can define the time evolution of correlation functions via the duality relation:

$$\int_{\Gamma_0} G_t(\eta)k_0(\eta)\lambda_\sigma(d\eta) = \int_{\Gamma_0} G_0(\eta)k_t(\eta)\lambda_\sigma(d\eta). \tag{1.15}$$

Then the adjoint operator $(\hat{H})^* =: H_B$ in $L^2(\Gamma_0,\lambda_\sigma)$ is the generator of the evolution semigroup for the correlation functional, i.e.,

$$\frac{\partial}{\partial t}k_t = -H_B k_t. \tag{1.16}$$

Using (1.2) and (1.13)–(1.15) we can show that (1.16) has the following form (see Ref. 14):

$$\frac{\partial}{\partial t}k_t(\eta) = \sum_{x\in\eta} \nabla_x\left[\frac{1}{\beta}\nabla_x k_t(\eta) + \nabla_x U_\phi(\eta)k_t(\eta) + \int dx' \nabla\phi(x-x')k_t(x'\cup\eta)\right], \quad \eta\in\Gamma_0. \tag{1.17}$$

In terms of the correlation functions this equation has the following hierarchical structure:

$$\begin{aligned}\frac{\partial}{\partial t}k_t^{(m)}(x)_m = {}& \frac{1}{\beta}\sum_{j=1}^m \Delta_{x_j}k_t^{(m)}(x)_m + \sum_{j=1}^m \sum_{k\neq j}^m ((\nabla\phi)(x_j-x_k)\nabla_{x_j} + (\Delta\phi)(x_j-x_k))k_t^{(m)}(x)_m \\ &+ \sum_{j=1}^m \int_{\mathbb{R}^d} dx'[(\nabla\phi)(x_j-x')\nabla_{x_j} + (\Delta\phi)(x_j-x')]k_t^{(m+1)}((x)_m,x').\end{aligned} \tag{1.18}$$

This equation appeared for the first time on a heuristic level in Ref. 16. In the present context the hierarchy (1.16)–(1.18) is a direct consequence of the Kolmogorov equation (1.4). It is called sometimes the Bogoliubov diffusion hierarchy and it is analogous to the BBGKY hierarchy for Hamiltonian dynamics. Note that some approaches to investigating this chain of equations in the case of a smooth interaction potential ϕ were proposed in Refs. 17–20.

The problem of existence of solutions of hierarchy (1.18) is additionally complicated by the fact that one should check that the obtained solution k_t corresponds to some state μ_t. Otherwise we cannot prove that we construct the evolution of some initial state μ_0. This problem was not discussed in Refs. 17–20. More precisely, for regular types of interaction potentials, low particle density and sufficiently small interval of time evolution the solution of the diffusion hierarchy (1.18) in the thermodynamic limit was obtained without any analysis of the existence of the dynamics for the corresponding states. In this article we obtain the existence of state μ_t using a general theorem about the connection between states and positive-definiteness of the corresponding correlations functions.[14,21]

Remark 1.2: Note that in theoretical physics a hierarchical system of equations is accepted very often as the definition of the dynamics of an infinite particle system. Such situation takes place, e.g., in Hamiltonian dynamics where the BBGKY hierarchy is considered as the definition of the evolution. Let us mention that, in general, connections between the BBGKY hierarchy approach and the state evolution are not investigated enough as pointed out in Ref. 22, Sec. 3.3. From the physical point of view the property of positivity for correlation functions is very important. But it is not enough to reconstruct the corresponding state.[11] A constructive condition which guarantees such reconstruction was proposed in Ref. 12 (see also Ref. 21). This is the positive-definiteness of the sequence of correlation functions [see below (2.25)–(2.27)]. The situation is the same as in the classical problem of momentum (see, e.g., Ref. 23). From this point of view many results on existence of solutions of the BBGKY hierarchy (see, e.g., Refs. 24 and 25) should be completed by a proof for the existence of the dynamics of states for every particular class of models, as it has been done for stationary solutions in Ref. 26 and for the one-dimensional nonstationary case in Ref. 27.

In this article we consider some class of singular superstable interactions (see Sec. III). Our main strategy is based on a construction of the semigroup p_B^t which corresponds to the evolution equation (1.16). For appropriate initial data k_0 it provides a global solution to the diffusion hierarchy (1.16)–(1.18):

$$k_t(\eta)=(p_B^t k_0)(\eta). \tag{1.19}$$

We obtain an expression for the operator p_B^t which can be easily defined on the Banach space $L_\beta^1(\Gamma_0,\lambda_{\xi\sigma})$ of $\lambda_{\xi\sigma}$-integrable functionals for some appropriate weight ξ (see Sec. II). But for most cases, which are interesting from a physical point of view, the correlation functionals of initial particle distributions do not belong to $L_\beta^1(\Gamma_0,\lambda_{\xi\sigma})$. Typically, they are only bounded and we need to extend the domain of the operator p_B^t to some class of bounded functionals. But from our point of view it would be very naive to hope that a global solution to the hierarchy (1.16)–(1.18) can be obtained for all initial data. Even in the case of a finite number of particles the singularity of the interaction potential does not allow us to define the evolution of arbitrary initial state. Generally speaking one can expect existence of a solution only if the initial correlation functions are chosen in a proper way. In this article we consider a class of initial functionals which correspond to some perturbations of the equilibrium state $\mu=\mu_\phi$ which is constructed by ϕ. The idea is that, as the equilibrium state is a perturbation of the free state (Poisson ideal gas), the nonequilibrium state should be some perturbation of the equilibrium Gibbs state. Such a choice of the initial state is very natural from a physical point of view and were used by many authors. Our main result (Theorem 4.1) is the following. Consider an initial state μ_0 which corresponds to a superstable potential

$$V=\phi+\psi, \tag{1.20}$$

where ϕ is the potential by which our dynamics (1.1) is governed and ψ is a superstable, lower regular interaction potential (see Sec. III). For the given measure μ_0 we consider a family of finite volume measures μ_0^Λ [for bounded $\Lambda\in\mathfrak{B}(\mathbb{R}^d)$] and the corresponding family of initial correlation functionals $k_0^\Lambda(\eta)$:[15]

$$k_0^\Lambda(\eta)=\int_{\Gamma_0}\lambda_\sigma(d\gamma)\mathcal{D}_0^\Lambda(\eta\cup\gamma) \tag{1.21}$$

with

$$\mathcal{D}_0^\Lambda(\eta)=Z_\Lambda^{-1}\mathbb{1}_{\Gamma_\Lambda}(\eta)e^{-\beta U_V(\eta)},\quad \eta\in\Gamma_0, \tag{1.22}$$

$$Z_\Lambda=\int_{\Gamma_0}\lambda_\sigma(d\eta)\mathbb{1}_{\Gamma_\Lambda}(\eta)e^{-\beta U_V(\eta)}. \tag{1.23}$$

For such initial correlation functionals we construct a solution of (1.16) as

$$k_t^\Lambda(\eta)=(p_B^t k_0^\Lambda)(\eta). \tag{1.24}$$

Our main technical result consists in the proof of a bound, uniform in Λ, for these correlation functionals:

$$k_t^\Lambda(\eta)\leqslant c_1^{|\eta|},\quad \eta\in\Gamma_0\quad c_1=c_1(z,\beta,T),\quad t\in[0,T], \tag{1.25}$$

for any time interval $[0,T]$. We obtain this result using the well-known technique of superstability estimates in classical statistical mechanics[28] and its generalization to the quantum case with Boltzman statistics.[29]

In our case this technique needs some modification. We should also note that as in Ref. 29 we need the restriction $d\leqslant 3$ on the dimension d of the system. It is connected with estimating the contribution of long Wiener trajectories in a functional integral representation for the correlation functions. Using (1.25) and the continuity in time (uniformly in Λ) [see (4.27)] we conclude that there exists a thermodynamic limit for k_t^Λ and these limit functionals k_t satisfy the equation (1.16) in a weak sense.

The structure of this paper is as follows: In Sec. II we construct the operator p_B^t and derive a representation for k_t^Λ. In Sec. III we discuss the class of interactions. In Sec. IV we recall some auxiliary constructions and formulate our main result. The proof of the main theorem is presented in Sec. V. The basic technical lemmas are outlined in the Appendix.

II. CORRELATION FUNCTIONS

In this section we construct the semigroup connected with the diffusion hierarchy (1.16)–(1.18) and derive a representation for the finite volume correlation functions (1.24). Let us define an operator K_0 on quasi-observables as

$$K_0 G := (KG) \upharpoonright \Gamma_0. \tag{2.1}$$

Then we can construct the operator

$$H_F := K_0 \hat{H} K_0^{-1}. \tag{2.2}$$

On smooth quasi-observables the operator H_F acts by the following formulas

$$H_F G = ((H_F G)^{(n)}(x)_n)_{n=0}^{\infty}, \quad G = ((G)^{(n)}(x)_n)_{n=0}^{\infty}, \tag{2.3}$$

where

$$(H_F G)^{(n)}(x)_n = (H_F^{(n)} G^{(n)})(x)_n = \sum_{j=1}^{n} \left(-\frac{1}{\beta} \Delta_{x_j} + \nabla_{x_j} U_\phi(x)_n \cdot \nabla_{x_j} \right) G^{(n)}(x)_n. \tag{2.4}$$

$H_F^{(n)}$ is the generator of the stochastic dynamics for an n-particle system:

$$dx_i(t) = -\sum_{i \neq j=1}^{n} \nabla \phi(x_i(t) - x_j(t)) dt + \sqrt{\frac{2}{\beta}} \quad dw_i(t), \quad i = 1, \ldots, n. \tag{2.5}$$

The problem of existence of the stochastic dynamics (2.5) with a singular potential is analyzed in Refs. 30 and 31. The operator $H_F^{(n)}$ is generated by the Dirichlet form

$$(H_F^{(n)} G^{(n)}, G^{(n)})_{L^2(\Gamma_0^{(n)}, \mu_\phi^{(n)})} = \frac{1}{\beta} \sum_{j=1}^{n} \int_{\Gamma_0^{(n)}} |\nabla_j G^{(n)}|^2 d\mu_\phi^{(n)}, \tag{2.6}$$

where

$$\mu_\phi^{(n)} := e^{-\beta U_\phi(\cdot)_n} \sigma^{(n)} \tag{2.7}$$

and $\sigma^{(n)}$ is defined in (1.7).

Then using (1.14)–(1.15) and (2.1)–(2.2) one can write for p_B^t the following representation:

$$p_B^t = e^{-tH_B} = K_0^* e^{-tH_F^*} (K_0^{-1})^* = D e^{-tH_F^*} D^{-1}. \tag{2.8}$$

Here the operators $D := K_0^*$ and D^{-1} are defined in $L^1(\Gamma_0, \lambda_{\xi\sigma})$, $\xi > 2$, by the following formulas (see Ref. 12 for details):

$$(DF)(\eta) = \int_{\Gamma_0} \lambda_\sigma(d\gamma) F(\eta \cup \gamma), \tag{2.9}$$

$$(D^{-1}F)(\eta) = \int_{\Gamma_0} \lambda_\sigma(d\gamma)(-1)^{|\gamma|} F(\eta \cup \gamma). \tag{2.10}$$

Remark 2.1: In our approach the representation (2.8) for p_B^t is a direct consequence of the Kolmogorov equation (1.4) without any reference to the hierarchical structure of (1.17) and (1.18). In the same way from the Liouville equation one can obtain the corresponding representation for the Hamiltonian dynamics.[14] Representations like (2.8) appeared earlier. They were obtained by application of the method of ''creation'' and ''annihilation'' operators for classical statistical mechanics (see Refs. 32–35). For connections with the diffusion hierarchy see Ref. 19.

Remark 2.2: It will be clear from the considerations below that in the case of a stable interaction potential ϕ for which $-\Delta\phi$ is also stable, the operator p_B^t can be defined on a Banach space $L_{\beta,\xi}$ with the norm

$$\|G\|_{\beta,\xi}=\int_{\Gamma_0}\lambda_{\xi\sigma}(d\eta)e^{(1/2)\,\beta U_\phi(\eta)}|G(\eta)|<\infty,\quad G\in L^1_{\beta,\xi}. \tag{2.11}$$

To obtain a representation for finite volume correlation functions k_t^Λ we take into account the definition of the operator D and rewrite (1.21) in the form

$$k_0^\Lambda(\eta)=(D\mathcal{D}_0^\Lambda)(\eta). \tag{2.12}$$

Then due to (2.8) and (1.20) the following representation is true:

$$k_t^\Lambda(\eta)=\left(D\left[e^{-tH_F^*}\left(\frac{1}{Z_\Lambda}e^{-\beta U_\phi}e^{-\beta U_\psi}\mathbb{1}_{\Gamma_\Lambda}\right)\right]\right)(\eta)=\left(D\left[\frac{1}{Z_\Lambda}e^{-tH_F}(e^{-\beta U_\psi}\mathbb{1}_{\Gamma_\Lambda})e^{-\beta U_\phi}\right]\right)(\eta), \tag{2.13}$$

where we use the fact that the operator H_F [see (2.4)] is a self-adjoint operator in $L^2(\Gamma_0,e^{-\beta U_\phi}\lambda_\sigma)$.

Now, to get an integral representation for k_t^Λ we use a functional integral representation for the operator e^{-tH_F} in (2.13). First of all, note that the operator H_F has the Fock structure (2.3) and (2.4), so we only need a representation for $e^{-tH_F^{(n)}}$ in $L^2(\Gamma_0^{(n)},\mu_\phi^{(n)})$. To get it we use the well-known ground state transformation (see, e.g., Ref. 36, Sec. 2, and Ref. 23, Chap. 7)

$$L^2(\Gamma_0^{(n)},\mu_\phi^{(n)})\ni f\mapsto e^{-(1/2)\,\beta U_\phi}f\in L^2(\Gamma_0^{(n)},\sigma^{(n)}). \tag{2.14}$$

The corresponding generator $\tilde{H}_F^{(n)}$ in $L^2(\Gamma_0^{(n)},\sigma^{(n)})$ has the following form,

$$\tilde{H}_F^{(n)}=-\frac{1}{\beta}\sum_{j=1}^{n}\Delta_{x_j}+\tilde{V}(x)_n, \tag{2.15}$$

with the effective potential

$$\tilde{V}(x)_n=\tilde{V}^+(x)_n+\tilde{V}^{(-\Delta\phi)}(x)_n:=\sum_{j=1}^{n}\left(\frac{\beta}{4}|\nabla_{x_j}U_\phi(x)_n|^2-\frac{1}{2}\Delta_{x_j}U_\phi(x)_n\right). \tag{2.16}$$

For the domain of the operator $\tilde{H}_F^{(n)}$ we have $\mathfrak{D}(\tilde{H}_F^{(n)})\supset C_0^\infty(\Gamma_0^{(n)})$, where $C_0^\infty(\Gamma_0^{(n)})$ denotes C^∞-functions on $\Gamma_0^{(n)}$ with compact supports. It can be shown that for any superstable potential ϕ the effective potential $\tilde{V}(x)_n$ is bounded from below for any fixed n. But for the class of potentials under consideration it is even superstable (see Sec. III). Therefore, we can apply the Feynman–Kac formula for the kernel of the semigroup $e^{-t\tilde{H}_F^{(n)}}$, where $\tilde{H}_F^{(n)}$ is considered in the sense of a form sum (2.15) Ref. 37, Chap. 2. As a result, we get

$$(e^{-tH_F^{(n)}})((x)_n;(y)_n)=e^{(1/2)\,\beta U_\phi(x)_n-(1/2)\,\beta U_\phi(y)_n}\int_{(\Omega^{t_\beta})^n}\prod_{j=1}^{n}W_{x_j;y_j}^{t_\beta}(d\omega_j)e^{-\beta\int_0^{t_\beta}d\tau\tilde{V}(\omega(\tau))_n}, \tag{2.17}$$

where

$$\Omega^{t_\beta}:=C([0,t_\beta]\to\mathbb{R}^d), \quad t_\beta=t\beta^{-1}, \tag{2.18}$$

$W^{t_\beta}_{x;y}$ is the conditional Wiener measure on the space Ω^{t_β} with conditions $\omega_j(t_\beta)=x_j$ and $\omega_j(0)=y_j$. It implies the following useful representation:

$$k_t^\Lambda(\eta)=\int_{\mathbb{R}^{dm}}d\xi\int_{(\Omega^{t_\beta})^m}W^{t_\beta}_{\eta;\xi}(d\omega)_m\rho_t^\Lambda(\omega)_m, \tag{2.19}$$

where $\eta=\{x_1,\ldots,x_m\}$, $\xi=\{y_1,\ldots,y_m\}$, $d\xi=dy_1\cdots dy_m$, and

$$\rho_t^\Lambda(\omega)_m=\sum_{n\geqslant 0}\frac{1}{n!}\int\tilde{\sigma}_\Lambda(\omega_{m+1})\cdots\int\tilde{\sigma}_\Lambda(\omega_{m+n})\tilde{D}_0^\Lambda(\omega(0))_{m+n}e^{-\tilde{U}(\omega)_{m+n}}. \tag{2.20}$$

In (2.20) we use the following notations:

$$\tilde{U}(\omega)_{m+n}:=\frac{1}{2}\beta U_\phi(x)_{m+n}+\int_0^{t_\beta}d\tau[U_{-\Delta\phi}(\omega(\tau))_{m+n}+U^+_{\nabla\phi}(\omega(\tau))_{m+n}], \tag{2.21}$$

$$U^+_{\nabla\phi}(x)_{m+n}:=\sum_{j=1}^{m+n}\left[\frac{1}{4}\beta|\nabla_{x_j}U_\phi(x)_{m+n}|^2\right]\geqslant 0, \tag{2.22}$$

$$\tilde{D}_0^\Lambda(\omega(0))_{m+n}:=e^{(1/2)\,\beta U_\phi(\omega(0))_{m+n}}\mathcal{D}_0^\Lambda(\omega(0))_{m+n}, \tag{2.23}$$

and

$$\int\tilde{\sigma}_\Lambda(d\omega)(\cdots)=z\int dx\int_\Lambda dy\int W^{t_\beta}_{x;y}(d\omega)(\cdots). \tag{2.24}$$

A representation like (2.19) and (2.20) was obtained also in Refs. 17 and 18 as the generalized solution of the finite volume diffusion hierarchy (1.18).

Remark 2.3: It is not hard to show that the sequence $k_t^\Lambda(\eta)$ is positive-definite in the sense of Refs. 12 and 13, which is the following.

Definition 2.1:[13] The sequence $k_t^\Lambda(\eta)$ is positive-definite if

$$\int_{\Gamma_0}(G\star\overline{G})(\eta)k_t^\Lambda(\eta)\lambda_\sigma(d\eta)\geqslant 0 \quad \textit{for} \quad \textit{all} \;\; G\in B_{bs}(\Gamma_0). \tag{2.25}$$

Here $B_{bs}(\Gamma_0)$ is the set of all bounded measurable functions with bounded support, $\overline{G}$ denotes the complex conjugate of G, and $\star$-star is the convolution, which is defined in the following way:

$$K(G_1\star G_2)=KG_1\cdot KG_2. \tag{2.26}$$

(See Ref. 13 for details.)

Now, inserting (1.24) into (2.25) and using the representation (2.8), (2.12), (2.17) and the property (2.26) (which is true, also, for K_0 [see (2.1)] we obtain

$$\int_{\Gamma_0}(G\star\overline{G})(\eta)p_t D\mathcal{D}_0^{\Lambda}(\eta)\lambda_\sigma(d\eta)$$

$$=\int_{\Gamma_0}(G\star\overline{G})(\eta)K_0^* e^{-tH_F^*}D^{-1}D\mathcal{D}_0^{\Lambda}(\eta)\lambda_\sigma(d\eta)$$

$$=\int_{\Gamma_0}e^{-tH_F}(K_0G)(\eta)(K_0\overline{G})(\eta)\mathcal{D}_0^{\Lambda}(\eta)\lambda_\sigma(\eta)$$

$$=\int_{\Gamma_0}\lambda_\sigma(d\eta)\int_{(\mathbb{R}^d)^{|\eta|}}d\eta'(e^{-tH_F^{|\eta|}})(\eta|\eta')|(K_0G)(\eta')|^2\mathcal{D}_0^{\Lambda}(\eta)\geqslant 0 \tag{2.27}$$

Positive definiteness together with the bound (1.25) gives a possibility to reconstruct the corresponding sequence of states (measures) μ_t^{Λ}.

Representation (2.19) and (2.20) is reminiscent to the representation of reduced density matrices by correlation functionals in quantum statistical mechanics with Boltzman statistics (see Refs. 38 and 39). This analogy enables us to apply powerful techniques from quantum statistical mechanics in the considered model. Following Ref. 40, we construct the configuration space $\Gamma_{\Omega^{t_\beta}}$ over the space Ω^{t_β} of Wiener trajectories in $\mathbb{R}^d$. Define configuration $\tilde{\gamma}$ as the infinite set of trajectories $\omega\in\Omega^{t_\beta}$ such that the set of values of these trajectories at time $\tau=0$ is a configuration $\gamma^0\in\Gamma$. Then define the configuration space of trajectories with initial points $\omega(0)$ in $\Lambda\in\mathfrak{B}_c(\mathbb{R}^d)$:

$$\Gamma_{\Omega_\Lambda^{t_\beta}}:=\{\tilde{\gamma}\in\Gamma_{\Omega^{t_\beta}}|\gamma^0\in\Gamma_\Lambda\}.$$

In the same way the Lebesgue–Poisson measure $\lambda_{\tilde{\sigma}}^{\Lambda}$ with intensity measure $\tilde{\sigma}_\Lambda$ is defined by (1.7). Then for the "correlation functionals" $\rho_t^{\Lambda}(\tilde{\eta})$, $\tilde{\eta}\in\Gamma_{\Omega_\Lambda^{t_\beta}}$, the following representation is true:

$$\rho_t^{\Lambda}(\tilde{\eta})=\int_{\Gamma_{\Omega_\Lambda^{t_\beta}}}\lambda_{\tilde{\sigma}}^{\Lambda}(d\tilde{\gamma})\tilde{D}_0^{\Lambda}(\eta^0\cup\gamma^0)e^{-\tilde{U}(\tilde{\eta}\cup\tilde{\gamma})}, \tag{2.28}$$

where $\tilde{U}(\tilde{\eta}\cup\tilde{\gamma})$ is defined by (2.21) with $\tilde{\eta}\cup\tilde{\gamma}=\{\omega_1,\ldots,\omega_{m+n}\}$ and $\tilde{D}_0^{\Lambda}$ as defined in (2.23).

Remark 2.4: In the following we write $\eta,\gamma,\xi,\ldots,\Omega,\Omega_\Lambda$, instead of $\tilde{\eta},\tilde{\gamma},\tilde{\xi},\ldots,\Omega^{t_\beta},\Omega_\Lambda^{t_\beta}$ and $\eta^\tau,\gamma^\tau,\xi^\tau,\ldots,\ldots,\tau\in[0,t_\beta]$ for the sets of values of the corresponding configurations at time τ. Note that these sets for $\tau>0$ are not configurations in Γ_0 because some points of their values can coincide (intersection of trajectories).

III. THE CLASS OF INTERACTION POTENTIALS

In this section we describe a class of interaction potentials which allow us to solve the problem formulated in the Introduction. As it was mentioned in Remark 2.2, even to define the operator p_B^t on $L^1_{\beta,\xi}$ we have to consider a rather narrow class of interaction potentials. The restrictions on the potential imposed below are, however, rather dictated by the technique of the superstability estimates for functional integrals. We hope that the stochastic dynamics, actually, exists for a more wide class of potentials and initial states. This point of view is supported by the fact that the equilibrium stochastic dynamics exists for a wide class of physically reasonable potentials (see, e.g., Ref. 2).

To define the said class of superstable interactions we denote by $\overline{\Delta}$ a partition of $\mathbb{R}^d$ into half open unit cubes Δ centered at the points $r\in\mathbb{Z}^d$ (see Ref. 28 for details):

$$\mathbb{R}^d=\bigcup_{\Delta\subset\overline{\Delta}}\Delta,\quad\forall\Delta,\Delta'\in\overline{\Delta},\quad\Delta\cap\Delta'=\emptyset.$$

We assume the following conditions to hold:

(A1) Smoothness:

$$\phi,\quad \psi\in C^3(\mathbb{R}^d\setminus\{0\}).$$

(A2) Superstability:

$$\forall i=1,2,3\quad \exists A_i>0,\quad B_i\geqslant 0:\quad U_{v_i}(\gamma)\geqslant \sum_{\Delta\in\bar{\Delta}}(A_i|\gamma_\Delta|^2-B_i|\gamma_\Delta|),$$

with $v_1=\phi$, $v_2=-\Delta\phi$ and $v_3=\psi$.

(A3) Lower-regularity:

For any $X,Y\subset\mathbb{R}^d$ and configurations $\gamma_X\in\Gamma_X$, $\gamma_Y\in\Gamma_Y$, define

$$W_{v_i}(\gamma_X|\gamma_Y)=\sum_{x\in\gamma_X,y\in\gamma_Y}v_i(x-y).\tag{3.1}$$

Then

$$-W_{v_i}(\gamma_X|\gamma_Y)\leqslant\sum_{\Delta,\Delta'\in\bar{\Delta}}\Psi_{v_i}(\Delta,\Delta')|\gamma_\Delta||\gamma_{\Delta'}|,\tag{3.2}$$

where

$$\Psi_v(\Delta,\Delta')=\sup_{x\in\Delta,x'\in\Delta'}v_-(x-x'),$$

and $v_-=-\min(v,0)$. We also require the existence of positive decreasing functions $\Psi_i(k)$ on positive integers such that

$$\Psi_i(k)\geqslant\sup_{\Delta,\Delta'\in\bar{\Delta};d(\Delta,\Delta')=k}\Psi_{v_i}(\Delta,\Delta')\tag{3.3}$$

and

$$\sum_{k=0}^{\infty}\Psi_i(k)k^{d+\mu_i-1}=F_{\mu_i}<+\infty\tag{3.4}$$

with $\mu_1=\mu_3>\frac{1}{2}$, $\mu_2>-\frac{3}{2}$ and

$$d(\Delta,\Delta')=\max_{1\leqslant\alpha\leqslant d}\inf_{x\in\Delta,x'\in\Delta'}|x^{(\alpha)}-x'^{(\alpha)}|.$$

Now we describe some class Φ of potentials which satisfy (A.1)–(A.3). Let

$$\begin{gathered}\phi(x)=\phi_+(x)+\phi_{st}(x),\\ \phi_+(0)=+\infty,\quad \phi_+(x)>0,\end{gathered}\tag{3.5}$$

$$\begin{gathered}-\Delta\phi_+(0)=+\infty,\quad -\Delta\phi_+(x)>0,\\ \phi_{st}(x)=\frac{1}{(2\pi)^d}\int dk e^{ixk}\tilde{\phi}(k),\quad \phi(k)\geqslant 0,\end{gathered}\tag{3.6}$$

$$\int dk\tilde{\phi}(k)<\infty, \quad \int dkk^2\tilde{\phi}(k)<\infty. \tag{3.7}$$

As an example (for $d=3$) of such a potential we can choose some function which has an asymptotic behavior near the origin like

$$\phi_+(x)\sim\frac{C_1}{|x|^{\alpha}}, \quad 0<\alpha\leqslant 1 \quad \textit{for} \quad |x|<r_0, \quad r_0>0,$$

and is sufficiently fast decreasing as $|x|\nearrow+\infty$. This is clear from a direct calculation which gives

$$-\Delta\frac{1}{|x|^{\alpha}}=\frac{\alpha(1-\alpha)}{|x|^{\alpha+2}}, \tag{3.8}$$

which is positive for $\alpha<1$.

IV. RUELLE'S CONSTRUCTIONS. MAIN RESULT

Our main technical tool is the technique of superstability estimates proposed by Ruell[28] for classical statistical mechanics. Later Esposito *et al.*[29] generalized this technique for the case of quantum statistical mechanics and proved the boundedness of the reduced density matrices (RDMs) for the Maxwell–Boltzmann statistics.

In this section we briefly recall some basic constructions which were made in Refs. 28 and 29.

A. Λ_q-cubs

For some $\alpha>0$ (to be fixed later) let

$$l_q=[e^{\alpha q}], \quad q\in\mathbb{N}, \tag{4.1}$$

where $[x]$ is the integer part of $x\in\mathbb{R}^+$ and

$$\Lambda_q=[-l_q-\tfrac{1}{2},l_q+\tfrac{1}{2}]^d, \quad |\Lambda_q|=(2l_q+1)^d. \tag{4.2}$$

We also set the origin in the center of some cube $\Delta_1\in\overline{\Delta}$. So, every cube Λ_q is the union of cubes Δ from $\overline{\Delta}$. And for convenience we suppose that for one of the trajectories $\omega\in\eta$ we have $\omega(0)=0\in\Delta_1\subset\mathbb{R}^d$. Following Ref. 29 we also introduce the sequence

$$\varphi(q)=q|\Lambda_q|. \tag{4.3}$$

We extensively use the following properties (see Ref. 29 for details). For given $\varepsilon>0$, $\alpha>0$ $\exists s_0$ such that for $s\geqslant s_0$

$$1+\alpha<\frac{l_{s+1}}{l_s}<e^{\alpha(1+\varepsilon)},$$

$$\frac{\varphi(s+1)}{\varphi(s)}<e^{\alpha(d+\varepsilon(d+1))}, \tag{4.4}$$

or for $\varepsilon<\varepsilon_0=1-2\alpha e^{2\alpha}$ (α is sufficiently small)

$$\frac{l_{s+1}}{l_s}<1+2\alpha,$$

$$\frac{\varphi(s+1)}{\varphi(s)}<(1+2\alpha)^{d+1}. \tag{4.5}$$

We also need the following lemma:

Lemma 4.1 (see Ref. 29): Let

$$r(k)=\min\{r\in\mathbb{N}|l_{q+r}>l_q+k,\quad \forall q\geqslant 1,\quad k\in\mathbb{N}\}. \tag{4.6}$$

Then

$$r(k)\leqslant 1+\frac{1}{\alpha}\log(k+2). \tag{4.7}$$

B. An extension of the functionals $\rho_t^\Lambda(\eta)$

The functionals ρ_t^Λ are defined on the configuration space Γ_{Ω_Λ}, where Ω_Λ is the space of continuous trajectories $\omega(\tau)$ [see (2.18)] with $\omega(0)\in\Lambda$. But in the construction we are going to apply we need to consider functionals ρ_t^Λ on the trajectories which take values in the bounded cube Λ_q or in its compliment Λ_q^c. In this case some trajectories can be discontinuous because they take their values in Λ_q not for all $\tau\in[0,t_\beta]$ but only in some intervals $[\tau_1,\tau_2]$, $[\tau_3,\tau_4]$, etc. So for $\tau\in(\tau_2,\tau_3),\ldots$ they are not defined. In this case the definition of $\tilde{U}(\eta_{\Lambda_q}\cup\gamma)$ and therefore $\rho_t^\Lambda(\eta_{\Lambda_q})$ becomes ambiguous. To avoid these difficulties we repeat the construction proposed in Ref. 29.

Let $\mathfrak{B}([0,t_\beta])$ be the σ-algebra of Borel sets in $[0,t_\beta]$ with Ω_B the set of all measurable functions:

$$\tilde{\omega}:B\to\mathbb{R}^d,\quad B\in\mathfrak{B}([0,t_\beta]). \tag{4.8}$$

Now we define a new configuration space by

$$\Gamma_{\tilde{\Omega}}=\bigcup_{n\geqslant 0}\Gamma_{\tilde{\Omega}}^{(n)}, \tag{4.9}$$

where for $n=0$ $\Gamma_{\tilde{\Omega}}^{(0)}$ is a nonempty set which, however, consists of the trajectories $\tilde{\omega}$ whose domain B have zero Lebesgue measure and

$$\Gamma_{\tilde{\Omega}}^{(n)}=\tilde{\Omega}_1^{\otimes n-symm},\quad \tilde{\Omega}_1=\bigcup_{B\in\mathfrak{B}([0,t_\beta])}\Omega_B. \tag{4.10}$$

Then, instead of $\tilde{U}(\omega)_{m+n}=\tilde{U}(\eta\cup\gamma)$, $\eta=\{\omega_1,\ldots,\omega_m\}$, $\gamma=\{\omega_{m+1},\ldots,\omega_{m+n}\}$ we define $\tilde{U}(\tilde{\eta}\cup\gamma)$ by the same formula (2.21) and (2.22), but instead of $U_{v_k}(\omega(\tau))_{m+n}$, $k=1,2$ ($v_1=\phi, v_2=-\Delta\phi$) we set for $\tilde{\omega}_j\in\tilde{\Omega}_1$

$$\sum_{1\leqslant i<j\leqslant m+n}\chi_{D(\tilde{\omega}_i)\cap D(\tilde{\omega}_j)}(\tau)v_k(\tilde{\omega}_i(\tau)-\tilde{\omega}_j(\tau)), \tag{4.11}$$

where $D(\tilde{\omega})$ is the domain of $\tilde{\omega}$. For fixed $\tilde{\eta}\in\Gamma_{\tilde{\Omega}}$ the function

$$\tilde{U}(\tilde{\eta}\cup\cdot):\Gamma_\Omega\to\mathbb{R}$$

is measurable on Γ_Ω w.r.t. the σ-algebra of Borel sets corresponding to the topology of point-wise convergence (see Ref. 29 for details). Finally, we define

$$\tilde{\rho}_t^\Lambda(\tilde{\eta})=\int_{\Gamma_{\Omega_\Lambda}}\lambda_{\tilde{\sigma}}^\Lambda(d\tilde{\gamma})\tilde{D}_0^\Lambda(\tilde{\eta}^0\cup\gamma^0)e^{-\tilde{U}(\tilde{\eta}\cup\gamma)}, \tag{4.12}$$

which is an extension of definition (2.28) [here $\tilde{\eta}$ has a different sense than in (2.28) (see Remark 2.4)].

It is clear that for $\eta \in \Gamma_{\Omega_\Lambda}$

$$\rho_t^\Lambda(\eta) = \tilde{\rho}_t^\Lambda(\tilde{\eta}). \tag{4.13}$$

Then for any bounded measurable $X \subset \Lambda$ we define the map $\pi_X : \Gamma_{\Omega_\Lambda} \to \Gamma_{\tilde{\Omega}_\Lambda}$, such that for any $\gamma \in \Gamma_{\Omega_\Lambda}$

$$\pi_X \gamma = (\pi_X \omega_1, \ldots, \pi_X \omega_{|\gamma|}), \tag{4.14}$$

where $\pi_X \omega_j \in \tilde{\Omega}_\Lambda$ and its domain is the measurable set

$$B = \{\tau \in [0, t_\beta] | \omega(\tau) \in X\}.$$

If B has nonzero Lebesgue measure, then $(\pi_X \omega)(\tau) = \omega(\tau)$, $\tau \in B$, and if B has zero measure, then $\pi_X \omega \in \Gamma_{\tilde{\Omega}}^{(0)}$. We need also the map $s : \Gamma_{\Omega_\Lambda} \to \Gamma_{\tilde{\Omega}_\Lambda}$, which is

$$\gamma \to s(\gamma) = \bigcup_{\Delta \in \overline{\Delta}} \pi_\Delta \gamma. \tag{4.15}$$

The union is taken over all $\Delta \in \overline{\Delta}$, such that $\pi_\Delta \gamma$ has a domain of nonzero Lebesgue measure. It is clear that

$$\tilde{\rho}_t^\Lambda(\tilde{\eta}) = \tilde{\rho}_t^\Lambda(s(\eta)). \tag{4.16}$$

C. Partitions of Γ_{Ω_Λ}

For every $\tau \in [0, t_\beta]$ and a given configuration $\tilde{\eta} \in \Gamma_{\tilde{\Omega}_\Lambda}$ we introduce some characteristics of a given configuration $\gamma \in \Gamma_{\Omega_\Lambda}$:

$$E_q^\tau(\xi) = E_{\Lambda_q}^\tau(\xi) = \sum_{\Delta \subset \Lambda_q} |\xi_\Delta^\tau|^2, \quad \xi = \tilde{\eta} \cup \gamma, \tag{4.17}$$

where $|\xi_\Delta^\tau|$ is the number of all trajectories from ξ which take values in Δ at time τ. We also denote $E^\tau(\xi)$ by the same expression (4.17) with summation over all $\Delta \in \overline{\Delta}$.

Then we define three factors

$$E_q^{(1)}(\xi) = E_q^{t_\beta}(\xi), \tag{4.18}$$

$$E_q^{(2)}(\xi) = \int_0^{t_\beta} d\tau E_q^\tau(\xi), \tag{4.19}$$

$$E_q^{(3)}(\xi) = E_q^0(\xi), \tag{4.20}$$

which correspond to the factors $\exp\{-1/2\beta U_\phi(\tilde{\eta}^{t_\beta} \cup \gamma^{t_\beta}\}$, $\exp\{\beta U_{\Delta\phi}(\tilde{\eta} \cup \gamma)\}$ and $\tilde{D}_0^\Lambda(\tilde{\eta}^0 \cup \gamma^0)$, respectively, to be controlled, and define

$$E_q(\xi) = \sum_{i=1}^{3} E_q^{(i)}(\xi). \tag{4.21}$$

Following Ref. 29 we can now construct a partition of Γ_{Ω_Λ} in the following way. For some large integer q_0 (to be fixed later) we introduce

$$\Gamma_{q_0-1}=\{\gamma\in\Gamma_{\Omega_\Lambda}|E_m(\xi)\leqslant\varphi(m)\quad\text{for all}\quad m\geqslant q_0-1\},\tag{4.22}$$

and for $q\geqslant q_0$

$$\Gamma_q=\{\gamma\in\Gamma_{\Omega_\Lambda}|E_{q-1}(\xi)>\varphi(q-1),\quad\text{but}\quad E_m(\xi)\leqslant\varphi(m)\quad\text{for}\quad m\geqslant q\}.\tag{4.23}$$

Then for $q\geqslant q_0-1$ let $\chi_q(\gamma)$ be an indicator function of the set Γ_q and for the partition

$$\Gamma_{\Omega_\Lambda}=\bigcup_{q\geqslant q_0-1}\Gamma_q\tag{4.24}$$

we consider the partition of the unity:

$$1=\sum_{q\geqslant q_0-1}\chi_q(\gamma).\tag{4.25}$$

D. Main result

Theorem 4.1: *For the interactions* $\phi(x)$ $(x\in\mathbb{R}^d,\ d\leqslant 3)$, *which satisfy (A1)–(A3), and for initial distributions (1.21)–(1.23) there exist constants* $c_1=c_1(z,\beta,T)$ *and* $c_2=c_2(z,\beta,T)$ *such that*

$$k_t^\Lambda(x)_m\leqslant c_1^m,\quad \eta\equiv\{x_1,\ldots,x_m\}\in\Gamma_\Lambda.\tag{4.26}$$

For any $t_1,t_2\in[0,T]$

$$|\langle k_{t_1}^\Lambda,\varphi\rangle-\langle k_{t_2}^\Lambda,\varphi\rangle|\leqslant c_2^m\|\varphi\|_m|t_1-t_2|,\tag{4.27}$$

uniformly in Λ *and* $t\in[0,T]$. *Here*

$$\langle k_t^\Lambda,\varphi\rangle=\int_{\mathbb{R}^{dm}}k_t^\Lambda(x)_m\varphi(x)_m(dx)^m,\tag{4.28}$$

where $(dx)^m=dx_1\cdot\cdots\cdot dx_m$, *for* $(x)_m=\{x_1,\ldots,x_m\}$, $\varphi\in C_0^\infty(\mathbb{R}^{dm})$,

$$c_2^m=\max\{m\xi_1^m\beta^{-1},\quad m(m-1)\xi_1^m,\quad m\xi_1^{m+1}\},\tag{4.29}$$

and

$$\|\varphi\|_m=\max_{1\leqslant j\leqslant m}\|\Delta_j\varphi\|_{L^1(\mathbb{R}^{md})}+\max_{j\neq k}\|\nabla_j\varphi\cdot\nabla_k\phi\|_{L^1(\mathbb{R}^{md})}+\max_j\|\nabla\phi\|_{L^1(\mathbb{R}^d)}\|\nabla_j\varphi\|_{L^1(\mathbb{R}^{md})}.\tag{4.30}$$

Remark 4.1: For potentials $\phi\in\Phi$ it is clear that $\|\nabla\phi\|_{L^1(\mathbb{R}^d)}<\infty$ (see Sec. III).

Remark 4.2: From (4.26), by compactness, we can choose a sequence $(\Lambda_n)_{n=1}^\infty$, $\Lambda_n\subset\Lambda_{n+1}$, $\Lambda_n\nearrow\mathbb{R}^d$, so that we get a limit point (in the weak sense) for $k_t^{\Lambda_n}$. Hence by diagonal argument we obtain the existence of a corresponding limit for any rational $t\in[0,T]$. Then from the continuity property (4.27) the existence of a weak-limit for k_t^Λ follows for all $t\in[0,T]$. And, finally, using positive-definiteness [see Remark 2.3, (2.25)–(2.27)] of k_t^Λ for any $\Lambda\subset\mathbb{R}^d$ we get a limit state μ_t (not unique), such that $K^*\mu_t=\rho_{\mu_t}=k_t\lambda_\sigma$.

V. PROOF OF THE MAIN THEOREM

First we note that neglecting the positive part of the effective potential $\tilde{V}(x)_n$ [see (2.16)] we get

$$\tilde{\rho}^{\Lambda}_{t_\beta}(\tilde{\eta}_\Lambda) \leqslant \tilde{\tilde{\rho}}^{\Lambda}_{t_\beta}(\tilde{\eta}_\Lambda), \tag{5.1}$$

where

$$\tilde{\tilde{\rho}}^{\Lambda}_{t_\beta}(\tilde{\eta}_\Lambda) = \tilde{\rho}^{\Lambda}_{t_\beta}(\tilde{\eta}_\Lambda)|_{\tilde{V}^+\equiv 0}.$$

Then, the main technical point of the proof is the following proposition.

Proposition 5.1: Under the same hypothesis as in Theorem 4.1 there exist a small $\alpha>0$, *a sufficiently large integer* $q_0=q_0(\alpha)$, *constants* $h(q_0),K(q_0)$, *and a positive decreasing function* $\varepsilon(q)$ *such that*

$$\tilde{\tilde{\rho}}^{\Lambda}_{t_\beta}(\tilde{\eta}_\Lambda) \leqslant C_0 e^{-(1/4)\,\beta A E(\eta_{\Lambda_{q_0}})} \tilde{\tilde{\rho}}^{\Lambda}_{t_\beta}(\tilde{\eta}_{\Lambda^c_{q_0}}) + \sum_{q\geqslant q_0} C_q e^{-(1/4)\,\beta A E(\tilde{\eta}_{\Lambda_q})} \tilde{\tilde{\rho}}^{\Lambda}_{t_\beta}(\tilde{\eta}_{\Lambda^c_q}), \tag{5.2}$$

with

$$A=\min\{A_1, \quad 2A_2, \quad A_3\},$$

$$C_0=e^{h(q_0)+K(q_0)}, \quad C_q=e^{-(1/8)\,\beta A\varphi(q-1)+\varepsilon(q)\varphi(q-1)+K(q_0)}, \quad q\geqslant q_0.$$

The proof of Theorem 4.1 follows from the next lemma.

Lemma 5.1: Let

$$S(\tilde{\eta}) = \{\Delta\in\overline{\Delta}\,|\,\exists\,\tau\in[0,t_\beta] \quad \textit{and} \quad \omega\in\tilde{\eta} \quad \textit{such that} \quad \omega(\tau)\in\Delta\}.$$

Then

$$\rho^{\Lambda}_{t_\beta}(\tilde{\eta}) \leqslant e^{-(1/4)\,\beta A E(\tilde{\eta})+\delta|S(\tilde{\eta})|}, \tag{5.3}$$

with $\delta>\log D$, $D=C_0+\Sigma_{q\geqslant q_0}C_q$.

Proof: We shall proceed by induction. Let $\tilde{\eta}'$ be a subconfiguration of $\tilde{\eta}$. We assume that (5.2) is true for any such $\tilde{\eta}'$. Then from (5.2)

$$\tilde{\tilde{\rho}}^{\Lambda}_{t_\beta}(\tilde{\eta}_\Lambda) \leqslant C_0 e^{-AE(\tilde{\eta}_{\Lambda_{q_0}})-AE(\tilde{\eta}_{\Lambda^c_{q_0}})+\delta|S(\tilde{\eta}_{\Lambda^c_{q_0}})|} + \sum_{q\geqslant q_0} C_q e^{-AE(\tilde{\eta}_{\Lambda_q})-AE(\tilde{\eta}_{\Lambda^c_q})+\delta|S(\tilde{\eta}_{\Lambda^c_q})|}$$
$$\leqslant e^{-AE(\tilde{\eta}_\Lambda)+\delta|S(\tilde{\eta}_\Lambda)|},$$

since $|S(\tilde{\eta}_{\Lambda^c_q})| \leqslant |S(\tilde{\eta}_\Lambda)|-1$. Taking into account (5.1) we get (5.3). □

Proof of Theorem 4.1: Using (5.3) from (2.19) we get

$$k^{\Lambda}_t(x)_m \leqslant z^m \prod_{j=1}^{m} \int dy_j \int W^{t_\beta}_{x_j,y_j}(d\omega_j) e^{-AE(\omega_j)+\delta|S(\omega_j)|}.$$

Using the Schwartz inequality, the estimate [see Ref. 29, (A.21)]

$$\int W^{t_\beta}_{x,y}(d\omega) e^{2\delta|S(\omega)|} \leqslant (2\pi t_\beta)^{-d/2} I(2\delta), \quad I(2\delta)=\int W^{t_\beta}_{0,0}(d\omega) e^{2\delta|S(\omega)|},$$

and the trivial estimate

$$\int dy \left(\int W^{t_\beta}_{x,y}(d\omega) \right)^{1/2} \leqslant 2^{3d/4}\pi^{d/4}t^{d/4}$$

we obtain (4.26) with

$$c_1 = z((4\pi)^d I(2\delta))^{1/2}.$$

In the same way one can prove Eq. (4.27). Indeed, we have by (1.18) that

$$\langle k_{t_1}^{\Lambda}, \varphi\rangle - \langle k_{t_2}^{\Lambda}, \varphi\rangle = \int_{t_1}^{t_2} d\tau \left\langle \frac{dk_\tau^\Lambda}{d\tau}, \varphi \right\rangle$$
$$= \int_{t_1}^{t_2} d\tau \sum_{j=1}^{m} [\langle k_\tau^\Lambda, \Delta_j \varphi\rangle - \langle k_\tau^\Lambda, \nabla_j \varphi \cdot \nabla_j U_\phi\rangle - \langle\langle k_\tau^\Lambda, \nabla_j \phi\rangle \nabla_j \varphi\rangle].$$

Using (4.26) we get (4.27) with the constants (4.29) and (4.30). □

Proof of Proposition 5.1: Inserting (4.25) into (4.12) we get

$$\tilde{\tilde{\rho}}_t^{\Lambda}(\tilde{\eta}_\Lambda) = \sum_{q \geq q_0 - 1} I_q(\tilde{\eta}_\Lambda), \tag{5.4}$$

with

$$I_q(\tilde{\eta}_\Lambda) = \int \lambda_{\tilde{\sigma}}^{\Lambda}(d\gamma) \chi_q(\gamma) \tilde{D}_0^{\Lambda}(\tilde{\eta}_\Lambda^0 \cup \gamma^0) e^{-\tilde{\tilde{U}}(\tilde{\eta} \cup \gamma)}, \tag{5.5}$$

where

$$\tilde{\tilde{U}}(\tilde{\eta} \cup \gamma) = \tilde{U}(\tilde{\eta} \cup \gamma) - \tilde{V}^{+}(\eta \cup \gamma). \tag{5.6}$$

To estimate (5.5) we construct a further partition of Γ_q:[29]

$$\Gamma_q = \Gamma_{q,\Lambda} = \Gamma_{q,\Lambda_q} \cup \Gamma_{q,\Lambda_q^c} \cup \Gamma_{q,\partial\Lambda_q}^{s} \cup \Gamma_{q,\partial\Lambda_q}^{l},$$

where Γ_{q,Λ_q} is the configuration of those trajectories which are completely contained in Λ_q, Γ_{q,Λ_q^c} are trajectories completely outside of Λ_q, i.e., in Λ_q^c, $\Gamma_{q,\partial\Lambda_q}^{s}$ are short trajectories, which cross the boundary of Λ_q but do not leave Λ_{q+2} and $\Gamma_{q,\partial\Lambda_q}^{l}$ are long trajectories, which cross $\partial\Lambda_q$ and leave Λ_{q+2}. By the infinite-divisibility of the Poisson–Lebesgue measure $\lambda_{\tilde{\sigma}}$, for any function $F(\gamma) \in L^1(\Gamma_{\Omega_\Lambda}, \lambda_{\tilde{\sigma}}^{\Lambda})$ which can be represented as

$$F(\gamma) = F_1(\gamma_{\Lambda_q}) F_2(\overline{\gamma}_{\Lambda_q^c}) F_3(\zeta_{\partial\Lambda_q}) F_4(\overline{\zeta}_{\partial\Lambda_q}) \tag{5.7}$$

for

$$\gamma = \gamma_{\Lambda_q} \cup \overline{\gamma}_{\Lambda_q^c} \cup \zeta_{\partial\Lambda_q} \cup \overline{\zeta}_{\partial\Lambda_q}, \quad \gamma_{\Lambda_q} \in \Gamma_{\Lambda_q}, \quad \overline{\gamma}_{\Lambda_q^c} \in \Gamma_{\Lambda_q^c}, \quad \zeta_{\partial\Lambda_q} \in \Gamma_{\partial\Lambda_q}^{s}, \quad \overline{\zeta}_{\partial\Lambda_q} \in \Gamma_{\partial\Lambda_q}^{l}$$

the following formula is true:

$$\int \lambda_{\tilde{\sigma}}^{\Lambda}(d\gamma) F(\gamma) = \int_{\Gamma_{q,\Lambda_q}} \lambda_{\tilde{\sigma}}^{\Lambda_q}(d\gamma) F_1(\gamma) \int_{\Gamma_{q,\Lambda_q^c}} \lambda_{\tilde{\sigma}}^{\Lambda_q^c}(d\overline{\gamma}) F_2(\overline{\gamma})$$
$$\times \int_{\Gamma_{q,\partial\Lambda_q}^{s}} \lambda_{\tilde{\sigma}}^{\Lambda}(d\zeta) F_3(\zeta) \int_{\Gamma_{q,\partial\Lambda_q}^{l}} \lambda_{\tilde{\sigma}}^{\Lambda}(d\overline{\zeta}) F_4(\overline{\zeta}).$$

But in our case the function

$$F(\gamma)=\chi_q(\gamma)\tilde{D}_0^{\Lambda}(\tilde{\eta}_\Lambda^0\cup\gamma_\Lambda^0)e^{-\tilde{\tilde{U}}(\tilde{\eta}\cup\gamma)} \tag{5.8}$$

does not satisfy (5.7). So, our next step is to estimate (5.8) as a product of some functions as in (5.7).

Due to (5.6), (2.21) and (2.22) we have

$$\tilde{\tilde{U}}(\tilde{\eta}\cup\gamma)=\frac{1}{2}\beta U_{v_1}(\tilde{\eta}^{t_\beta}\cup\gamma^{t_\beta})+\beta\int_0^{t_\beta}d\tau U_{v_2}(\tilde{\eta}^\tau\cup\gamma^\tau),$$

with $v_1=\phi$, and $v_2=-\Delta\phi$. Then for any $\tau\in[0,t_\beta]$ we can write for the energy $U_{v_i}(\tilde{\eta}^\tau\cup\gamma^\tau)$ the following decomposition:

$$\begin{aligned}U_{v_i}(\tilde{\eta}^\tau\cup\gamma^\tau)&=U_{v_i}(\tilde{\eta}^\tau_{\Lambda_q}\cup\gamma^\tau_{\Lambda_q}\cup(\pi_{\Lambda_q}\zeta)^\tau\cup(\pi_{\Lambda_q}\overline{\zeta})^\tau)+U_{v_i}(\tilde{\eta}^\tau_{\Lambda_q^c}\cup\overline{\gamma}^\tau_{\Lambda_q^c})+U_{v_i}((\pi_{\Lambda_q^c}\zeta)^\tau\cup(\pi_{\Lambda_q^c}\overline{\zeta})^\tau)\\&\quad+W_{v_i}(\tilde{\eta}^\tau_{\Lambda_q}\cup\gamma^\tau_{\Lambda_q}\cup(\pi_{\Lambda_q}\zeta)^\tau\cup(\pi_{\Lambda_q}\overline{\zeta})^\tau|\tilde{\eta}^\tau_{\Lambda_q^c}\cup\overline{\gamma}^\tau_{\Lambda_q^c}\cup(\pi_{\Lambda_q^c}\zeta)^\tau\cup(\pi_{\Lambda_q^c}\overline{\zeta})^\tau)\\&\quad+W_{v_i}(\tilde{\eta}^\tau_{\Lambda_q^c}\cup\overline{\gamma}^\tau_{\Lambda_q^c}|(\pi_{\Lambda_q^c}\zeta)^\tau)+W_{v_i}(\tilde{\eta}^\tau_{\Lambda_q^c}\cup\overline{\gamma}^\tau_{\Lambda_q^c}|(\pi_{\Lambda_q^c}\overline{\zeta})^\tau),\end{aligned} \tag{5.9}$$

where in the same way as for $\tilde{U}$ we use the substitution (4.11) to define the interaction energy W_{v_i} for $\tilde{\omega}\in\tilde{\Omega}_1$. Now to estimate the various terms in (5.9) we prove some lemmas.

Lemma 5.2: For each positive, integer q and $\tau\in[0,t_\beta]$

$$U_{v_i}(\tilde{\eta}^\tau_{\Lambda_q}\cup\gamma^\tau_{\Lambda_q}\cup(\pi_{\Lambda_q}\zeta)^\tau\cup(\pi_{\Lambda_q}\overline{\zeta})^\tau)\geqslant\frac{1}{4}A_iE_q^\tau(\xi)+\frac{1}{2}A_iE_q^\tau(\tilde{\eta}_{\Lambda_q})-\frac{B_i^2}{A_i}|\Lambda_q|. \tag{5.10}$$

Proof: The proof follows from Lemma A.1 (see the Appendix) and the definition of $E_q^\tau(\xi)$. □

By the stability condition we have

$$U_{v_i}((\pi_{\Lambda_q^c}\zeta)^\tau\cup(\pi_{\Lambda_q^c}\overline{\zeta})^\tau)\geqslant -B_i(|\zeta|+|\overline{\zeta}|). \tag{5.11}$$

Lemma 5.3: Let ξ_a and ξ_a' be subconfigurations of $\xi\in\Gamma_q$ and ξ_a contained in Λ_{q+a} and ξ_a' in Λ_q^c. Then for any $\tau\in[0,t_\beta]$ there exist a small enough α, sufficiently large $q_0^{(1)}$, a constant $h_i^{(a)}(q_0^{(1)})$ and a decreasing function $\varepsilon_i^{(a)}(q)$ on the integers, such that for each $q\geqslant q_0^{(1)}-1$

$$-W_{v_i}(\xi_a^\tau|\xi_a'^{\tau})\leqslant\begin{cases}h_i^{(a)}(q_0^{(1)}) & \text{for } q=q_0^{(1)}-1,\\ \varepsilon_i^{(a)}(q)\varphi(q-1) & \text{for } q\geqslant q_0^{(1)}.\end{cases} \tag{5.12}$$

Remark 5.1: We use Lemma 5.3 with $a=0$ for the fourth term in (5.9) ($\xi_0=\tilde{\eta}_{\Lambda_q}\cup\gamma_{\Lambda_q}$, $\xi_0'=\tilde{\eta}_{\Lambda_q^c}\cup\overline{\gamma}_{\Lambda_q^c}\cup\pi_{\Lambda_q^c}\xi\cup\pi_{\Lambda_q^c}\overline{\zeta})$ and with $a=2$ for the fifth term of (5.9) ($\xi_2=\pi_{\Lambda_q^c}\zeta$, $\xi_2'=\tilde{\eta}_{\Lambda_q^c}\cup\overline{\gamma}_{\Lambda_q^c}$).

Proof: Let us prove the lemma for $a=0$, $i=1$ and for $q\geqslant q_0^{(1)}$ which we chose later. Taking $i=1$, $a=0$ and $\tau=t_\beta$ in Lemma A.2 and using the fact that

$$E^{(1)}_{q+1\backslash q-1}(\xi^{t_\beta})\leqslant E_{q+1\backslash q-1}(\xi)=E_{q+1}(\xi)-E_{q-1}(\xi)$$

and because of $\xi\in\Gamma_q$, $E_{q-1}(\xi)>\varphi(q-1)$ and $E_m(\xi)\leqslant\varphi(m)$ for $m\geqslant q$, we have

$$-W_{v_1}(\xi_0^{t_\beta}|\xi_0'^{t_\beta}) \leqslant \frac{1}{2}F_{\mu_1}[\varphi(q+1)-\varphi(q-1)]+\frac{1}{2}\beta\varepsilon_1^{(1)}(q)\varphi(q) + \frac{1}{2}\beta\varepsilon_2^{(2)}(q)\varphi(q)\sum_{k=1}^{\infty}(a_k^{(1)}-a_{k+1}^{(1)})\varphi(q+k+1), \tag{5.13}$$

where $a_k^{(1)}=\Psi_1(l_{q+k}-l_q)$. Then from (4.5) we have

$$\varphi(q+1)-\varphi(q-1)<[(1+2\alpha)^{2d+2}-1]\varphi(q-1), \tag{5.14}$$

$$\varphi(q)<(1+2\alpha)^{d+1}\varphi(q-1). \tag{5.15}$$

From the definitions (4.1)–(4.3), choosing α sufficiently small and

$$q_0^{(1)}>-\frac{2}{\alpha}\log(1-e^{-2\alpha}), \tag{5.16}$$

it is easy to calculate that $q+k+1<(2/\alpha)\log(l_{q+k+1}-l_q+1)$ and

$$\sum_{k=1}^{\infty}(a_k^{(1)}-a_{k+1}^{(1)})\varphi(q+k+1)\leqslant c(\alpha)\sum_{l_{q+1}-l_q}^{\infty}\Psi_1(m)(2m+1)^d\log(m+1)=c(\alpha)\varepsilon_0(q) \tag{5.17}$$

with $c(\alpha)=2\alpha^{-1}(1+\alpha)$. Now from (5.13)–(5.16) we have (5.12) for $q\geqslant q_0^{(1)}$ with

$$\varepsilon_1^{(0)}(q)=\tfrac{1}{2}\beta F_{\mu_1}[(1+2\alpha)^{2d+2}-1]+\tfrac{1}{2}\beta\varepsilon_1^{(1)}(q)(1+2\alpha)^{d+1} + \tfrac{1}{2}\beta\varepsilon_2^{(1)}(q)(1+2\alpha)^{d+1}c(\alpha)\varepsilon_0(q). \tag{5.18}$$

The proofs for the cases $i=2$ and $a=2,i=1,2$ are the same.

Now we consider the case $q=q_0^{(1)}-1$. The only difference in the proof lies in estimating the first term in Eq. (A2). As $\gamma\in\Gamma_{q_0-1}$ this term cannot be very small and we estimate it by

$$E^{t_\beta}_{q_0^{(1)}\setminus q_0^{(1)}-2}(\xi^{t_\beta})\leqslant E_{q_0}(\xi)\leqslant\varphi(q_0^{(1)}).$$

Then inequality (5.12) for $q=q_0^{(1)}-1$ holds with

$$h_i^{(a)}=[\tfrac{1}{2}\beta F_{\mu_i}+\tfrac{1}{2}\beta\varepsilon_1^{(i)}(q_0^{(1)})+\tfrac{1}{2}\beta\varepsilon_2^{(i)}(q_0^{(i)})]\varepsilon(q_0^{(1)}).$$

□

And finally the sixth term of (5.9) can be estimated by

Lemma 5.4: Let $\overline{\zeta}$ *be some subconfiguration of* $\gamma\in\Gamma_q$, *contained in* Λ_m, $m>q$. *Then there exist a* $q_0^{(2)}$ *large enough and some constant* $b=b(\alpha,d)$, *such that for each* $q\geqslant q_0^{(2)}$

$$-\tfrac{1}{2}\beta W_{v_1}((\pi_{\Lambda_q^c}\overline{\zeta})^{t_\beta}|\tilde{\eta}^{t_\beta}_{\Lambda_q^c}\cup(\pi_{\Lambda_q^c}\gamma)^{t_\beta})\leqslant\tfrac{1}{2}\beta bF_{\mu_1}|\overline{\zeta}|\varphi(m)^{1/2}, \tag{5.19}$$

$$-\beta W_{v_2}(\pi_{\Lambda_q^c}\overline{\zeta}|\tilde{\eta}_{\Lambda_q^c}\cup\pi_{\Lambda_q^c}\gamma)\leqslant tbF_{\mu_2}|\overline{\zeta}|\varphi(m)^{1/2}. \tag{5.20}$$

Proof: The proofs of (5.19) and (5.20) are almost the same. Let us prove for example (5.19). Using Lemma A.3. with $\tau=t_\beta$, $\xi_1=\pi_{\Lambda_q^c}\overline{\zeta}$, $\xi_2=\tilde{\eta}_{\Lambda_q^c}\cup\pi_{\Lambda_q^c}\gamma$ and taking into account that for $m>q$

$$E^{t_\beta}_{m+r(k)}(\xi)\leqslant E_{m+r(k)}(\xi)\leqslant\varphi(m+r(k)),$$

as $\xi=\tilde{\eta}\cup\gamma$ and $\gamma\in\Gamma_q$ we get

$$-\frac{1}{2}\beta W_{v_1}(\cdots|\cdots)\leqslant\frac{1}{2}\beta b_1|\overline{\zeta}|\sum_{k=1}^{\infty}\Psi_1(k)(k+1)^{(d-1)/2}\varphi(m+r(k))^{1/2}.$$

[See for $r(k)$ Eqs. (4.6) and (4.7).] Using (4.4), (4.7) and (3.4) we get (5.19) with $b=b_1\exp[\alpha/2(d+\varepsilon(d+1))]2^{d-1+\mu}$, $\varepsilon=\min\{1-2\alpha e^{2\alpha},(2\mu-1)(\alpha+1)^{-1}\}$ and $q_0^{(2)}=s_0$ (see (4.4), (4.5) for definition of s_0). □

And finally we present some inequalities for the initial density distribution (1.21)–(1.23).

Lemma 5.5: Let the potentials ϕ and ψ satisfy (A.1)–(A.3). *Then*

1^o. *For any $\Lambda_q\subset\Lambda$, $q\geqslant q_0-1$, and $\xi^0_\Lambda\in\Gamma_q$ there exist $A_3>0$ and $B_3\geqslant 0$ such that*

$$\tilde{D}^\Lambda_0(\xi^0_\Lambda)\leqslant e^{[-A_3\beta\Sigma_{\Delta\subset\Lambda_q}|\xi^0_\Delta|^2+B_3\beta\Sigma_{\Delta\subset\Lambda_q}|\xi^0_\Delta|]}\tilde{D}^\Lambda_0(\xi^0_{\Lambda^c_q}) \tag{5.21}$$

2^o. *For any partition*

$$\xi_{\Lambda^c_q}=\overline{\xi}^0_{\Lambda_s\cap\Lambda^c_q}\cup\tilde{\xi}^0_{\Lambda^c_q},\quad \overline{\xi}^0_{\Lambda_s\cap\Lambda^c_q}\cap\tilde{\xi}^0_{\Lambda^c_q}=\varnothing,\quad s>q, \tag{5.22}$$

there exists $C_3\geqslant 0$ such that

$$\tilde{D}^\Lambda_0(\xi^0_{\Lambda^c_q})\leqslant e^{C_3\beta|\overline{\xi}^0_{\Lambda_s\cap\Lambda^c_q}|\varphi(s)^{1/2}}\tilde{D}^\Lambda_0(\tilde{\xi}^0_{\Lambda^c_q}). \tag{5.23}$$

Proof: The proof is a direct consequence of Ruelle's technique.[28]

Now we collect all the estimates (5.10)–(5.12), (5.19), (5.20), (5.21), and (5.23) to obtain for $q\geqslant q_0=\max\{q_0^{(1)},q_0^{(2)}\}$ [see (5.16)]:

$$\begin{aligned}I_q(\tilde{\eta}_\Lambda)\leqslant{}& e^{-(1/4)\beta AE_q(\tilde{\eta}_{\Lambda_q})-(1/8)\beta A\varphi(q-1)+\varepsilon_0(q)\varphi(q-1)+\varepsilon_1(q)\varphi(q-1)}\int\lambda^{\Lambda_q}_{\tilde{\sigma}}(d\gamma)\int\lambda^{\Lambda_{q+2}}_{\tilde{\sigma}}(d\zeta)e^{B|\zeta|}\\&\times\int\lambda^{\Lambda^c_q}_{\tilde{\sigma}}(d\overline{\gamma})e^{-\tilde{\tilde{U}}(\tilde{\eta}_{\Lambda^c_q}\cup\overline{\gamma})}\tilde{D}^{\Lambda^c_q}_0(\tilde{\eta}^0_{\Lambda^c_q}\cup\overline{\gamma}^0)\prod_{m\geqslant q+3}\int\lambda^{\Lambda_m}_{\tilde{\sigma}}(d\overline{\zeta}^{(m)})e^{B|\overline{\zeta}^{(m)}|+C|\overline{\zeta}^{(m)}|\varphi(m)^{1/2}}\end{aligned} \tag{5.24}$$

with

$$A=\min\{A_1,\quad 2A_2,\quad 2A_3\},$$

$$B=\max\{\tfrac{1}{2}\beta B_1,\quad TB_2,\quad \beta B_3\},$$

$$C=\max\{\tfrac{1}{2}\beta bF_{\mu_1},\quad TbF_{\mu_2},\quad C_3\},$$

and

$$\epsilon_0(q)=\frac{(1+2\alpha)^{d+1}}{q}\left[\frac{\beta B_1^2}{2A_1}+\frac{TB_2^2}{A_2}+\frac{\beta B_3^2}{A_3}\right],$$

$$\varepsilon_1(q)=\sum_{i\in\{1,2\}}\sum_{a\in\{0,2\}}\varepsilon_i^{(a)}(q). \tag{5.25}$$

Now from the definition of the Poisson–Lebesgue measure we get

$$\int \lambda_{\tilde{\sigma}}^{\Lambda_q}(d\gamma) \leqslant e^{\varepsilon_2(q)\varphi(q-1)}, \quad \varepsilon_2(q) = \frac{z(1+2\alpha)^{d+1}}{q}, \tag{5.26}$$

$$\int \lambda_{\tilde{\sigma}}^{\Lambda_{q+2}}(d\zeta) e^{B|\zeta|} \leqslant e^{\varepsilon_3(q)\varphi(q-1)}, \quad \varepsilon_3(q) = \frac{z(1+2\alpha)^{3d+3}}{q} e^B, \tag{5.27}$$

$$\int \lambda_{\tilde{\sigma}}^{\Lambda_q^c}(d\overline{\gamma}) e^{-\tilde{\tilde{U}}(\tilde{\eta}_{\Lambda_q^c} \cup \overline{\gamma})} \tilde{D}_0^{\Lambda_q^c}(\tilde{\eta}^0_{\Lambda_q^c} \cup \overline{\gamma}^0) \leqslant \tilde{\tilde{\rho}}^{\Lambda}_{t_\beta}(\tilde{\eta}_{\Lambda_q^c}). \tag{5.28}$$

Applying the same arguments as in (5.26) and (5.27), making a partition of all long trajectories $\overline{\zeta}$ in (5.24) according to their lengths and using the resummation formula we get

$$I_q^{(4)} \equiv \prod_{m \geqslant q+3} \int \lambda_{\tilde{\sigma},\partial\Lambda_q}^{\Lambda_m}(d\overline{\zeta^{(m)}}) e^{B|\overline{\zeta^{(m)}}| + C|\overline{\zeta^{(m)}}|\varphi(m)^{1/2}}$$

$$= \sum_{k=0}^{\infty} \frac{e^{Bk}}{k!} \sum_{m_1,\ldots,m_k \geqslant q+3} \int \tilde{\sigma}_{\Lambda_{m_1}}(d\overline{\zeta^{(m_1)}}) \cdots \tilde{\sigma}_{\Lambda_{m_k}}(d\overline{\zeta^{(m_k)}}) e^{C\Sigma_{j=1}^{k}\varphi(m_j)^{1/2}}.$$

Then we use (see Ref. 29 or 38)

$$\int \tilde{\sigma}_{\Lambda_m}(d\overline{\zeta^{(m)}}) \leqslant z \frac{|\Lambda_m|}{(2\pi t_\beta)^{d/2}} e^{-c\,(l_m^2/2t_\beta)(\alpha^2/(1+2\alpha)^4}$$

with $c = c(d)$ and get

$$I_q^{(4)} \leqslant e^{ze^B f(T_\beta)}, \quad T_\beta = T/\beta, \tag{5.29}$$

where

$$f(T_\beta) = \sum_{m \geqslant q_0+3} (2l_m+1)^d \exp\left\{ -c \frac{l_m^2}{2t_\beta} \frac{\alpha^2}{(1+2\alpha)^4} + Cm^{1/2}(2l_m+1)^{d/2} \right\}.$$

Obviously we have $f(T_\beta) < \infty$ for $d \leqslant 3$.

As a result we get (5.1) for $q \geqslant q_0$ with $\varepsilon(q) = \Sigma_{0 \leqslant j \leqslant 3} \varepsilon_j(q)$ and $K = ze^B f(T_\beta)$.

In the same way we have for $q = q_0 - 1$

$$I_{q_0-1} \leqslant e^{-1/4\,\beta A E_{q_0}(\tilde{\eta}_{\Lambda_{q_0}}) + h(q_0) + K} \tilde{\tilde{\rho}}^{\Lambda}(\tilde{\eta}_{\Lambda_{q_0}^c}),$$

with

$$h(q_0) = \sum_{i \in \{1,2\}} \sum_{a \in \{0,2\}} h_i^{(a)}(q_0).$$

As a result we get (5.1) from (5.24), (5.26)–(5.29) with $\varepsilon(q) = \Sigma_{0 \leqslant j \leqslant 3} \varepsilon_j(q)$ and $K(q_0) = ze^B f(T_\beta)$. □

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APPENDIX: PROOF OF THE BASIC LEMMAS

Lemma A.1: For any $\tau\in[0,t_\beta]$ *and* $q\in\mathbb{N}$ *and* $\tilde{\eta}^\tau,\gamma^\tau\subset\Lambda_q$,

$$U_{v_i}(\tilde{\eta}^\tau_{\Lambda_q}\cup(\pi_{\Lambda_q}\gamma)^\tau)\geqslant\frac{1}{4}A_iE_q^\tau(\xi^\tau)+\frac{1}{2}A_iE_q^\tau(\tilde{\eta}^\tau_{\Lambda_q})-\frac{B_i^2}{A_i}|\Lambda_q|. \tag{A1}$$

Proof: From the superstability condition (A.2) (see Sec. III) we obtain

$$U_{v_i}(\tilde{\eta}^\tau_{\Lambda_q}\cup(\pi_{\Lambda_q}\gamma)^\tau)\geqslant A_i\sum_{\Delta\subset\Lambda_q}|\xi^\tau_\Delta|^2-B_i\sum_{\Delta\subset\Lambda_q}|\xi^\tau_\Delta|.$$

Then (A1) follows from the inequalities $-n\geqslant -A_in^2/4B_i-B_i/A_i$ and $E_q^\tau(\xi)\geqslant E_q^\tau(\tilde{\eta})$. $\square$

Lemma A.2: Let $\tau\in[0,t_\beta]$, $q\in\mathbb{N}$ *and* $\xi_1\in\Gamma_{\tilde{\Omega}_{\Lambda_{q+a}}}$ *with* $a=0$ *or* $a=2$, $\xi\in\Gamma_{\tilde{\Omega}_{\Lambda_q^c}}$, $\xi_1\cup\xi_2\subset\xi\in\Gamma_q$. *Then there exist positive, decreasing functions* $\varepsilon_1^{(i)}(q),\varepsilon_2^{(I)}(q)$ *such that*

$$-W_{v_i}(\xi_1^\tau|\xi_2^\tau)\leqslant F_{\mu_i}E^\tau_{q+a+1\backslash q-1}(\xi)+\varepsilon_1^{(i)}(q)E^\tau_{q+a}(\xi)+\varepsilon_2^{(i)}(q)\varphi(q)\sum_{k=1}^\infty(a_k^i-a_{k+1}^i)E^\tau_{q+a+k+1}(\xi), \tag{A2}$$

with $a_k^i=\Psi_i(l_{q+a+k}-l_{q+a})$.

Proof: From the regularity assumption (A3),

$$-W_{v_i}(\xi_1^\tau|\xi_2^\tau)\leqslant\frac{1}{2}\sum_{\Delta\subset\Lambda_{q+a}}\sum_{\Delta'\subset\Lambda_q^c}\Psi_i(\Delta,\Delta')[|\xi^\tau_\Delta|^2+|\xi^\tau_{\Delta'}|^2]. \tag{A3}$$

According to the partitions $\Lambda_{q+a}=\Lambda_{q+a-1}\cup(\Lambda_{q+a}\backslash\Lambda_{q+a-1})$, $\Lambda_q^c=(\Lambda_{q+a+1}\backslash\Lambda_q)\cup\Lambda^c_{q+a+1}$ we write down the r.h.s. of (A3) as

$$\begin{aligned}
-W_{v_i}(\xi_1^\tau|\xi_2^\tau)\leqslant{}&\frac{1}{2}\sum_{\Delta\subset\Lambda_{q+a}\backslash\Lambda_{q-1}}\sum_{\Delta'\subset\Lambda_{q+a+1}\backslash\Lambda_q}\Psi_i(\Delta,\Delta')|\xi^\tau_\Delta|^2\\
&+\frac{1}{2}\sum_{\Delta\subset\Lambda_{q-1}}\sum_{\Delta'\subset\Lambda_{q+a+1}\backslash\Lambda_q}\Psi_i(\Delta,\Delta')|\xi^\tau_\Delta|^2+\frac{1}{2}\sum_{\Delta\subset\Lambda_{q+a}}\sum_{\Delta'\subset\Lambda^c_{q+a+1}}\Psi_i(\Delta,\Delta')|\xi^\tau_\Delta|^2\\
&+\frac{1}{2}\sum_{\Delta\subset\Lambda_{q+a}}\sum_{\Delta'\subset\Lambda_{q+a+1}\backslash\Lambda_q}\Psi_i(\Delta,\Delta')|\xi^\tau_\Delta|^2\\
&+\frac{1}{2}\sum_{\Delta\subset\Lambda_{q+a}}\sum_{\Delta'\subset\Lambda^c_{q+a+1}}\Psi_i(\Delta,\Delta')|\xi^\tau_{\Delta'}|^2\\
\leqslant{}&\frac{1}{2}F_{\mu_i}E^\tau_{q+a\backslash q-1}(\xi)+\frac{1}{2}F^{(i)}(l_q-l_{q-1})E^\tau_{q-1}(\xi)+\frac{1}{2}F^{(i)}(l_{q+a+1}-l_{q+a})E^\tau_{q+a}(\xi)\\
&+\frac{1}{2}F_{\mu_i}E^\tau_{q+a+1\backslash q}(\xi)+\frac{1}{2}\sum_{\Delta\in\Lambda_{q+a}}\sum_{k=1}^\infty\Psi_i(l_{q+a+k}-l_{q+a})E_{q+a+k+1\backslash q+a+k}(\xi)\\
\leqslant{}&F_{\mu_i}E^\tau_{q+a+1\backslash q-1}(\xi)+F^{(i)}(l_q-l_{q-1})E^\tau_{q+a}(\xi)\\
&+\frac{1}{2}|\Lambda_{q+a}|\sum_{k=1}^\infty(a_k-a_{k+1})E^\tau_{q+a+k+1}(\xi),
\end{aligned}$$

where we have used the summation over layers $\Lambda_{q+a+k+1}\backslash\Lambda_{q+a+k}$, some resummation formula. As a result we obtain (A2) with $\varepsilon_1^{(i)}(q)=F^{(i)}(l_q-l_{q-1})=\sup_{\Delta\in\Lambda_{q-1}}\Sigma_{\Delta'\in\Lambda_q^c}\Psi_i(\Delta,\Delta')$ and $\varepsilon_2^{(i)}=2^{-1}(q+a)^{-1}$. □

Lemma A.3: Let $\tau\in[0,t_\beta]$, $q\in\mathbb{N}$, ξ_1 *contained in* Λ_s, $s>q$, ξ_2, *contained in* Λ_q^c *and* $\xi_1\cup\xi_2\subset\xi\in\Gamma_q$. *Then there exists* $q_0^{(2)}$ *such that for* $q\geqslant q_0^{(2)}$

$$-W_{v_i}(\xi_1^\tau|\xi_2^\tau)\leqslant b_1|\xi_1^\tau|\sum_{k=0}^{\infty}\Psi_i(k)(k+1)^{(d-1)/2}(E_{s+r(k)}^\tau(\xi))^{1/2}, \tag{A4}$$

with $b_1=b_1(d)$ *and* $r(k)$, *which is defined in (4.6).*

Proof: Using the regularity assumption (A.3) and Schwartz inequality we get

$$-W_{v_i}(\xi_1^\tau|\xi_2^\tau)\leqslant\sum_{\Delta\in\Lambda_s}\sum_{\Delta'\in\Lambda_q^c}\Psi(\Delta,\Delta')|\xi_{1,\Delta}^\tau||\xi_{2,\Delta'}^\tau|$$

$$\leqslant\sum_{k=0}^{\infty}\Psi_i(k)\sum_{\Delta\in\Lambda_s}|\xi_{1,\Delta}^\tau|\left(\sum_{\Delta',d(\Delta,\Delta')=k}1^2\right)^{1/2}\left(\sum_{\Delta',d(\Delta,\Delta')=k}|\xi_{2,\Delta'}^\tau|^2\right)^{1/2}.$$

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