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Euclidean Gibbs States for Quantum Continuous Systems via Cluster Expansion. II. Bose and Fermi Statistics.

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Abstract

This work is the continuation of the article [6,7], in which the investigation of Euclidean Gibbs states for Quantum Continuouse Systems was started. In this article we construct a thermodynamic limit for a finite volume Euclidean Gibbs state for Bose and Fermi statistics at arbitrary fixed temperature, $\mu < -B$(stability condition constant) and sufficientely small mass of particles using cluster expansion method, assumin g integrability of interaction potential.

Key words: Continuous Bose-Fermi system, Cluster expansion, Poisson measure, Euclidean Gibbs state.

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1 Introduction

In quantum statistical mechanics of the continuous systems a finite-volume Gibbs state is determined on the algebra of local observables $\mathbb{U}_{\pm}(\Lambda), \Lambda \subset \mathbb{R}^d$ as [1]:

$$\omega^{\beta}_{\Lambda}(A) = \frac{Tr_{\mathcal{F}_{\pm}(*)}(e^{-\beta(H_{\Lambda}-\mu N_{\Lambda})}A)}{Tr_{\mathcal{F}_{\pm}(*)}(e^{-\beta(H_{\Lambda}-\mu N_{\Lambda})})}, \tag{1.1}$$

where $A \in \mathbb{U}_{\pm}(\Lambda)$, H_{Λ} is interacting Hamiltonians on Bose $(+)$ or Fermi (-) Fock space $\mathcal{F}_{\pm}(*)$ with some selfadjoint boundary conditions on the boundary of Λ, which acts in the N-particle subspace $\mathcal{F}^{(\mathcal{N})}_{\pm}(*)$ as operator

$$H^{(N)}_{\Lambda} \;=\; -\frac{\hbar^2}{2m_0}\sum_{i=1}^{N}\Delta^{\Lambda}_{x_i} + U(x)_N, \quad (x)_N \equiv (x_1, ..., x_N),$$

N_{Λ} is the operator of number of particles and paramiters β, μ, m_0 are correspondingly inverse temperature, chemical potential and mass of particles.

For the opertators $a(f)$ of CAR or CCR algebras (see [1] for details) the states (1.1) can be expressed by the elements of reduced dencity matrices(RDM):

$$\omega^{\beta}_{\Lambda}(a^*(f_1)...a^*(f_m)a(g_m)...a(g_1)) = \int (dx)^m (dy)^m \bar{g}_1(y_1)...\bar{g}_m(y_m) f_1(x_1)...f_m(x_m)\widetilde{\rho}^{\Lambda}((y)_m \tag{1.2}$$

where

$$\widetilde{\rho}^{\Lambda}((x)_m;(y)_m) = Z^{-1}_{\varepsilon,\Lambda}\sum_{n\geq 0}\frac{z^{m+n}}{n!}\int_{\Lambda^n}(du)^n \sum_{\pi} \varepsilon^{|\pi|} e^{-\beta H^{\Lambda}}((x)_m,(u)_n;\pi[(y)_m,(u)_n]) \tag{1.3}$$

and $Z_{\varepsilon,\Lambda}$ is grand partition function which provides the r.h.s. of (1.3) to be unity at $m = 0$.Here,also, $\varepsilon = +1$ for Bose systems and $\varepsilon = -1$ for Fermi systems and π is a permutation of $m+n$ variables.

Putting the Feynman-Kac formula (see, for example, [2]) for the kernel of $\exp\{-\beta H_N^\Lambda\}$ one can rewrite (1.3) in the form

$$\widetilde{\rho}^\Lambda((x)_m;(y)_m) = Z_{\varepsilon,\Lambda}^{-1} \sum_{n\geq 0} \frac{z^{m+n}}{n!} \int_{\Lambda^n} (du)^n \sum_{\pi} \varepsilon^{|\pi|} \int W^{\beta,m_0}_{(x)_m,(u)_n;\pi[(y)_m,(u)_n]}(d\omega)_{m+n} e^{-U(\omega)_{m+n}} \tag{1.4}$$

with the notation

$$U(\omega)_N = \sum_{1\leq i<j\leq N} \int_0^\beta d\tau U(\omega(\tau))_N \tag{1.5}$$

and paramiter $z = e^{\mu\beta}$, which calls chemical activity and $W^{\beta,m_0}(...)$ is the product of conditional Wiener measures.

J. Ginibre [3.4] was the first who constructed infinite volume reduced density matrices for both statistics. The main technical step is the representation of $\widetilde{\rho}^\Lambda$ by correlation functionals:

$$\widetilde{\rho}^\Lambda((x)_m;(y)_m) = \sum_{j_1,\ldots,j_m\geq 1} \sum_{\pi} \varepsilon^{|\pi|} \int W^{\beta,m_0}_{(x)_m;\pi[(y)_m]}(d\omega)_m \rho^\Lambda(\omega)_m, \tag{1.6}$$

where

$$\rho^\Lambda(\omega)_m = z^q Z_\Lambda^{-1} \sum_{n=0}^\infty \frac{1}{n!} \sum_{j_{m+1}=1}^\infty \varepsilon^{j_{m+1}-1} \frac{z^{j_{m+1}}}{j_{m+1}} \int_\Lambda du_1 \int W^{j_{m+1}\beta,m_0}_{u_1;u_1}(d\widetilde{\omega}_{m+1})...\times$$
$$\times ... \sum_{j_{m+n}=1}^\infty \varepsilon^{j_{m+n}-1} \frac{z^{j_{m+n}}}{j_{m+n}} \int_\Lambda du_n \int W^{j_{m+n}\beta,m_0}_{u_n;u_n}(d\widetilde{\omega}_{m+n}) e^{-U(\omega^*)_{m+n}}, \tag{1.7}$$

with $q = j_1 + ... + j_m$, $\varepsilon = \pm 1$ corresponds Bose and Fermi statistics respectively and $\omega_k^* = \omega_j$ for $k = 1,...,m$, $\omega_k^* = \widetilde{\omega}_j$ for $k = m+1,...,m+n$.

In (1.7) $\omega_j(\widetilde{\omega}_j)$ are so called composite Wiener trajectories(loops) with time-length $j_k\beta$, $j_k = 1,2,...;k = 1,...,m+n$ and $U(\omega^*)_{m+n}$ is the integral from 0 to β of the potential energy of the $j_1 + ... + j_{m+n}$ points on the $m+n$ trajectories $(\omega)_m, (\widetilde{\omega})_n$. Every trajectory ω_k is represented in U by its values in j_k points $\omega_k(\tau), \omega_k(\tau+\beta), ..., \omega_k(\tau+(j_k-1)\beta)$(see (2.11).

By analogy with the classical case [5] J. Ginibre proved that $\rho^{\Lambda}(\omega)_m$ satisfy Kirkwood-Salzburg(KS) type equations which have (uniformely in Λ) some unique solutions in some Banach space for small values of chemical activity z and the limit functions satisfy KS equations.

Having limit functions $\rho((x)_m;(y)_m)$ and their properties we can reconstruct Gibbs state (1.1) in the thermodynamic limit for some class of local observables.

In our previouse article [6] we consider the problem from another side. We introduced so-called Euclidean Gibbs State(EGS) as a Gibbs measure(EGM) on the configuration space of Wiener loops for Maxvell-Boltzman statistics and constructed its limit (for suffisiently small β) using cluster expansion method. Such approach has its own right and may be very useful for the cases, where the method of KS equations does not work, in particular for the construction of Gibbs states at an arbitrary values of paramiters β, z, m_0.

In the following article [7] the next step in this direction was made. We constructed correlation functions as Wick moments of EGM and using Ginibre's (or KS) approach constructed correlation functions in thermodynamic limit for sufficientely small mass of particles.

In this work we extend the construction and results of [6] on the case of Bose and Fermi Statistics.

The contents of this work is as follows. In Section 2 we define configuration spase of composite Wiener loops and construct finite-volume EGM on it as a spesifications of Poisson measure on the nonlocally compact space of composite Wiener loops. In the Section 3 we extend the result of paper[6] and construct a thermodynamic limit of EGS for QS in the case of integrable interaction potentials using cluster expansion method.

2 Euclidean Gibbs state for Bose and Fermi statistics.

To define Euclidean Gibbs state we remind for the readers some definitions of the mentioned paper [6] with some generalization on the case of quantum statistics.

Let for any integer $j \geq 1$ $\Omega^{j\beta} := C\big([0, j\beta] \mapsto \mathbb{R}^d\big)$ be the Banach space of continuous functions

$$\omega : [0, j\beta] \mapsto \mathbb{R}^d, \quad \omega(0) = x, \quad \omega(j\beta) = y$$

We note the subspace of loops ($\omega(0) = \omega(j\beta)$) by $\widetilde{\Omega}^{j\beta}$, and besides define

$$\Omega \;=\; \bigcup_{j\geq 1} \Omega^{j\beta}, \qquad \widetilde{\Omega} \;=\; \bigcup_{j\geq 1} \widetilde{\Omega}^{j\beta},$$

which are the spaces of composite Wiener trajectories and loops correspondingly.

For every $\omega \in \Omega$ with $\omega(0) = x$ and $\omega(j\beta) = y$ we can write the following representation

$$\omega(\tau) \;=\; \widetilde{\omega}_x(\tau) + \frac{\tau}{j\beta}(y - x)$$

where $\widetilde{\omega}_x \in \widetilde{\Omega}^{j\beta}$.

This formula gives the mapping

$$\Omega \ni \omega \mapsto (\omega(0), \omega(j\beta)) \in \mathbb{R}^d \times \mathbb{R}^d$$

and we can define the one-to-one mapping

$$\mathbb{R}^d \times \widetilde{\Omega} \ni (y, \widetilde{\omega}_x) \mapsto \omega \in \Omega$$

by some family of nonlinear operators P_y,for $y \in \mathbb{R}^d$:

$$P_y : \widetilde{\Omega} \mapsto \Omega, \quad (P_y\widetilde{\omega})(\tau) \;=\; (P_y\widetilde{\omega}_x)(\tau) \;=\; \widetilde{\omega}_x(\tau) + \frac{\tau}{j\beta}(y - x) \qquad (2.1)$$

Let $W^{j\beta,m_0}_{x,y}(d\omega)$ denote the conditional Wiener measure of the Brownian particle with mass m_0 on the Borel σ-algebra $\mathfrak{B}(\Omega)$(here and later $\mathfrak{B}(X)$ denote

Borel σ-algebra of X). But in both cases of BS and QS the measure on the Wiener trajectories can be reduced to the measure on the loops. To define the measure on $\widetilde{\Omega}$ consider the mapping

$$\widetilde{\Omega} \ni \widetilde{\omega} \mapsto \widetilde{\omega}(0) \in \mathbb{R}^d, \tag{2.2}$$

which gives a representation of an arbitrary $\widetilde{\omega} \in \widetilde{\Omega}$ in the form

$$\widetilde{\omega} = \widetilde{\omega}(0) + \hat{\omega}, \qquad (2.2^*),$$

where $\hat{\omega} \in \widetilde{\Omega}_0$ and $\widetilde{\Omega}_0$ is the space of composite Wiener loops with $\widetilde{\omega}(0) = 0$. To define a Poisson measure on $\widetilde{\Omega}$ first of all we should choose a proper localization, since our underlying spaces $\widetilde{\Omega}^{j\beta}$ are not locally compact. Let $\mathfrak{L}$ be the system of all Borel, bounded subsets in $\mathbb{R}^d$. Then for any $\Lambda \subset \mathfrak{L}$ we define the set

$$\widetilde{\Omega}_\Lambda := \left\{ \widetilde{\omega} \in \widetilde{\Omega}^{j\beta} \mid \widetilde{\omega}(0) \in \Lambda \, j \in \mathbb{N} \right\}, \qquad \Lambda \in \mathfrak{L} \tag{2.3}$$

Following [8] we construct Poisson measure on the configuration space of Wiener loops from $\widetilde{\Omega}$ as a marked Poisson measure on $\mathbb{R}^d \times \widetilde{\Omega}_0$. For this we define the space of marked configurations as follows

$$\Gamma^\beta = \Gamma^\beta_{\widetilde{\Omega}} = \Gamma^\beta_{\mathbb{R}^d \times \widetilde{\Omega}_0} := \left\{ \gamma = (\widetilde{\gamma}, \widetilde{\omega}_{\widetilde{\gamma}}) \mid \widetilde{\gamma} \in \Gamma_{\mathbb{R}^d}, \widetilde{\omega}_{\widetilde{\gamma}} \in \widetilde{\Omega}^{j\beta}_{\widetilde{\gamma}}, j \in \mathbb{N} \right\},$$

where $\widetilde{\Omega}^{j\beta}_{\widetilde{\gamma}}$ stands for the set of all maps $\widetilde{\gamma} \ni x \mapsto \widetilde{\omega}_x \in \widetilde{\Omega}^{j\beta}$ and

$$\Gamma_{\mathbb{R}^d} = \left\{ \widetilde{\gamma} = \{x_1, ..., x_n, ...,\}, x_i \in \mathbb{R}^d \mid x_i \neq x_j \quad for \quad i \neq j, \quad and \quad |\widetilde{\gamma} \cap \Lambda| < \infty \quad if \quad |$$

For any $\Lambda \in \mathfrak{L}$, define the configuration space Γ^β_Λ over $\widetilde{\Omega}_\Lambda$ as

$$\Gamma^\beta_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma^\beta_{\Lambda,n}, \qquad \Gamma^\beta_{\Lambda,n} := \left\{ \gamma \in \Gamma^\beta \mid \, |\gamma| = |\widetilde{\gamma} \cap \Lambda| = n \right\},$$

$\sqcup$ denoting the union of mutually disjoint sets.

We define a measure on $(\widetilde{\Omega}^{j\beta}, \mathfrak{B}(\widetilde{\Omega}^{j\beta}))$ as the image of product measure $W_{0,0}^{j\beta,m_0}(d\widetilde{\omega}_0)dx$ under the mapping $(2.2) - (2.2^*)$ by the formula:

$$\sigma^{j\beta}(d\widetilde{\omega}) \;=\; \sigma^{j\beta,m_0}(d\widetilde{\omega}) \;=\; W_{x,x}^{j\beta,m_0}(d\widetilde{\omega})dx \tag{2.4}$$

and finally we construct the measure on $\mathfrak{B}(\widetilde{\Omega})$ by:

$$\hat{\sigma}^{\beta,z}(\Delta) \;=\; \sum_{j\geq 1} \frac{z^j}{j}\sigma^{j\beta}(\Delta_j), \tag{2.5}$$

where $\Delta \in \mathfrak{B}(\widetilde{\Omega})$. and $\hat{\sigma}_\Lambda^{\beta,z}$ denote the restriction of the measure $\hat{\sigma}^{\beta,z}$ to the set $\widetilde{\Omega}_\Lambda$.

We also define measure $(\hat{\sigma}^\beta)^{\hat{\otimes}n}(d\widetilde{\omega})_n$ in the following way (see also [6]). Consider the measurable space $\big((\Omega_\Lambda^\beta)^n, \mathfrak{B}((\Omega_\Lambda^\beta)^n)\big)$, where $(\Omega_\Lambda^\beta)^n$ is the n-fold topological product of Ω_Λ^β by itself, and $\mathfrak{B}\big((\Omega_\Lambda^\beta)^n\big) = \mathfrak{B}(\Omega_\Lambda^\beta)^{\otimes n}$ is the Borel σ-algebra on $(\Omega_\Lambda^\beta)^n$. Define

$$(\Omega_\Lambda^\beta)^{\sim n} := \begin{cases} \Omega_\Lambda^\beta, & n = 1 \\ (\Omega_\Lambda^\beta)^n \setminus \{\, (\omega_1, \ldots, \omega_n) \in (\Omega_\Lambda^\beta)^n \mid \exists\, i, j \; i \neq j : \omega_i = \omega_j \,\}, & n = 2, 3, \ldots \end{cases} \tag{2.6}$$

For every $n \in \mathbb{N}$, we define the σ-algebra $\mathfrak{B}\big((\Omega_\Lambda^\beta)^{\sim n}\big)$ on $(\Omega_\Lambda^\beta)^{\sim n}$ as the one induced by $\mathfrak{B}\big((\Omega_\Lambda^\beta)^n\big)$ and denote the sub-σ-algebra of $\mathfrak{B}((\Omega_\Lambda^\beta)^{\sim n})$ consisting of all its symmetric sets by $\mathfrak{B}_{\mathrm{sym}}\big((\Omega_\Lambda^\beta)^{\sim n}\big)$.

Define the mapping

$$(\Omega_\Lambda^\beta)^{\sim n} \ni (\omega_1, \ldots, \omega_n) \mapsto S_n(\omega_1, \ldots, \omega_n) = \{\omega_1, \ldots, \omega_n\} \in \Gamma_{\Lambda,n}^\beta, \qquad n \in \mathbb{N}.$$

Then $(\sigma^\beta)^{\hat{\otimes}n}(d\widetilde{\omega})_n$ is the image under the mapping S_n of the measure $(\sigma^\beta)^{\otimes n}(d\widetilde{\omega})_n$ on $(\Gamma_{\Lambda,n}^\beta, \mathfrak{B}(\Gamma_{\Lambda,n}^\beta))$.

Now the Poisson measure $\pi_\Lambda^{z,\beta} = \pi_{\hat{\sigma}_\Lambda^{\beta,z}}$ on $\big(\Gamma_\Lambda^\beta, \mathfrak{B}(\Gamma_\Lambda^\beta)\big)$ with intensity $\hat{\sigma}_\Lambda^{\beta,z}$ (for $0 < z < 1$) is defined as

$$\pi_\Lambda^{z,\beta}(\Delta) = e^{-\hat{\sigma}_\Lambda^{\beta,z}(\widetilde{\Omega}_\Lambda)} \sum_{n=0}^{\infty} \frac{z^n}{n!}\, (\hat{\sigma}_\Lambda^{\beta,z})^{\hat{\otimes}n}(\Delta \cap \Gamma_{\Lambda,n}^\beta), \qquad \Delta \in \mathfrak{B}(\Gamma_\Lambda^\beta), \tag{2.7}$$

where $\sigma^{\beta}_{\Lambda}(\widetilde{\Omega}_{\Lambda}) = (m/2\pi\beta)^{d/2}|\Lambda|\sum_{j\geq 1}\frac{z^j}{j^{j+1/2}},|\Lambda|$, denoting the Lebesgue measure of Λ(we have omited technical details in this construction refering the reader to the articles [7,8][1]for the details).

We can also define Poisson measure $\pi^{z,\beta}$ for any z and $\mathbb{R}^d$ instead of Λ by the Laplas transformation

$$\int_{\Gamma^\beta} \pi^{z,\beta}(d\gamma)e^{<f,\gamma>} = e^{\int_\Omega \hat{\sigma}^{\beta,z}(d\widetilde{\omega})(e^{f(\widetilde{\omega})}-1)}$$

for any $f \in C_0(\widetilde{\Omega})$(:= the set of all continuous functions on $\widetilde{\Omega}$.

Remark 2.1. *Due to localization of (2.3) the defined Poisson measure is infinite divisible measure with respect to initial points in Λ, i.e. for any $\Lambda_1, \Lambda_2 \in \mathbb{R}^d$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and any integrable, $\mathfrak{B}(\Gamma^\beta_{\Lambda_1})-$ and $\mathfrak{B}(\Gamma^\beta_{\Lambda_2})-$measurable functions $F_1(\gamma)$ and $F_2(\gamma)$ the following formula are true*

$$\int_{\Gamma^\beta} F_1(\gamma)F_2(\gamma)\pi^{z,\beta}(d\gamma) = \int_{\Gamma^\beta_{\Lambda_1}} F_1(\gamma)\pi^{z,\beta}(d\gamma)\int_{\Gamma^\beta_{\Lambda_2}} F_2(\gamma)\pi^{z,\beta}(d\gamma) \qquad (2.8)$$

This immediately follows from the fomular for Laplas transformation.

Then for an arbitrary bounded Borel set $\Lambda \subset \mathbb{R}^d$, and for any $F \in L^1(\Gamma^\beta_\Lambda, \pi^{z,\beta}_\Lambda(d\gamma))$ the following formula is true[1]:

$$\int_{\Gamma^\beta_\Lambda} F(\gamma)\pi^{z,\beta}_\Lambda(d\gamma) = e^{-\hat{\sigma}^{\beta,z}_\Lambda(\widetilde{\Omega}_\Lambda)}\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{(\widetilde{\Omega}_\Lambda)^n} F(\widetilde{\omega}_1,...,\widetilde{\omega}_n)(\hat{\sigma}^{\beta,z}_\Lambda)^{\hat{\otimes}n}(d\widetilde{\omega}_1,...,d\widetilde{\omega}_n). \qquad (2.9)$$

To define Gibbs measure for the continuous system of quantum particles in $\mathbb{R}^d$ with interaction described by a pair potential $v(x,x')(v : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\})$ we impose the following conditions on the potential:

[1]Note that in [1] the constant $\lambda = (2\pi\beta/m_0)^{1/2} = 1$.

(i) stable:

$$\exists B > 0: \quad \forall\, n \geq 2,\ \forall\, (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n, \qquad \sum_{1 \leq i < j \leq n} v(x_i, x_j) \geq -Bn, \tag{2.10}$$

(ii) symmetric: $v(x, x') = v(x', x)$;

(iii) translation-invariant:

$$v(x, x') = \hat{v}(x - x'),$$

and such that $\hat{v}(x) < \infty$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

(iv) integrable:

$$\int |\hat{v}(x)| dx < \infty$$

To define EGS we should make some remarks.

Remark 2.2. *For Bose case ($\varepsilon = +1$) there is no any difficalty but technical one. Having in mind the construction of interaction energy $U(\widetilde{\omega})_N$ on the N composite loops(see explanation after eq.(1.7) we rewright (1.5) in the form*

$$\widetilde{U}(\widetilde{\omega})_N = \frac{1}{2} \int_0^\beta d\tau \left[\sum_{1 \leq i,k \leq N} \sum_{s=1}^{j_i} \sum_{p=1}^{j_k} \widetilde{v}(\omega_i(\tau + (s-1)\beta) - \omega_k(\tau + (p-1)\beta)) \right], \tag{2.11}$$

where $\widetilde{v}(x) = \hat{v}(x)$ if $x \neq 0$, $\widetilde{v}(0) = 0$ and with a stability condition, which in this case looks like:

$$\widetilde{U}(\widetilde{\omega})_N \geq -\beta B \sum_{k=1}^{N} j_k$$

Remark 2.3. *For Fermi case ($\varepsilon = -1$) an uderlying Gibbs measure can not be positive defined and apriori it is not clear weather it exist even in finite volume. But cluster expansion, which we construct in the next section gives the right to define it regorously at least for small values of paramiter*

m_0(or β, z). To define it as a nonpositive specification of Poisson measure we introduce a function

$$j(\widetilde{\omega}) \ = \ j \qquad if \qquad \widetilde{\omega} \in \widetilde{\Omega}_{\Lambda}^{j\beta} \tag{2.12}$$

and for $\gamma \in \Gamma_{\Lambda}^{\beta}$

$$j(\gamma) \ = \ \sum_{\widetilde{\omega}\in\gamma}(j(\widetilde{\omega})-1), j(\widetilde{\omega}) = j_k, \quad for \quad \widetilde{\omega} \in \widetilde{\Omega}_{\Lambda}^{j_k\beta}.$$

Then a negative factor $\varepsilon^{j_{m+1}+\ldots+j_{m+n}-n}$ in (1.7) can be written as $\exp[i\pi\theta(-\varepsilon)j(\gamma)]$, whe $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x < 0$.

So, we have

Definition 2.1. An Euclidean Gibbs state for a finite volume $\Lambda \in \mathfrak{L}$ with boundary condition γ is defined as the measure on $\left(\Gamma^{\beta}, \mathfrak{B}(\Gamma^{\beta})\right)$ given by the following formula

$$G_{\Lambda}^{z,\beta}(\Delta \mid \gamma) = \varepsilon_{\gamma_{\Lambda^c}}(\Delta_{\Lambda^c}) \begin{cases} \left(\Xi_{\Lambda}(\gamma)\right)^{-1} \int\limits_{\Delta_{\Lambda}} \exp[-U(\gamma'_{\Lambda} \mid \gamma_{\Lambda^c}) + i\pi\theta(-\varepsilon)j(\gamma'_{\Lambda})]\, \pi^{z,\beta}(d\gamma'_{\Lambda}), \\ 0, \end{cases}$$

for any $\Delta \in \mathfrak{B}(\Gamma^{\beta})$, where $\Lambda^{c} := \mathbb{R}^d \backslash \Lambda$, $\Delta_{\Lambda} := P_{\mathbb{R}^d,\Lambda}(\Delta)$, $\varepsilon_{\gamma}(\Delta) := \chi_{\Delta}(\gamma)$, $\gamma_{\Lambda} = (\widetilde{\gamma}_{\Lambda}, \widetilde{\omega}_{\widetilde{\gamma}_{\Lambda}})$,

$$U(\gamma' \mid \gamma) := \sum_{\widetilde{\omega}' \in \gamma'} V(\widetilde{\omega}') + \sum^{*}_{\{\widetilde{\omega}, \widetilde{\omega}'\} \subset \gamma'} V(\widetilde{\omega}, \widetilde{\omega}') + \sum^{*}_{\widetilde{\omega}' \in \gamma', \widetilde{\omega} \in \gamma} V(\widetilde{\omega}, \widetilde{\omega}'), \tag{2.13}$$

$$V(\widetilde{\omega}) = \int_0^\beta d\tau \sum_{1 \le s < p \le j} \hat{v}(\widetilde{\omega}(\tau + (s-1)\beta) - \widetilde{\omega}(\tau + (p-1)\beta)), \widetilde{\omega} \in \widetilde{\Omega}^{j\beta}_\Lambda, \tag{2.14}$$

$$V(\widetilde{\omega}, \widetilde{\omega}') = \sum_{k=1}^{j} \sum_{k'=1}^{j'} \int_0^\beta d\tau \hat{v}(\widetilde{\omega}(\tau + (k-1)\beta) - \widetilde{\omega}'(\tau + (k'-1)\beta)), \tag{2.15}$$

$$\Xi_\Lambda(\gamma_{\Lambda^c}) := \int_{\Gamma^\beta} \exp[-U(\gamma'_\Lambda, \gamma_{\Lambda^c}) + i\pi\theta(-\varepsilon) j(\gamma'_\Lambda)]\, \pi^{z,\beta}(d\gamma'_\Lambda).$$

The star attached to the sum in (2.13) means that, by definition, we put the series equal to infinity if it does not converge absolutely.

Using this definition, the definitions of Posson measure on Γ^β_Λ and having in mind Ginibre's correlation functions we can define correlation functions of EGM on $\widetilde{\Omega}_\Lambda$ as (see[7] for details).

$$\rho^\Lambda(\widetilde{\omega})_m = \Xi_\Lambda^{-1} \int_{\hat{\Gamma}^\beta_\Lambda} \hat{\pi}^{z,\beta}_\Lambda(d\gamma) e^{-\widetilde{U}((\widetilde{\omega})_m \cup \gamma) + i\pi\theta(-\varepsilon) j(\gamma)}, \tag{2.16}$$

where

$$\Xi_{\varepsilon,\Lambda} = e^{-\hat{\sigma}^{\beta,z}_\Lambda(\widetilde{\Omega}_\Lambda)} Z_{\varepsilon,\Lambda}$$

But these correlation functions do not correspond those in the construction of RDM. We should slightly correct them. This correction is very simple. We should consider the function ρ^Λ not only on the loops $\widetilde{\omega} \in \widetilde{\Omega}^{j\beta}$, but as the functions of trajectories ω with $\omega(0) \neq \omega(j\beta)$, i.e. in the space $\Omega^{j\beta}$. This extention can be constructed by the family of linear operators T_{y^m}, wich are defined through nonlinear operators $P_{y_i}, i = 1, ..., m$ (see (2.1):

$$\rho^\Lambda_{BS}(\omega)_m = \left(T_{y^m} \rho^\Lambda\right)(\omega)_m = \rho^\Lambda(P_{y_1}\widetilde{\omega}_1, ..., P_{y_m}\widetilde{\omega}_m), \tag{2.17}$$

Remark 2.4. *The correlation functions (2.16) are slightly different from those were introduced by J. Ginibre in [2,3]. The Ginibre correlation functions were defined for trajectories which take their values in* Λ. *The correlation functions (2.16) are defined on the trajectories, which take their values in all* $\mathbb{R}^d$*, but their initial points should be in* Λ. *But in thermodynamic limit both sequences of the correlation functions should coincide due to the uniqness of the solution of the KS chain of equations, the solutions of which they are.*

Remark 2.5. *Due to this constructions we can investigate the Euclidean correlation functions (2.16) and then reconstruct the functions* $\rho_{BS}(\omega^m)$ *by the formula (2.17).*

3 EGS for QS in Thermodynamic Limit. Cluster Expansion.

In the previouse article [6] we constructed the thermodynamic limit of EGS for

Maxwell-Boltzman statistics using cluster expansion which was proposed [9] for the case of classical continuouse systems. The general scheme of construction of cluster expansion is the same as in [6], but in the case of QS we should more carefully estimate every term in the tree-graph representation of cluster expansion. The problem appears due to additional factorials, which are the consiquence of additional sums in the construction of $\widetilde{V}(\widetilde{\omega}, \widetilde{\omega}')$(see (2.15)). Now we breafly repeat the construction of the paper [6], refering the reader to [6] for technical details.

Let $\mathcal{F}_{\mathcal{M}} = \mathcal{F}_{\mathcal{M}}(\widetilde{\otimes}^{\beta})$ denote the class of all $\mathfrak{B}(\widetilde{\Omega}^{\beta})$-measurable functions

$$f\colon \widetilde{\Omega}^{\beta} \mapsto [-M, M] \tag{3.1}$$

such that $\operatorname{supp} f \in \mathfrak{L}$.

For any $f \in \mathcal{F}_{\mathcal{M}}$, $M > 1$, such that $\operatorname{supp} f \subseteq \Lambda \in \mathfrak{L}$, construct the Laplace transform of the measure $G_\Lambda^{z,\beta}(\cdot \mid \varnothing)$:

$$L_\Lambda^{z,\beta}(f) := \int_{\hat{\Gamma}^\beta} e^{-\langle\gamma,f\rangle}\, G_\Lambda^{z,\beta}(d\gamma \mid \varnothing). \tag{3.2}$$

Remark 3.1. *For any $\mathfrak{B}(\widetilde{\Omega}^\beta)$-measurable function $f_0 : \widetilde{\Omega}^\beta \mapsto [-M, M]$ and any $\Lambda \in \mathfrak{L}$ the function $f(\widetilde{\omega}) = 1_\Lambda(\widetilde{\omega}(0)) f_0(\widetilde{\omega})$ belongs to $\mathcal{F}_{\mathcal{M}}$. Hence, the functions from $\mathcal{F}_{\mathcal{M}}$ separate points in $\widetilde{\Omega}^\beta$, which yields that the Laplace transform of a probability measure on $\hat{\Gamma}^\beta$ defined for functions from $\mathcal{F}_{\mathcal{M}}$ uniquely determines the measure.*

Taking into account the definition of a finite volume Gibbs state, we can rewrite (3.2) as follows:

$$L_\Lambda^{z,\beta}(f) = \big(\Xi_\Lambda\big)^{-1} \int_{\hat{\Gamma}^\beta} \exp\Big[-\frac{1}{2}\widetilde{U}(\gamma_\Lambda; f)\Big]\, \hat{\pi}^{z,\beta}(d\gamma), \tag{3.3}$$

where(see [7])

$$\widetilde{U}(\gamma; f) := 2\langle\gamma, f\rangle + \langle :\gamma^{\otimes 2}:, \widetilde{V}\rangle$$

and

$$\Xi_\Lambda := \Xi_\Lambda(\varnothing) = \int_{\Gamma^\beta} \exp\Big[-\frac{1}{2}\widetilde{U}(\gamma_\Lambda; 0)\Big] \hat{\pi}^{z,\beta}(d\gamma).$$

Now we formulate the main theorem of this work.

Theorem 3.1. *Let the potential is such that conditions* (i)-(iv) *are fulfilled. Then, for any $0 < z < e^{-\beta B}$ and $\beta > 0$, there is $m_* > 0$ such that, for any $m_0 < m_*$, there exists a thermodynamic limit for finite volume Euclidean Gibbs state corresponding to the quantum continuous Bose and Fermi systems with inverse temperature β, activity z and potential $\hat{v}$.*

To prove the Theorem we are going to show that $L_\Lambda^{z,\beta}(f)$ converges as $\Lambda \nearrow \mathbb{R}^d$ to some function $L^{z,\beta}(f)$ which is Gâteaux analytic at zero. By general results on the moment problem, this implies the existence of the weak local limit $G^{z,\beta}(\cdot)$ of $G_{\Lambda}^{z,\beta}(\cdot,\varnothing)$, $\Lambda \nearrow \mathbb{R}^d$, as a measure on the linear space of all point measures $\mathcal{M}_{\surd}$ over Ω^{β}, i.e., on the "configuration" space over $\widetilde{\Omega}^{\beta}$, with multiple points permitted[10-12].

Then, we hope the obtained limiting measure $G^{z,\beta}$ satisfy the DLR equations and $\hat{\Gamma}^{\beta}(\subset \mathcal{M}_{\surd})$ as a set of full measure.

To prove the convergence of $L_{\Lambda}^{z,\beta}(f)$, we introduce a function

$$\Phi_{\Lambda}^{z,\beta}(X;\delta f) := \begin{cases} (\Xi_{\Lambda})^{-1} \int\limits_{\hat{\Gamma}^{\beta}} \exp\Big[-\frac{1}{2}\widetilde{U}(\gamma_{\Lambda\setminus X};\delta f)\Big]\,\hat{\pi}^{z,\beta}(d\gamma), & \text{if } X \subset \Lambda, \\ 0, & \text{otherwise} \end{cases}$$

and show that point-wise with respect to X and uniformly with respect to

$$\delta \in O(\rho) := \{\delta \in \mathbb{C} \mid |\delta| < \rho\}$$

it converges to some function $\Phi^{z,\beta}(\cdot;\delta f)$ for some $\rho > 0$. Since the function $\Phi_{\Lambda}^{z,\beta}(X;\delta f)$ is obviously analytic in δ, the analyticity of the function $\Phi^{z,\beta}(X;\delta f)$ follows, and so that of $L_{\Lambda}^{z,\beta}(\delta f)$, because

$$\Phi_{\Lambda}^{z,\beta}(\varnothing;f) = L_{\Lambda}^{z,\beta}(f).$$

But first, we need to construct a cluster expansion for the function $\Phi_{\Lambda}^{z,\beta}$. To do this, let us take a partition $\mathfrak{D}$ of $\mathbb{R}^d$ consisting of unit cubes $\{\Delta\}$ which are half-opened and half-closed in such a way that they are disjoint. For any $\widetilde{X} \in \mathfrak{L}$, the subset of $\mathfrak{D}$ consisting of all the cubes which are in $\widetilde{X}$ is denoted by $\mathfrak{D}_{\widetilde{X}}$,.

Remark 3.2. *Here and below, Λ and all the other sets in $\mathbb{R}^d$ we are working with are supposed to be finite unions of cubes from $\mathfrak{D}$, if not stated otherwise.*

For any $X \subset \Lambda$ and an arbitrary $X_1 \in \mathfrak{D}_{\Lambda}$, set $\Lambda' := \Lambda \setminus (X \setminus X_1) = (\Lambda \setminus X) \cup X_1$ and consider all possible sequences of sets $\mathfrak{Y}_n = \{Y_1,\ldots,Y_n\}$

and $\mathfrak{X}_n = \{X_1, \dots, X_n\}$, $n = 1, \dots, |\Lambda'|$, constructed in such a way that $Y_1 = X_1$ and

$$X_k = X_{k-1} \cup Y_k, \quad Y_k \in \mathfrak{D}_{\Lambda' \setminus X_{k-1}}, \qquad k = 2, \dots, |\Lambda'|.$$

Now, using standard technique (most closele to [13-15]) of switching-off (step by step) the interaction between X_k and $\Lambda \setminus X_k$(or $\widetilde{X} \setminus X_k$) and using formula (2.8) we get (see [7] for details):

$$\Phi_\Lambda^{z,\beta}(X \setminus X_1; \delta f) = K_1(X_1; \delta f)\Phi_\Lambda^{z,\beta}(X \cup X_1; \delta f) + \sum_{n=2}^{|\Lambda'|} \sum_{2,\dots,n}^{\Lambda'} K_n(\mathfrak{X}_n; \delta f)\Phi_\Lambda^{z,\beta}(X \cup X_n; \delta f), \tag{3.4}$$

where

$$\begin{aligned} K_n(\mathfrak{X}_n; \delta f) \; :=& (-1)^{n-1} \int_0^1 \dots \int_0^1 ds_1 \cdots ds_{n-1} \int_{\hat{\Gamma}^\beta} \prod_{j=2}^{n} \Big(\sum_{i=1}^{j-1} s_i \dots s_{j-2} \widetilde{U}(\gamma_{Y_i}, \gamma_{Y_j}) \Big) \times \\ & \times \exp \Big[-\frac{1}{2} \widetilde{U}_{X_n}(\gamma; \delta f; \mathfrak{X}_{n-1}; (s)_{n-1}^1) \Big] \hat{\pi}^{z,\beta}(d\gamma), \end{aligned} \tag{3.5}$$

with

$$\widetilde{U}_{X_n}\big(\gamma; \delta f; \mathfrak{X}_{n-1}; (s)_{n-1}^1\big) = \sum_{i=1}^{n} \widetilde{U}(\gamma_{Y_i}; \delta f) + \sum_{1 \le i < j \le n} s_i \cdots s_{j-1} \widetilde{U}(\gamma_{Y_i}, \gamma_{Y_j}).$$

and

$$\widetilde{U}\big(\gamma_{Y_i}; \gamma_{Y_j}\big) = \sum_{\widetilde{\omega} \in \gamma_{Y_i}, \widetilde{\omega}' \in \gamma_{Y_j}} \widetilde{V}(\widetilde{\omega}, \widetilde{\omega}').$$

We also use in (3.4) the following notation

$$\sum_{2,\dots,k}^{\Lambda'} = \sum_{Y_k \in \mathfrak{D}_{\Lambda' \setminus X_{k-1}}} \cdots \sum_{Y_3 \in \mathfrak{D}_{\Lambda' \setminus X_2}} \sum_{Y_2 \in \mathfrak{D}_{\Lambda' \setminus X_1}}, \qquad 2 \le k \le |\Lambda'|.$$

Changing the order of summation and multiplication in (3.5), we can rewrite it in the following form (which is more convenient for further estimates):

$$K_n(\mathfrak{X}_n;\delta f) = \sum_{\eta:|\eta|=n} (-1)^{n-1} \int_0^1 \dots \int_0^1 ds_1 \cdots ds_{n-1} \prod_{i=2}^{n} (s_{\eta(i)} \cdots s_{i-2}) \times$$

$$\times \int_{\hat{\Gamma}^\beta} \prod_{i=2}^{n} \widetilde{U}(\gamma_{Y_{\eta(i)}}, \gamma_{Y_i}) \exp\Big[-\frac{1}{2} \widetilde{U}_{X_n}\big(\gamma; \delta f; \mathfrak{X}_{n-1}; (s)_{n-1}^1\big)\Big] \hat{\pi}^{z,\beta}(d\gamma), \quad (3.6)$$

where $\sum\limits_{\eta:|\eta|=n}$ means summation over all the tree graphs with n vertices, i.e., over all functions $\eta : \{2,\dots,n\} \mapsto \{1,\dots,n-1\}$ such that $\eta(i) < i$ for all $i = 2,\dots,n$.

Proof of the Theorem 3.1

The strategy of the proof is the following. Using cluster expansion construction (3.4) we can derive for the function $\Phi_\Lambda^{z,\beta}(X;\delta f)$ some kind of operator equation in some Banach space. Then using the same procedure as for the analysis of KS equations we construct the limit function$\Phi^{z,\beta}(X;\delta f)$ as the solution of these equations in thermodynamic limit.

Suppose now that $X \neq \varnothing$, $X \subset \Lambda$, and choose $X_1 = \Delta_X \subset X$. Then, solving (3.4) with respect to $\Phi_\Lambda^{z,\beta}(X;\delta f)$, we get

$$\Phi_\Lambda^{z,\beta}(X;\delta f) = \frac{\Phi_\Lambda^{z,\beta}(X \setminus \Delta_X;\delta f) - \sum\limits_{n=2}^{|\Lambda'|} \sum\limits_{2,\dots,n}^{\Lambda'} K_n(\mathfrak{X}_n;\delta f)\Phi_\Lambda^{z,\beta}(X \cup X_n;\delta f)}{K_1(X_1;\delta f)}.$$

Defining an operator

$$(Q(\delta f)\Phi)(X) = \begin{cases} \big(K_1(\Delta_X;\delta f)\big)^{-1}\Big[\Phi(X\setminus\Delta_X) - \sum\limits_{n=2}^{\infty} \sum\limits_{2,\dots,n}^{\mathbb{R}^d\setminus X} K_n(\mathfrak{X}_n;\delta f)\Phi(X \cup X_n)\Big], & \text{if} \\ 0, & \text{ot} \end{cases}$$

in the Banach space $\mathcal{B}_\alpha$ of functions $\Phi = \Phi(X)$ on bounded subsets of $\mathbb{R}^d$

with the norm

$$\|\Phi\|_\alpha = \sup_{X\subset\mathbb{R}^d} e^{-\alpha|X|}|\Phi(X)|$$

and $\alpha > 0$ to be specified later, we can regard $\Phi_\Lambda^{z,\beta}(X;\delta f)$ as a solution of the following operator equation of the Kirkwood–Salsburg type:

$$\Phi = \Phi_\Lambda^{z,\beta}(\varnothing;\delta f)\varkappa_\varnothing + \hat{\varkappa}_\Lambda Q(\delta f)\hat{\varkappa}_\Lambda\Phi, \tag{3.7}$$

where, for any $\widetilde{X} \in \mathfrak{L}$, $\varkappa_{\widetilde{X}}$ denotes the characteristic function of all the subsets of $\widetilde{X}$, i.e.,

$$\varkappa_{\widetilde{X}}(X) = \begin{cases} 1, & X \subset \widetilde{X}, \\ 0, & \text{otherwise}, \end{cases}$$

and $\hat{\varkappa}_{\widetilde{X}}$ is the operator of multiplication by $\varkappa_{\widetilde{X}}$.

Let us estimate the norm of $Q(\delta f)$ in $\mathcal{B}_\alpha$ and show that, choosing m_* and ρ sufficiently small and α sufficiently large, we can make it less than some constant $C < 1$ for all $m_0 < m_*$ and $|\delta| < \rho$:

$$\begin{aligned}
\|Q(\delta f)\| &\le \sup_{\|\Phi\|_\alpha=1}\sup_{X\subset\mathbb{R}^d} \frac{e^{-\alpha|X|}}{|K_1(\Delta_X;\delta f)|}\Big[|\Phi(X\setminus\Delta_X)| + \sum_{n=2}^{\infty}\sum_{2,\dots,n}^{\mathbb{R}^d\setminus X}|K_n(\mathfrak{X}_n;\delta f)|\,|\Phi(X\cup X_n)| \\
&\le \sup_{X\subset\mathbb{R}^d} \frac{e^{-\alpha|X|}}{|K_1(\Delta_X;\delta f)|}\Big[e^{\alpha(|X|-1)} + \sum_{n=2}^{\infty}\sum_{2,\dots,n}^{\mathbb{R}^d\setminus X}|K_n(\mathfrak{X}_n;\delta f)|e^{\alpha(|X|+n-1)}\Big] \\
&\le 2\Big[e^{-\alpha} + \sup_{X\subset\mathbb{R}^d}\sum_{n=2}^{\infty}\sum_{2,\dots,n}^{\mathbb{R}^d\setminus X}|K_n(\mathfrak{X}_n;\delta f)|e^{\alpha(n-1)}\Big] \\
&\le 2\Big[e^{-\alpha} + \sum_{n=2}^{\infty}\sum_{2,\dots,n}^{\mathbb{R}^d}|K_n(\mathfrak{X}_n;\delta f)|e^{\alpha(n-1)}\Big].
\end{aligned} \tag{3.8}$$

Here we used the fact that because of the analyticity of $K_1(\Delta_X;\delta f)$ with respect to δ at zero and the following inequality

$$K_1(\Delta_X;0) = \int_{\hat{\Gamma}^\beta} \exp\big[-\frac{1}{2}\langle\gamma_{\Delta_X}^{\otimes 2},\widetilde{V}\rangle\big]\,\hat{\pi}^{z,\beta}(d\gamma) \ge e^{-c(z)\lambda^{-d}} = e^{-c(z)(m_0/2\pi\beta)^{d/2}}, \tag{3.9}$$

where

$$c(z) = \sum_{j\geq 1} \frac{z^j}{j^{d/2+1}} \quad \textit{and} \quad \lambda = (\frac{m_0}{2\pi\beta})^{1/2}.$$

(which can be easily obtained from the r.h.s. of (3.9) using the definition (2.9)) we can always find (for fixed $\beta > 0$ and $z < 1$) a sufficiently small ρ and m_0 such that

$$K_1(\Delta_X;\delta f) \geq \frac{1}{2}$$

for all $\delta \in O(\rho)$. By (3.6), we have

$$\sum_{2,\ldots,n}^{\mathbb{R}^d} |K_n(\mathfrak{X}_n;\delta f)| \leq \sum_{\eta:|\eta|=n} \int_0^1 \ldots \int_0^1 ds_1 \cdots ds_{n-1} \prod_{i=2}^{n} (s_{\eta(i)} \cdots s_{i-2}) \times$$
$$\times \sum_{2,\ldots,n}^{\mathbb{R}^d} \int_{\hat{\Gamma}^\beta} \hat{\pi}^{z,\beta}_{X_n}(d\gamma) \prod_{i=2}^{n} |\widetilde{U}(\gamma_{Y_{\eta(i)}},\gamma_{Y_i})| \exp\Big[-\frac{1}{2}\widetilde{U}_{X_n}\big(\gamma;(\mathrm{Re}\,\delta)f;\mathfrak{X}_{n-1};(s)^1_{n-1}\big)\Big], \tag{3.10}$$

where $\mathrm{Re}\,\delta$ denotes the real part of $\delta \in \mathbb{C}$.

Noting that, for any $\widetilde{X} \supset X_n$, $U_{\widetilde{X}}\big(\gamma;\delta f;\mathfrak{X}_n;(s)^1_n\big)$ is recursively given by the formula

$$\widetilde{U}_{\widetilde{X}}\big(\gamma;\delta f;\mathfrak{X}_n;(s)^1_n\big) = (1-s_n)\big[\widetilde{U}_{X_n}\big(\gamma;\delta f;\mathfrak{X}_{n-1};(s)^1_{n-1}\big) + \widetilde{U}(\gamma_{\widetilde{X}\setminus X_n};\delta f)\big]$$
$$+ s_n\widetilde{U}_{\widetilde{X}}\big(\gamma;\delta f;\mathfrak{X}_{n-1};(s)^1_{n-1}\big),$$

where $\widetilde{U}_{X'}(\gamma;\delta f;\mathfrak{X}_0;s_0) := \widetilde{U}(\gamma_{X'};\delta f)$, and taking into account that, by virtue of (2.10) and (3.1),

$$\widetilde{U}_{X'}\big(\gamma;(\mathrm{Re}\,\delta)f\big) \geq -(2\beta B + \rho M)\sum_{\widetilde{\omega}\in\gamma} j(\widetilde{\omega})$$

for all $X' \in \mathfrak{L}$, $\delta \in O(\rho)$ and $f \in \mathcal{F}_{\mathcal{M}}$, we deduce by induction (cf. [13]) that

$$\widetilde{U}_{\widetilde{X}}\big(\gamma;(\mathrm{Re}\,\delta)f;\mathfrak{X}_{n-1};(s)^1_{n-1}\big) \geq -(2\beta B + \rho M)\sum_{\widetilde{\omega}\in\gamma} j(\widetilde{\omega}). \tag{3.11}$$

Using (2.9) and the fact that, for any $X', X'' \in \mathfrak{L}$,

$$\big|\widetilde{U}(\gamma_{X'}, \gamma_{X''})\big| = \big|\langle \gamma_{X'} \otimes \gamma_{X''}, \widetilde{V}\rangle\big| \le \langle \gamma_{X'} \otimes \gamma_{X''}, |\widetilde{V}|\rangle,$$

we get

$$\begin{aligned}
\int_{\hat{\Gamma}^\beta} \hat{\pi}^{z,\beta}_{X_n}(d\gamma) \prod_{i=2}^n |\widetilde{U}(\gamma_{Y_{\eta(i)}}, \gamma_{Y_i})| \exp\Big[-\frac{1}{2}\widetilde{U}_{X_n}(\gamma; (\mathrm{Re}\,\delta)f; \mathfrak{X}_{n-1}; (s)^1_{n-1})\Big] \\
\le e^{-nc(z)\lambda^{-d}} \sum_{k=0}^\infty \frac{1}{k!} \int_{(\widetilde{\Omega}^{com}_{X_n})^{\otimes k}} \prod_{l=1}^k (\hat{\sigma}^{\beta,z'}_{X_n}(d\widetilde{\omega}_l) \prod_{i=2}^n |\widetilde{U}(\gamma_{Y_{\eta(i)}}, \gamma_{Y_i})| \\
= \exp\big[n\lambda^{-d}\big(c(z') - c(z)\big)\big] \int_{\hat{\Gamma}^\beta_{X_n}} \hat{\pi}^{z',\beta}_{X_n}(d\gamma) \prod_{i=2}^n |\widetilde{U}(\gamma_{Y_{\eta(i)}}, \gamma_{Y_i})| \\
\le \exp\big[n\lambda^{-d}\big(c(z') - c(z)\big)\big] \int_{\hat{\Gamma}^\beta_{X_n}} \hat{\pi}^{z',\beta}_{X_n}(d\gamma) \prod_{i=2}^n \langle \gamma_{Y_{\eta(i)}} \otimes \gamma_{Y_i}, |\widetilde{V}|\rangle,
\end{aligned} \tag{3.12}$$

where $z' := ze^{\beta B + \rho M} < 1$ and for $\widetilde{U}(\gamma_{Y_{\eta(i)}}, \gamma_{Y_i})$ in the second line of (3.12) we keep the same notation, but it means here exactly the following sum:

$$\sum_{\widetilde{\omega} \in \widetilde{\Omega}^{com}_{Y_{\eta(i)}}, \widetilde{\omega}' \in \widetilde{\Omega}^{com}_{Y_i}} \widetilde{V}(\widetilde{\omega}, \widetilde{\omega}').$$

As follows from Lemma 2.3 of [6]

$$\int_{\hat{\Gamma}^\beta_{X_n}} \hat{\pi}^{z',\beta}_{X_n}(d\gamma) \prod_{i=2}^n \langle \gamma_{Y_{\eta(i)}} \otimes \gamma_{Y_i}, |\widetilde{V}|\rangle = \sum_\xi \mathbb{E}^{z',n}_{\widetilde{\omega}}(\xi) \Big[\prod_{i=2}^n |\widetilde{V}_{Y_{\eta(i)}, Y_i}(\widetilde{\omega}_i, \widetilde{\omega}'_i)|\Big], \tag{3.13}$$

where for any $X', X'' \subset \Lambda$,

$$\widetilde{V}_{X', X''}(\omega', \omega'') := \widetilde{V}(\omega', \omega'')\chi_{\Omega^\beta_{X'}}(\omega')\chi_{\Omega^\beta_{X''}}(\omega''), \tag{3.14}$$

ξ in the sum runs through all the possible combinations of chains, and $\mathbb{E}^{z',n}_{\widetilde{\omega}}(\xi)$ denotes the integration w.r.t. the measure $\hat{\sigma}^{\beta,z'}(d\widetilde{\omega}_i)$ of the $\widetilde{\omega}_i$-variables to which the combination of chains ξ is applied. Evidently, any chain connecting $\widetilde{\omega}_i$-variables from different $\widetilde{\Omega}^{\beta}_{Y_i}$-sets will give zero, and so the summation in (3.13) is actually taken over all the combinations of chains ξ' which connect $\widetilde{\omega}_i$-variables from the same $\widetilde{\Omega}^{\beta}_{Y_i}$-sets:

$$\int_{\hat{\Gamma}^{\beta}_{X_n}} \hat{\pi}^{z',\beta}_{X_n}(d\gamma) \prod_{i=2}^{n} \langle \gamma^{\otimes 2}, |\widetilde{V}_{Y_{\eta(i)},Y_i}|\rangle = \sum_{\xi'} \mathbb{E}^{z',n}_{\widetilde{\omega}}(\xi') \Big[\prod_{i=2}^{n} |\widetilde{V}_{Y_{\eta(i)},Y_i}(\widetilde{\omega}_i, \widetilde{\omega}'_i)| \Big]. \tag{3.15}$$

In fact the sums over ξ means the sum over all possible moments of the measure $\hat{\pi}^{z',\beta}(d\gamma)$.So, the moments like $\int \hat{\pi}^{z',\beta}(d\gamma)\langle \gamma_{Y_1}, f_1\rangle\langle \gamma_{Y_2}, f_2\rangle = 0$ for $Y_1 \cap Y_2 = \varnothing$ and due to this fact the sum over ξ' appears.

Substituting (3.15)in (3.12) and then in (3.10), we can write

$$\sum_{2,\dots,n}^{\mathbb{R}^d} |K_n(\mathfrak{X}_n;\delta f)| \leq \sum_{\eta:|\eta|=n} \int_0^1 \dots \int_0^1 ds_1 \cdots ds_{n-1} \prod_{i=2}^{n} (s_{\eta(i)} \cdots s_{i-2}) \times$$
$$\times\, e^{n\lambda^{-d}[c(z')-c(z)]} \sum_{\xi'} \sum_{2,\dots,n}^{\mathbb{R}^d} \mathbb{E}^{z',n}_{\widetilde{\omega}}(\xi') \Big[\prod_{i=2}^{n} |\widetilde{V}_{Y_{\eta(i)},Y_i}(\widetilde{\omega}_i, \widetilde{\omega}'_i)| \Big]. \tag{3.16}$$

Next, by (3.14),(2.15) and (2.2*)

$$\sum_{2,\dots,n}^{\mathbb{R}^d} \mathbb{E}^{z',n}_{\widetilde{\omega}}(\xi') \Big[\prod_{i=2}^{n} |\widetilde{V}_{Y_{\eta(i)},Y_i}(\widetilde{\omega}_i, \widetilde{\omega}'_i)| \Big] = \int_0^{\beta} \dots \int_0^{\beta} d\tau_2 \cdots d\tau_n \times$$
$$\times \sum_{2,\dots,n}^{\mathbb{R}^d} \mathbb{E}^{z',n}_{(x,\hat{\omega})}(\xi') \Big[\prod_{i=2}^{n} \chi_{Y_{\eta(i)}}(x_i)\chi_{Y_i}(x'_i) \Big| \sum_{k_i=1}^{j_i} \sum_{k'_i=1}^{j'_i} \hat{v}(x_i - x'_i + \hat{\omega}_i(t_i^{k_i}) - \hat{\omega}'_i(t_i^{k'_i}))\Big| \Big] \tag{3.17}$$

with $t_i^l = \tau_i + (l-1)\beta$. Taking into account that η is a tree graph, we have, for an arbitrary $\tau_2, \dots, \tau_n \in [0,\beta]$ and ξ',

$$\sum_{2,\dots,n}^{\mathbb{R}^d} \mathbb{E}^{z',n}_{(x,\hat{\omega})}(\xi') \Big[\prod_{i=2}^{n} \chi_{Y_{\eta(i)}}(x_i)\chi_{Y_i}(x'_i) \Big| \sum_{k_i=1}^{j_i} \sum_{k'_i=1}^{j'_i} \hat{v}(x_i - x'_i + \hat{\omega}_i(t_i^{k_i}) - \hat{\omega}'_i(t_i^{k'_i}))\Big| \Big]$$

$$
\begin{aligned}
&= \sum_{2,\dots,n-1}^{\mathbb{R}^d} \mathbb{E}^{\,z',n}_{(x,\hat\omega)}(\xi')\Big[\prod_{i=2}^{n-1}\chi_{Y_{\eta(i)}}(x_i)\chi_{Y_i}(x_i')\Big|\sum_{k_i=1}^{j_i}\sum_{k_i'=1}^{j_i'}\hat v(x_i-x_i'+\hat\omega_i(t_i^{k_i})-\hat\omega_i'(t_i^{k_i'}))\Big|\times \\
&\qquad \times \sum_{Y_n\in\mathfrak{D}_{\mathbb{R}^d\setminus X_{n-1}}}\chi_{Y_{\eta(n)}}(x_n)\chi_{Y_n}(x_n')\Big|\sum_{k_i=1}^{j_i}\sum_{k_i'=1}^{j_i'}\hat v(x_i-x_i'+\hat\omega_i(t_i^{k_i})-\hat\omega_i'(t_i^{k_i'}))\Big|\Big] \\
&\le \sum_{2,\dots,n-1}^{\mathbb{R}^d} \mathbb{E}^{\,z',n-1}_{(x,\hat\omega)}(\xi')\Big[\prod_{i=2}^{n-1}\chi_{Y_{\eta(i)}}(x_i)\chi_{Y_i}(x_i')\Big|\sum_{k_i=1}^{j_i}\sum_{k_i'=1}^{j_i'}\hat v(x_i-x_i'+\hat\omega_i(t_i^{k_i})-\hat\omega_i'(t_i^{k_i'}))\Big|\Big]\times \\
&\qquad \times \int_{Y_n}dx_n\sum_{j_n'\ge 1}\frac{z'^{j_n'}}{j_n'}\sum_{k_n=1}^{j_n}\sum_{k_n'=1}^{j_n'}\int_{\widetilde\Omega_0^{j_n'\beta}}W_{0,0}^{j_n'\beta,m}(d\hat\omega_n')\int_{\mathbb{R}^d}dx'\Big|\hat v\big(x-x'+\hat\omega(t_n^k)-\hat\omega'(\\
&= \|\hat v\|_{L^1(\mathbb{R}^d)}\sum_{2,\dots,n-1}^{\mathbb{R}^d} \mathbb{E}^{\,z',n}_{(x,\hat\omega)}(\xi')\Big[\prod_{i=2}^{n-1}\chi_{Y_{\eta(i)}}(x_i)\chi_{Y_i}(x_i')\Big|\sum_{k_i=1}^{j_i}\sum_{k_i'=1}^{j_i'}\hat v(x_i-x_i'+\hat\omega_i(t_i^{k_i})-\hat\omega_i'(t_i^{k_i'}))) \\
&\qquad \times j_{Y_{\eta(n)}}(\widetilde\omega_n)j_{Y_n}(\widetilde\omega_n') \le \cdots \le \big(\|\hat v\|_{L^1(\mathbb{R}^d)}\big)^{n-1}\mathbb{E}^{z',n}_{\widetilde\omega}(\xi')\Big[\prod_{i=2}^{n} j_{Y_{\eta(i)}}(\widetilde\omega_i)j_{Y_i}(\widetilde\omega_i')\Big],
\end{aligned}
\tag{3.18}
$$

where $j_{Y_i}(\widetilde\omega) = \chi_{Y_i}(x)j(\widetilde\omega) = \chi_{Y_i}(x)j \quad \text{for} \quad \widetilde\omega\in\widetilde\Omega^{j\beta}$ and $j(\widetilde\omega)$ is defind by (2.12).

Remark 3.3. *After summation over Y_n in the 5-th line of (3.18)(as a result we get $\|\hat v\|_{L^1(\mathbb{R}^d)}$) we again inserted $1=\int_{Y_n}\chi_{Y_n}(x_n')dx_n'$ with any $Y_n\subset\widetilde X\setminus X_{n-1}$, $|Y_n|=1$ and used the definition of measure $\hat\sigma^{z',\beta}_{Y_n}$. i.e.*

$$
\sum_{j\ge 1}\frac{z'^j}{j}\int_{Y_n}dx_n'\int_{\widetilde\Omega_0^{j\beta}}W_{0,0}^{j\beta}(d\hat\omega_n')j\chi_{Y_n}(x_n') \;=\; \int_{\widetilde\Omega^{com}_{Y_n}}\hat\sigma^{z',\beta}_{Y_n}(d\widetilde\omega_n)j_{Y_n}(\widetilde\omega_n).
$$

Now taking into account again Lemma 2.3 of [6], the estimate (3.18) can be rewritten in the form (for any $Y_i\subset\mathfrak{D}_{\widetilde X}$ with $Y_i\cup Y_j=\emptyset$ for $i\neq j$ and $X_n=\cup Y_i$)

$$\sum_{2,\dots,n}^{\mathbb{R}^d} \int_{\hat{\Gamma}^\beta} \hat{\pi}^{z,\beta}_{X_n}(d\gamma) \prod_{i=2}^{n} \left| \widetilde{U}(\gamma_{Y_{\eta(i)}}, \gamma_{Y_i}) \right| \le \big(\beta \|\hat{v}\|_{L^1(\mathbb{R}^d)}\big)^{n-1} \sum_{\xi'} \mathbb{E}^{z',n}_{\widetilde{\omega}}(\xi') \Big[\prod_{i=2}^{n} j_{Y_{\eta(i)}}(\widetilde{\omega}_i) j_{Y_i}(\widetilde{\omega}_i) \Big]$$

$$= \big(\beta \|\hat{v}\|_{L^1(\mathbb{R}^d)}\big)^{n-1} \prod_{i=1}^{n} \int \hat{\pi}^{z',\beta}_{Y_i}(d\gamma) \langle \gamma, j_{Y_i} \rangle^{d^*_\eta(i)}, \tag{3.19}$$

where $d^*_\eta(1) = d_\eta(1)$, and $d^*_\eta(i) = d_\eta(i) + 1$ for $i \ge 2$, and for a tree graph η, denote by $d_\eta(i)$ the number of edges entering the i-th vertex, i.e., $d_\eta(i) := \#\{\eta^{-1}(i)\}$.

To complete the estimate of the norm we prove the following proposition.

Proposition 3.1. *For any $Y \in \Lambda$, $|Y| = 1$, $z' < 1$ and function $j_Y(\widetilde{\omega})$, which is defind by (2.12)*

$$M_k = \int \hat{\pi}^{z',\beta}_Y(d\gamma) \langle \gamma, j_Y \rangle^k \ \le \ z' \lambda^{-d} \frac{1}{(1-z')^k} k!. \tag{3.20}$$

Proof. By induction for $k = 1$

$$M_1 = \int_{\widetilde{\Omega}^{com}_Y} \hat{\sigma}^{z',\beta}_Y(\widetilde{\omega}) j_Y(\widetilde{\omega}) = \sum_{j \ge 1} \frac{z'^j}{j} \int_Y dx \int_{\widetilde{\Omega}^{j\beta}} W^{j\beta,m}_{x,x}(d\widetilde{\omega}) j \le \lambda^{-d} \frac{z'}{1-z'}.$$

Let(3.20) is true for $k = 2, \dots, n-1$. Then using integration by parts formula ((3.1) of [7]) we get

$$M_n = \int \hat{\pi}^{z'\beta}_Y(d\gamma) \langle \gamma, j_Y \rangle^n = \int \hat{\pi}^{z',\beta}_Y(d\gamma) \int_{\widetilde{\Omega}^{com}} \hat{\sigma}^{\beta,z'}_Y(d\widetilde{\omega}) j_Y(\widetilde{\omega}) \big[\langle \gamma \cup \widetilde{\omega}, j_Y \rangle\big]^{n-1}$$

$$= \int_{\widetilde{\Omega}^{com}} \hat{\sigma}^{\beta,z'}_Y(d\widetilde{\omega}) j_Y(\widetilde{\omega}) \big[\langle \gamma, j_Y \rangle + j_Y(\widetilde{\omega})\big]^{n-1} \ = \ \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} \widehat{j_Y^{n-k}} M_k, \tag{3.21}$$

where

$$\widehat{j^l_Y} = \int \hat{\sigma}^{\beta,z'}_Y(d\widetilde{\omega})\big(j_Y(\widetilde{\omega})\big)^l = \sum_{j\geq 1}\frac{z'^j}{j}j^l\int_Y dx\int_{\widetilde{\Omega}^{j\beta}_Y} W^{j\beta,m}_{x,x}(d\widetilde{\omega}) \leq \lambda^{-d}(l-1)!\frac{z'}{(1-z')^l}. \tag{3.22}$$

Using (3.22) and (3.20) for $k \leq n-1$ we get from (3.21) the inequality (3.20) for $k = n$

Therefore, taking into account that $\lambda = (2\pi\beta/m_0)^{1/2}$ and $\sum_i d_\eta(i) = n-1$, from (3.19), (3.20) we have

$$\sum_{\xi'}\sum_{2,\dots,n}^{\mathbb{R}^d}\mathbb{E}^{z',n}_{\widetilde{\omega}}(\xi')\Big[\prod_{i=2}^n|\widetilde{V}_{Y_{\eta(i)},Y_i}(\omega_i,\omega'_i)|\Big] \leq \prod_{i=1}^{n-1}d_\eta(i)!\,\big(\beta^{1-d/2}z'(1-z')^{-2}m_0^{d/2}\|\hat{v}\|_{L^1(\mathbb{R}^d)}\big)^{n-1}. \tag{3.23}$$

Substituting (3.22 in (3.16) and using the Battle–Federbush estimate [16,17]

$$\sum_{\eta:|\eta|=n}\int_0^1\dots\int_0^1 ds_1\cdots ds_{n-1}\prod_{i=2}^n(s_{\eta(i)}\cdots s_{i-2})\prod_{i=1}^{n-1}d_\eta(i)! \leq 4^n,$$

by virtue of (3.8), we have that

$$\|Q(\delta f)\| \leq 2\Big[e^{-\alpha}+\sum_{n=2}^\infty\big(4\exp[(\frac{m}{2\pi\beta})^{d/2}\big(c(z')-c(z)\big)]\big)^n\big(\beta^{1-d/2}z'(1-z')^{-2}m^{d/2}e^\alpha\|\hat{v}\|_{L^1(\mathbb{R}^d)}\big)^{n-1}\Big]. \tag{3.24}$$

with $z' = ze^{\beta B+\rho M}$.

Thus, for an arbitrary fixed β, $z < e^{-\beta B}$, and choosing $\alpha > log4$ we can find sufficiently small $m_* > 0$ and $\rho > 0$ such that, for all $m_0 < m_*$, the right hand side of (3.24) (and so the norm of $Q(\delta f)$) is less than or equal to $C < 1$ uniformly for all $\delta \in O(r)$.

Then, it is obvious that $\|\hat{\varkappa}_\Lambda Q(\delta f)\hat{\varkappa}_\Lambda\| \leq C$, and so from (3.7), we get

$$\Phi^{z,\beta}_\Lambda(X;\delta f) = \big((\mathbf{1}-\hat{\varkappa}_\Lambda Q(\delta f)\hat{\varkappa}_\Lambda)^{-1}\Phi^{z,\beta}_\Lambda(\varnothing;\delta f)\varkappa_\varnothing\big)(X)$$
$$= \sum_{n=0}^\infty\big(\hat{\varkappa}_\Lambda Q(\delta f)\hat{\varkappa}_\Lambda\big)^n\Phi^{z,\beta}_\Lambda(\varnothing;\delta f)\varkappa_\varnothing(X). \tag{3.25}$$

Now, the end of the proof of Theorem 3.1 is the same as in [6].

Remark 3.4. *We have proved (as in [7] for MB statistics) that for sufficiently small z and m_0 an Euclidean Gibbs measure, which is defined for any finite volume Λ, for Quantum Statistics has thermodynamic limit. Now the main open problem is to prove that the limit measure is again Gibbsian, e.g. satisfies the DLR equation.*

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