УДК 517.9

A NEW HERMITE – HADAMARD TYPE INEQUALITY AND AN APPLICATION TO QUASI-EINSTEIN METRICS*

НОВА НЕРІВНІСТЬ ТИПУ ЕРМІТА – АДАМАРА ТА ЇЇ ЗАСТОСУВАННЯ ДО КВАЗІЕЙНШТЕЙНІВСЬКОЇ МЕТРИКИ

Xiang Gao

School Math. Sci. Ocean Univ. China
Lane 238, Songling Road, Laoshan District, Qingdao City,
Shandong Province, 266100, China
e-mail: gaoxiangshuli@126.com

We firstly establish a new generalization of the classical Hermite – Hadamard inequality for a real-valued convex function. Then the convexity of the matrix function $g(A) = f(\det A)$ is proved under certain conditions on the function f and the matrix A. Based on these, we derive a new Hermite – Hadamard type inequality, and finally present an application to the estimate of the volume of quasi-Einstein metrics.

Спочатку встановлено узагальнення класичної нерівності Ерміта – Адамара для дійсної опуклої функції. Далі доведено опуклість матричнозначної функції $g(A) = f(\det A)$ за певних умов на функцію f та матрицю A. На підставі цього результату отримано узагальнення нерівності типу Ерміта – Адамара та наведено застосування до оцінки об'єму квазіейнштейнівської метрики.

1. Introduction and main results. Inequalities with good symmetry are important and interesting in analysis and PDE, and among the inequality theory, the inequalities relating to convexity are extremely valuable. A well-known example is the famous Hermite – Hadamard inequality, which was firstly published in [1] and gives us an estimate of the mean value of a convex function:

Theorem 1.1 (Hermite – Hadamard inequality). *If $f : I \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

An account on the history of this inequality can be found in [2]. Surveys on various generalizations and developments can be found in [3] and [4].

Throughout this paper, we denote by I the closed interval $[a,b]$, then recall that a real-valued function f is said to be convex on I if

$$f(\mu x + (1-\mu)y) \le \mu f(x) + (1-\mu)f(y),$$

* This work is supported by the National Natural Science Foundation of China (NSFC) 11301493, 11101267 and Fundamental Research Funds for the Central Universities.

and concave on I if

$$f(\mu x + (1-\mu)y) \geq \mu f(x) + (1-\mu)f(y)$$

for all $x, y \in I$ and $0 \leq \mu \leq 1$.

In this paper we firstly establish a new generalization of the Hermite – Hadamard inequality, and prove that for an arbitrary nonnegative real-valued integrable function $\Phi : I \to \mathbb{R}$, there exist real numbers l, L such that:

$$f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right) \leq l \leq \frac{1}{b-a}\int\limits_a^b f\circ\Phi(x)dx \leq L \leq \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2},$$

where

$$l(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)f\left(\frac{1}{(\mu_{k+1}-\mu_k)(b-a)}\int\limits_{(1-\mu_k)a+\mu_k b}^{(1-\mu_{k+1})a+\mu_{k+1}b}\Phi(x)dx\right),$$

$$L(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)\,\frac{f\circ\Phi((1-\mu_k)a+\mu_k b)+f\circ\Phi((1-\mu_{k+1})a+\mu_{k+1}b)}{2}.$$

In fact, we prove the following theorem.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and $\Phi : I \to \mathbb{R}$ be a nonnegative real-valued integrable function such that $f\circ\Phi(x)$ is also convex. Then for arbitrary $n \in \mathbb{N}$, $\mu_0 = 0$, $\mu_{n+1} = 1$ and arbitrary $0 \leq \mu_1 \leq \ldots \leq \mu_n \leq 1$, we have*

$$f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right) \leq l(\mu_1,\ldots,\mu_n) \leq \frac{1}{b-a}\int\limits_a^b f\circ\Phi(x)dx \leq$$

$$\leq L(\mu_1,\ldots,\mu_n) \leq \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2},$$

where

$$l(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)f\left(\frac{1}{(\mu_{k+1}-\mu_k)(b-a)}\int\limits_{(1-\mu_k)a+\mu_k b}^{(1-\mu_{k+1})a+\mu_{k+1}b}\Phi(x)dx\right),$$

$$L(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)\,\frac{f\circ\Phi((1-\mu_k)a+\mu_k b)+f\circ\Phi((1-\mu_{k+1})a+\mu_{k+1}b)}{2}.$$

Corollary 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and $\Phi : I \to \mathbb{R}$ be a nonnegative real-valued integrable function such that $f\circ\Phi(x)$ is also convex, then for arbitrary $n \in \mathbb{N}$, $\mu_0 = 0$,*

$\mu_{n+1} = 1$ *and arbitrary* $0 \le \mu_1 \le \ldots \le \mu_n \le 1$, *we have*

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)dx\right) \le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} l(\mu_1,\ldots,\mu_n) \le \frac{1}{b-a}\int_a^b f\circ\Phi(x)dx \le$$

$$\le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} L(\mu_1,\ldots,\mu_n) \le \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2},$$

where $l(\mu_1,\ldots,\mu_n)$ *and* $L(\mu_1,\ldots,\mu_n)$ *are defined in Theorem 1.2.*

Then we consider the convexity of matrices, recall that a matrix $A \in M_n$ is said to be positive definite if $\mathrm{Re}\,(x^T Ax) > 0$, and is said to be nonnegative definite if $\mathrm{Re}\,(x^T Ax) \ge 0$ for all nonzero $x \in \mathbb{C}^n$. The convex set of positive definite matrices is denoted by M_n^+, and the convex set of nonnegative definite matrices is denoted by SM_n. Together with the definition of real-valued convex functions, we have the definition of convexity of matrix functions as follows:

Definition 1.1. *A real valued function f defined on* M_n^+ *or* SM_n *is said to be convex if*

$$f(\mu A + (1-\mu)B) \le \mu f(A) + (1-\mu)f(B),$$

and is said to be concave if

$$f(\mu A + (1-\mu)B) \ge \mu f(A) + (1-\mu)f(B)$$

for all $0 \le \mu \le 1$ *and all* $A, B \in M_n^+$ *or* SM_n, $A \ne B$.

Recall that it has been proved by Ky Fan in [5] (see also [6]) that the function $g(A) = \log(\det A)$ is a strictly concave function on the convex set of positive definite Hermitian matrices M_n^+. But in general, the function $g(A) = f(\det A)$ is not convex or concave for a general function f. In this paper, we will prove the convexity of the matrix function $g(A) = f(\det A)$ under certain conditions on the function f and matrix A as follows:

Theorem 1.3. *Let* A, B *be two nonnegative definite matrices such that* $AB = BA$, *and* $\lambda_i(A)$, $\lambda_i(B)$, *where* $i = 1,\ldots,n$, *be the eigenvalues of* A *and* B. *Then for arbitrary monotonic increasing and convex function* $f(x)$, *the inequality*

$$f(\det(\mu A + (1-\mu)B)) \le \mu f(\det A) + (1-\mu)f(\det B)$$

holds true for all $0 \le \mu \le 1$ *if one of the following conditions is satisfied:*
(i) $\lambda_i(A) \le \lambda_i(B)$ *for all* $i = 1,\ldots,n$;
(ii) $\lambda_i(A) \ge \lambda_i(B)$ *for all* $i = 1,\ldots,n$.

Thus by using Theorems 1.2 and 1.3, we derive our main result which is a new Hermite–Hadamard type inequality for the function $g(t) = f(\det A(t))$ as follows:

Theorem 1.4. *Let* $A(t) : I \to SM_n$ *be a family of nonnegative definite real-valued matrices with the eigenvalues* $\lambda_i(A(t))$, *where* $i = 1,\ldots,n$, *for corresponding* $t \in I$. *Suppose that for any* $t_1 \ne t_2 \in I$ *and* $0 \le \mu \le 1$, *we have* $A(t_1)A(t_2) = A(t_2)A(t_1)$ *and*

$$A(\mu t_1 + (1-\mu)t_2) \le \mu A(t_1) + (1-\mu)A(t_2),$$

then for $n \in \mathbb{N}$, $\mu_0 = 0$, $\mu_{n+1} = 1$ *and arbitrary* $0 \le \mu_1 \le \ldots \le \mu_n \le 1$, *the inequality*

$$f\left(\frac{1}{b-a}\int\limits_a^b \det A(t)dt\right) \le l_A(\mu_1,\ldots,\mu_n) \le \frac{1}{b-a}\int\limits_a^b f(\det A(t))dt \le$$

$$\le L_A(\mu_1,\ldots,\mu_n) \le \frac{f(\det A(a)) + f(\det A(b))}{2}$$

holds true for an arbitrary convex function $f : \mathbb{R} \to \mathbb{R}$ *if one of the following conditions is satisfied:*

(i) $\lambda_i(A(t_1)) \le \lambda_i(A(t_2))$ *for all* $i = 1,\ldots,n$ *and* $t_1 < t_2 \in I$;

(ii) $\lambda_i(A(t_1)) \ge \lambda_i(A(t_2))$ *for all* $i = 1,\ldots,n$ *and* $t_1 < t_2 \in I$, *where*

$$l_A(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k) f\left(\frac{1}{(\mu_{k+1}-\mu_k)(b-a)}\int\limits_{(1-\mu_k)a+\mu_k b}^{(1-\mu_{k+1})a+\mu_{k+1}b} \det A(t)dt\right),$$

$$L_A(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)\frac{f(\det A((1-\mu_k)a+\mu_k b)) + f(\det A((1-\mu_{k+1})a+\mu_{k+1}b))}{2}.$$

Remark 1.1. It is obvious that there exist many matrix families satisfying the conditions in Theorem 1.4 such that $\det A(t)$ and $f(\det A(t))$ integrable on the interval I.

Moreover by using Theorem 1.4, we can actually prove that there exist real numbers l_A, L_A such that we have the following.

Corollary 1.2. *Let* $A(t) : I \to SM_n$ *be a family of nonnegative definite real-valued matrices with the eigenvalues* $\lambda_i(A(t))$, *where* $i = 1,\ldots,n$, *for corresponding* $t \in I$. *Suppose that for any* $t_1 \ne t_2 \in I$ *and* $0 \le \mu \le 1$, *we have* $A(t_1)A(t_2) = A(t_2)A(t_1)$ *and*

$$A(\mu t_1 + (1-\mu)t_2) \le \mu A(t_1) + (1-\mu)A(t_2),$$

then for $n \in \mathbb{N}$, $\mu_0 = 0$, $\mu_{n+1} = 1$ *and arbitrary* $0 \le \mu_1 \le \ldots \le \mu_n \le 1$, *the inequalities*

$$f\left(\frac{1}{b-a}\int\limits_a^b \det A(t)dt\right) \le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} l_A(\mu_1,\ldots,\mu_n) \le \frac{1}{b-a}\int\limits_a^b f(\det A(t))dt \le$$

$$\le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} L_A(\mu_1,\ldots,\mu_n) \le \frac{f(\det A(a)) + f(\det A(b))}{2}$$

holds true for an arbitrary convex function $f : \mathbb{R} \to \mathbb{R}$ *if one of the following conditions is satisfied:*

(i) $\lambda_i(A(t_1)) \le \lambda_i(A(t_2))$ *for all* $i = 1,\ldots,n$ *and* $t_1 < t_2 \in I$;

(ii) $\lambda_i(A(t_1)) \ge \lambda_i(A(t_2))$ *for all* $i = 1,\ldots,n$ *and* $t_1 < t_2 \in I$,

where $l_A(\mu_1,\ldots,\mu_n)$ *and* $L_A(\mu_1,\ldots,\mu_n)$ *are defined in Theorem 1.4.*

Now let (M^n, g_0) be a Riemannian manifold. The Ricci flow on M^n is a process that deforms the metric of the Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out irregularities in the metric. See [7] or [8] for details. The Ricci flow is defined by the partial differential equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}\,g(t),$$

$$g(0) = g_0,$$

where $\mathrm{Ric}\,g(t)$ denotes the Ricci curvature of the metric $g(t)$. Ricci flow is a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings, which can be used to deform g into a metric distinguished by its curvature. For example, if the Riemannian manifold (M^2, g) is two-dimensional, the Ricci flow, once suitably renormalised, deforms a metric conformally to one of constant curvature.

The quasi-Einstein metric, also named as the Ricci soliton, which is the fixed point of the Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings.

In fact, let $X(t)$ be a time dependent family of smooth vector fields on M^n, generating a family of diffeomorphisms $\varphi(t)$, and the quasi-Einstein metric is actually the self-similar solution

$$g(t) = \varphi^*(t) g(0)$$

of Ricci flow, where $g(0)$ denotes the initial metric.

From the equation point of view, the quasi-Einstein metric is also a natural generalization of the Einstein metric.

Definition 1.2. (Quasi-Einstein metric). *A complete Riemannian manifold (M^n, g) is called a quasi-Einstein metric if there exists a smooth function $f : M^n \to \mathbb{R}$, such that*

$$Rc(g) + \nabla\nabla f + \frac{\varepsilon}{2} g = 0,$$

where Rc is the Ricci curvature tensor and ε is a real constant. Furthermore

(i) if $\varepsilon < 0$, then it is called a shrinking quasi-Einstein;

(ii) if $\varepsilon = 0$, then it is called a steady one;

(iii) if $\varepsilon > 0$, then it is called an expanding one.

In this paper, by using Theorem 1.4 and Jensen's inequality for the manifold, we derive an estimate of the volume of quasi-Einstein manifolds as follows:

Theorem 1.5. *Let (M^n, g) be a Riemannian manifold and $\varphi^*(t) : I \to \mathrm{Diff}\,(M^n)$ be a family of diffeomorphisms of M^n such that their corresponding matrices are nonnegative definite real-valued matrices, which are also denoted by $\varphi^*(t)$. Let $\lambda_i(\varphi^*(t))$, where $i = 1, \ldots, n$, denote the eigenvalues of $\varphi^*(t)$, for corresponding $t \in I$. Suppose that for any $t_1 \neq t_2 \in I$ and $0 \leq \mu \leq 1$, we have*

$$\varphi^*(\mu t_1 + (1-\mu)t_2) \leq \mu\varphi^*(t_1) + (1-\mu)\varphi^*(t_2).$$

Then for a quasi-Einstein metric $g(t) = \varphi^*(t)g(0)$ *defined on the interval I, and arbitrary* $n \in \mathbb{N}$, $\mu_0 = 0$, $\mu_{n+1} = 1$, $0 \le \mu_1 \le \ldots \le \mu_n \le 1$, *the inequality*

$$f\left(\frac{1}{b-a}\int\limits_a^b V(t)dt\right) \le l_{\phi^*}(\mu_1,\ldots,\mu_n),$$

$$\frac{1}{b-a}\int\limits_a^b f(V(t))dt \le L_{\varphi^*}(\mu_1,\ldots,\mu_n)$$

holds true for an arbitrary convex function $f:\mathbb{R}\to\mathbb{R}$ *if one of the following conditions is satisfied:*

(i) $\lambda_i(\varphi^*(t_1)) \le \lambda_i(\varphi^*(t_2))$ *for all* $i = 1,\ldots,n$ *and* $t_1 < t_2 \in I$;

(ii) $\lambda_i(\varphi^*(t_1)) \ge \lambda_i(\varphi^*(t_2))$ *for all* $i = 1,\ldots,n$ *and* $t_1 < t_2 \in I$, *where* $V(t)$ *is the volume of* M^n $\int_{M^n}\sqrt{\det(g(t))}dV$ *corresponding to the metric* $g(t)$ *and*

$$l_{\varphi^*}(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)f\left(\frac{1}{(\mu_{k+1}-\mu_k)(b-a)}\int\limits_{(1-\mu_k)a+\mu_k b}^{(1-\mu_{k+1})a+\mu_{k+1}b}\sqrt{\det(\varphi^*(t))}dt\right),$$

$$L_{\varphi^*}(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)\times$$
$$\times\frac{f\left(\sqrt{\det(\varphi^*((1-\mu_k)a+\mu_k b))}\right)+f\left(\sqrt{\det(\varphi^*((1-\mu_{k+1})a+\mu_{k+1}b))}\right)}{2}.$$

Corollary 1.3. *Let* (M^n, g) *be a Riemannian manifold and* $\varphi^*(t): I \to \mathrm{Diff}(M^n)$ *be a family of diffeomorphisms of* M^n *such that their corresponding matrices are nonnegative definite real-valued matrices, which are also denoted by* $\varphi^*(t)$. *Let* $\lambda_i(\varphi^*(t))$, *where* $i = 1,\ldots,n$, *denote the eigenvalues of* $\varphi^*(t)$, *for corresponding* $t \in I$. *Suppose that for any* $t_1 \ne t_2 \in I$ *and* $0 \le \mu \le 1$, *we have*

$$\varphi^*(\mu t_1 + (1-\mu)t_2) \le \mu\varphi^*(t_1) + (1-\mu)\varphi^*(t_2).$$

Then for a quasi-Einstein metric $g(t) = \varphi^*(t)g(0)$ *defined on the interval I, and arbitrary* $n \in \mathbb{N}$, $\mu_0 = 0$, $\mu_{n+1} = 1$, $0 \le \mu_1 \le \ldots \le \mu_n \le 1$, *the inequalities*

$$f\left(\frac{1}{b-a}\int\limits_a^b V(t)dt\right) \le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} l_{\varphi^*}(\mu_1,\ldots,\mu_n),$$

$$\frac{1}{b-a}\int\limits_a^b f(V(t))dt \le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} L_{\varphi^*}(\mu_1,\ldots,\mu_n)$$

holds true for an arbitrary convex function $f : \mathbb{R} \to \mathbb{R}$ if one of the following conditions is satisfied:

(i) $\lambda_i(\varphi^*(t_1)) \leq \lambda_i(\varphi^*(t_2))$ *for all* $i = 1, \ldots, n$ *and* $t_1 < t_2 \in I$;

(ii) $\lambda_i(\varphi^*(t_1)) \geq \lambda_i(\varphi^*(t_2))$ *for all* $i = 1, \ldots, n$ *and* $t_1 < t_2 \in I$,

where $V(t)$, $l_{\varphi^*}(\mu_1, \ldots, \mu_n)$ *and* $L_{\varphi^*}(\mu_1, \ldots, \mu_n)$ *are defined in Theorem 1.5.*

The paper is organized as follows. In Section 2, we firstly establish two useful lemmas, by which we then prove Theorem 1.2. In Section 3, we prove Theorem 1.3 by using an interesting lemma. In Section 4, based on Theorems 1.2 and 1.4, we prove our main result Theorem 1.4 and present an application to the estimate of the volume of quasi-Einstein metrics.

2. Lemmas and proof of Theorem 1.2. In order to prove Theorem 1.2, we also need some lemmas as follows:

Lemma 2.1. *If* $f : \mathbb{R} \to \mathbb{R}$ *and* $\Phi : I \to \mathbb{R}$ *are integrable functions, then we have*

$$\frac{1}{b-a}\int\limits_a^b f \circ \Phi(x)dx = \int\limits_0^1 f \circ \Phi(\mu a + (1-\mu)b)d\mu = \int\limits_0^1 f \circ \Phi(\mu b + (1-\mu)a)d\mu.$$

Proof. We could use the change of variables $x = \mu a + (1-\mu)b$ and $x = \mu b + (1-\mu)a$ to complete the proof.

Lemma 2.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex function, and* $\Phi : I \to \mathbb{R}$ *be a nonnegative real-valued integrable function such that* $f \circ \Phi(x)$ *is also convex, then we have*

$$f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right) \leq \frac{1}{b-a}\int\limits_a^b f \circ \Phi(x)dx \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}.$$

Proof. Observing that the first inequality is actually the famous Jensen's inequality (see [9]), thus we only need to prove the second one. Since $f \circ \Phi(x)$ is a convex function, we have, for arbitrary $\mu \in [0,1]$, that

$$\frac{f \circ \Phi(\mu a + (1-\mu)b) + f \circ \Phi((1-\mu)a + \mu b)}{2} \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}. \tag{2.1}$$

Integrating (2.1) over $[0,1]$ and using Lemma 2.1 we have

$$\frac{1}{b-a}\int\limits_a^b f \circ \Phi(x)dx \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}.$$

Remark 2.1. If $f : \mathbb{R} \to \mathbb{R}$ is a concave function, and $\Phi : I \to \mathbb{R}$ is a nonnegative real-valued integrable function such that $f \circ \Phi(x)$ is also concave, then as in the proof of Lemma 2.2 we have

$$\frac{f \circ \Phi(a) + f \circ \Phi(b)}{2} \leq \frac{1}{b-a}\int\limits_a^b f \circ \Phi(x)dx \leq f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right).$$

With the help of Lemmas 2.1 and 2.2, we now turn to a prove Theorem 1.2.

Proof of Theorem 1.2. It follows from the hypothesis that $f(x)$ and $f\circ\Phi(x)$ are both convex functions, and Lemma 2.2 that

$$f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right) \le \frac{1}{b-a}\int\limits_a^b f\circ\Phi(x)dx \le \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \tag{2.2}$$

By assumption $\lambda_0 = 0$, so

$$[a,(1-\lambda_1)a+\lambda_1 b] = [t(1-\lambda_0)a+\lambda_0 b,(1-\lambda_1)a+\lambda_1 b].$$

Then applying (2.2) to

$$[(1-\lambda_k)a+\lambda_k b,(1-\lambda_{k+1})a+\lambda_{k+1}b]$$

for $k = 0,1,\ldots,n$ we have

$$f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)dx\right) \le$$

$$\le \frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f\circ\Phi(x)dx \le$$

$$\le \frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}. \tag{2.3}$$

Multiplying each term in (2.3) by corresponding $(\lambda_{k+1}-\lambda_k)$, and adding the resulting inequalities, we get

$$\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)dx\right) \le$$

$$\le \sum_{k=0}^{n}\frac{1}{b-a}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f\circ\Phi(x)dx \le$$

$$\le \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2},$$

that is,

$$l(\lambda_1,\ldots,\lambda_n) \le \frac{1}{b-a}\int\limits_a^b f\circ\Phi(x)dx \le L(\lambda_1,\ldots,\lambda_n),$$

where $l(\lambda_1,\dots,\lambda_n)$ and $L(\lambda_1,\dots,\lambda_n)$ are defined as in Theorem 1.2.

To prove the remaining two inequalities,

$$f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right) \le l(\lambda_1,\dots,\lambda_n) \le L(\lambda_1,\dots,\lambda_n) \le \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2},$$

we use the fact that $f:\mathbb{R}\to\mathbb{R}$, $f\circ\Phi(x)$ are both convex functions and observe that $\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)=1$. We have

$$f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right) = f\left(\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)dx\right) \le$$

$$\le \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k) f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)dx\right) \le$$

$$\le \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2} \le$$

$$\le \frac{1}{2}\sum_{k=0}^{n}(((1-\lambda_k)-(1-\lambda_{k+1}))((1-\lambda_k)+(1-\lambda_{k+1}))f\circ\Phi(a))+$$

$$+\frac{1}{2}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)(\lambda_{k+1}+\lambda_k)f\circ\Phi(b) =$$

$$=\frac{1}{2}\sum_{k=0}^{n}\left(\left((1-\lambda_k)^2-(1-\lambda_{k+1})^2\right)f\circ\Phi(a)+\left(\lambda_{k+1}^2-\lambda_k^2\right)f\circ\Phi(b)\right) =$$

$$= \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}.$$

Theorem 1.2 is proved.

Remark 2.2. In fact the key point of our proof only relies on Lemma 2.2, thus by the following inequality in Remark 2.1:

$$\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2} \le \frac{1}{b-a}\int\limits_a^b f\circ\Phi(x)dx \le f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right),$$

we have the following theorem.

Theorem 2.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a concave function, and $\Phi: I\to\mathbb{R}$ be a nonnegative real-valued integrable function such that $f\circ\Phi(x)$ is also concave. Then for arbitrary $n\in\mathbb{N}$, $\mu_0=0$,*

$\mu_{n+1} = 1$ *and arbitrary* $0 \le \mu_1 \le \ldots \le \mu_n \le 1$, *we have*

$$\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2} \le l(\mu_1,\ldots,\mu_n) \le \frac{1}{b-a}\int\limits_a^b f\circ\Phi(x)dx \le$$

$$\le L(\mu_1,\ldots,\mu_n) \le f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right),$$

where

$$l(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)\frac{f\circ\Phi((1-\mu_k)a+\mu_k b)+f\circ\Phi((1-\mu_{k+1})a+\mu_{k+1}b)}{2},$$

$$L(\mu_1,\ldots,\mu_n) = \sum_{k=0}^{n}(\mu_{k+1}-\mu_k)f\left(\frac{1}{(\mu_{k+1}-\mu_k)(b-a)}\int\limits_{(1-\mu_k)a+\mu_k b}^{(1-\mu_{k+1})a+\mu_{k+1}b}\Phi(x)dx\right).$$

Corollary 2.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a concave function, and* $\Phi : I \to \mathbb{R}$ *be a nonnegative real-valued integrable function such that* $f\circ\Phi(x)$ *is also concave, then for arbitrary* $n \in \mathbb{N}$, $\mu_0 = 0$, $\mu_{n+1} = 1$ *and arbitrary* $0 \le \mu_1 \le \ldots \le \mu_n \le 1$, *we have*

$$\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2} \le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} l(\mu_1,\ldots,\mu_n) \le \frac{1}{b-a}\int\limits_a^b f\circ\Phi(x)dx \le$$

$$\le \sup_{0\le\mu_1\le\ldots\le\mu_n\le 1} L(\mu_1,\ldots,\mu_n) \le f\left(\frac{1}{b-a}\int\limits_a^b \Phi(x)dx\right),$$

where $l(\mu_1,\ldots,\mu_n)$ *and* $L(\mu_1,\ldots,\mu_n)$ *are defined in Theorem 2.1.*

3. Proof of Theorem 1.3. In order to prove Theorem 1.3, we shall need the following lemma, which is in fact the multiplicative analogue of Chebyshev's algebraic inequality.

Lemma 3.1. *If* $0 \le \alpha, \beta \le 1$ *satisfying* $\alpha+\beta = 1$, *and* $\mu_i \ge \nu_i$ *for arbitrary* $1 \le i \le n$ *or* $\mu_i \le \nu_i$ *for arbitrary* $1 \le i \le n$, *then*

$$\prod_{i=1}^{n}(\mu_i\alpha+\nu_i\beta) \le \alpha\prod_{i=1}^{n}\mu_i + \beta\prod_{i=1}^{n}\nu_i. \tag{3.1}$$

Proof. The approach we use is mathematical induction. Firstly we consider $n = 2$, since $0 \le \alpha, \beta \le 1$ and $\alpha+\beta = 1$, we have

$$(\mu_1\alpha+\nu_1\beta)(\mu_2\alpha+\nu_2\beta) = ((\mu_1-\nu_1)\alpha+\nu_1)(\mu_2-(\mu_2-\nu_2)\beta) =$$

$$= \mu_2(\mu_1-\nu_1)\alpha - \nu_1(\mu_2-\nu_2)\beta - (\mu_1-\nu_1)(\mu_2-\nu_2)\alpha\beta + \mu_2\nu_1 =$$

$$= \mu_1\mu_2\alpha + \nu_1\nu_2\beta - (\mu_1-\nu_1)(\mu_2-\nu_2)\alpha\beta.$$

It follows from the hypotheses that

$$(\mu_1\alpha + \nu_1\beta)(\mu_2\alpha + \nu_2\beta) \le \mu_1\mu_2\alpha + \nu_1\nu_2\beta.$$

Assume that (3.1) is true for $n = k$, we prove that it is also true for $n = k + 1$. Since (3.1) holds for $n = k$, we get

$$\prod_{i=1}^{k+1}(\mu_i\alpha + \nu_i\beta) = (\mu_{k+1}\alpha + \nu_{k+1}\beta)\prod_{i=1}^{k}(\mu_i\alpha + \nu_i\beta) \le$$

$$\le (\mu_{k+1}\alpha + \nu_{k+1}\beta)\left(\alpha\prod_{i=1}^{k}\mu_i + \beta\prod_{i=1}^{k}\nu_i\right).$$

As in the proof for $n = 2$ we obtain that

$$(\mu_{k+1}\alpha + \nu_{k+1}\beta)\left(\alpha\prod_{i=1}^{k}\mu_i + \beta\prod_{i=1}^{k}\nu_i\right) = \left(\alpha^2\mu_{k+1}\prod_{i=1}^{k}\mu_i + \beta^2\nu_{k+1}\prod_{i=1}^{k}\nu_i\right) -$$

$$- \alpha\beta(\mu_{k+1} - \nu_{k+1})\left(\prod_{i=1}^{k}\mu_i - \prod_{i=1}^{k}\nu_i\right). \qquad (3.2)$$

Since the second term in (3.2) is nonpositive for $\mu_i \ge \nu_i$ or for $\mu_i \le \nu_i$, it follows from the condition $0 \le \alpha, \beta \le 1$ that

$$\prod_{i=1}^{k+1}(\mu_i\alpha + \nu_i\beta) \le \alpha\prod_{i=1}^{k+1}\mu_i + \beta\prod_{i=1}^{k+1}\nu_i$$

and consequently inequality (3.1) holds for $n = k + 1$.

Lemma 3.1 is proved.

With the help of Lemma 3.1, we now turn to a prove Theorem 1.3.

Proof of Theorem 1.3. Since $f(x)$ is a convex function, we have

$$f(\mu x + (1-\mu)y) \le \mu f(x) + (1-\mu)f(y)$$

for arbitrary $x, y \in \mathbb{R}$ and $0 \le \mu \le 1$. Putting $x = \det A$ and $y = \det B$ it follows that

$$f(\mu\det A + (1-\mu)\det B) \le \mu f(\det A) + (1-\mu)f(\det B). \qquad (3.3)$$

On the other hand, since $AB = BA$, it is known from [6] that there exists an orthogonal matrix C such that $A = C^T\Lambda_A C$, $B = C^T\Lambda_B C$, where

$$\Lambda_A = \operatorname{diag}\{\lambda_1(A), \ldots, \lambda_n(A)\}, \quad \Lambda_B = \operatorname{diag}\{\lambda_1(B), \ldots, \lambda_n(B)\}.$$

Thus

$$\det(\mu A + (1-\mu)B) \le \mu\det A + (1-\mu)\det B$$

is equivalent to

$$\det(\mu\Lambda_A + (1-\mu)\Lambda_B) \le \mu \det \Lambda_A + (1-\mu) \det \Lambda_B,$$

that is

$$\prod_{i=1}^{n}(\mu\lambda_i(A) + (1-\mu)\lambda_i(B)) \le \mu \prod_{i=1}^{n} \lambda_i(A) + (1-\mu) \prod_{i=1}^{n} \lambda_i(B).$$

Therefore, if the condition (i) is satisfied, we have $\lambda_i(A) \le \lambda_i(B)$ for arbitrary $1 \le i \le n$, and if the condition (ii) is satisfied, we have $\lambda_i(A) \ge \lambda_i(B)$ for arbitrary $1 \le i \le n$. Then it follows from Lemma 3.1 that

$$\prod_{i=1}^{n}(\mu\lambda_i(A) + (1-\mu)\lambda_i(B)) \le \mu \prod_{i=1}^{n} \lambda_i(A) + (1-\mu) \prod_{i=1}^{n} \lambda_i(B),$$

which is equivalent to

$$\det(\mu A + (1-\mu)B) \le \mu \det A + (1-\mu) \det B.$$

Furthermore, since $f(x)$ is a monotone increasing function, together with (3.3) we have

$$f(\det(\mu A + (1-\mu)B)) \le \mu f(\det A) + (1-\mu) f(\det B).$$

Theorem 1.3 is proved.

4. Proof of the main results. In this section, with the help of Theorems 1.2 and 1.3, we prove our main results.

Proof of Theorem 1.4. If one of the following conditions is satisfied:
(i) $\lambda_i(A(t_1)) \le \lambda_i(A(t_2))$ for all $i = 1, \ldots, n$ and $t_1 < t_2 \in I$;
(ii) $\lambda_i(A(t_1)) \ge \lambda_i(A(t_2))$ for all $i = 1, \ldots, n$ and $t_1 < t_2 \in I$,
then by Theorem 1.3 we have for arbitrary monotone increasing and convex function $f(x)$, that the inequality

$$f(\det(\mu A(t_1) + (1-\mu)A(t_2))) \le \mu f(\det A(t_1)) + (1-\mu) f(\det A(t_2)) \tag{4.1}$$

holds for any $t_1 < t_2 \in I$ and $0 \le \mu \le 1$. Moreover since for any $t_1 \ne t_2 \in I$ and $0 \le \mu \le 1$, we have

$$A(\mu t_1 + (1-\mu)t_2) \le \mu A(t_1) + (1-\mu)A(t_2).$$

Since $A(t) : I \to M_n^+$ is a family of nonnegative definite real-valued matrices, it follows that $A(\mu t_1 + (1-\mu)t_2)$ and $\mu A(t_1) + (1-\mu)A(t_2)$ are both nonnegative definite real-valued matrices. By [6] we have that there exists an orthogonal matrix C such that

$$A(\mu t_1 + (1-\mu)t_2) = C^T \Lambda_{A(\mu t_1 + (1-\mu)t_2)} C$$

and

$$\mu A(t_1) + (1-\mu)A(t_2) = C^T \Lambda_{\mu A(t_1) + (1-\mu)A(t_2)} C,$$

where

$$\Lambda_{A(\mu t_1+(1-\mu)t_2)} = \mathrm{diag}\,\{\lambda_1(A(\mu t_1+(1-\mu)t_2)),\ldots,\lambda_n(A(\mu t_1+(1-\mu)t_2))\},$$

$$\Lambda_{\mu A(t_1)+(1-\mu)A(t_2)} = \mathrm{diag}\,\{\lambda_1(\mu A(t_1)+(1-\mu)A(t_2)),\ldots,\lambda_n(\mu A(t_1)+(1-\mu)A(t_2))\}.$$

Then it follows from $A(\mu t_1+(1-\mu)t_2) \le \mu A(t_1)+(1-\mu)A(t_2)$ that

$$C^T\Lambda_{A(\mu t_1+(1-\mu)t_2)}C \le C^T\Lambda_{\mu A(t_1)+(1-\mu)A(t_2)}C,$$

which is equivalent to

$$0 \le \lambda_i(A(\mu t_1+(1-\mu)t_2)) \le \lambda_i(\mu A(t_1)+(1-\mu)A(t_2))$$

for any $1 \le i \le n$. Thus

$$\det A(\mu t_1+(1-\mu)t_2) = \prod_{i=1}^n \lambda_i(A(\mu t_1+(1-\mu)t_2)) \le \prod_{i=1}^n \lambda_i(\mu A(t_1)+(1-\mu)A(t_2)) =$$

$$= \det(\mu A(t_1)+(1-\mu)A(t_2)).$$

Since $f(x)$ is monotone and increasing, it follows that

$$f(\det A(\mu t_1+(1-\mu)t_2)) \le f(\det(\mu A(t_1)+(1-\mu)A(t_2))).$$

Then by using (4.1) we have

$$f(\det A(\mu t_1+(1-\mu)t_2)) \le \mu f(\det A(t_1))+(1-\mu)f(\det A(t_2))$$

for any $t_1 \ne t_2 \in I$ and $0 \le \mu \le 1$, which implies that the function $f(\det A(t))$ is a convex function of t. Then by using Theorem 1.2 we have

$$f\left(\frac{1}{b-a}\int_a^b \det A(t)dt\right) \le l_A(\mu_1,\ldots,\mu_n) \le \frac{1}{b-a}\int_a^b f(\det A(t))dt \le$$

$$\le L_A(\mu_1,\ldots,\mu_n) \le \frac{f(\det A(a))+f(\det A(b))}{2},$$

where $l_A(\mu_1,\ldots,\mu_n)$ and $L_A(\mu_1,\ldots,\mu_n)$ are defined in Theorem 1.4.

Theorem 1.4 is proved.

Then we consider the quasi-Einstein metric, note that from the local coordinate formula

$$dV(t) = \sqrt{\det(g(t))}dx^1\wedge\ldots\wedge dx^n = \sqrt{\det(\varphi^*(t)g(0))}dx^1\wedge\ldots\wedge dx^n,$$

by using Theorem 1.4 and Jensen's inequality for the manifold, we derive an estimate of the volume of quasi-Einstein manifold as follows, and prove the estimate of the volume of the quasi-Einstein metric.

Proof of Theorem 1.5. Since $g(t)$ is a quasi-Einstein metric, we have $g(t_1+t_2) = g(t_2+t_1)$, which is equivalent to

$$\varphi^*(t_1)\varphi^*(t_2)g(0) = \varphi^*(t_1+t_2)g(0) = \varphi^*(t_2+t_1)g(0) = \varphi^*(t_2)\varphi^*(t_1)g(0).$$

Thus we have $\varphi^*(t_1)\varphi^*(t_2) = \varphi^*(t_2)\varphi^*(t_1)$ for any $t_1 \neq t_2 \in I$. Moreover for any $0 \leq \mu \leq 1$, it follows from the hypothesis that

$$\varphi^*(\mu t_1 + (1-\mu)t_2) \leq \mu\varphi^*(t_1) + (1-\mu)\varphi^*(t_2),$$

then if one of the following conditions is satisfied:

(i) $\lambda_i(\varphi^*(t_1)) \leq \lambda_i(\varphi^*(t_2))$ for all $i = 1,\ldots,n$ and $t_1 < t_2 \in I$;

(ii) $\lambda_i(\varphi^*(t_1)) \geq \lambda_i(\varphi^*(t_2))$ for all $i = 1,\ldots,n$ and $t_1 < t_2 \in I$,

as in the proof of Theorem 1.4, for the function $f(\sqrt{\det(\varphi^*(t))})$ we have that

$$f\left(\frac{1}{b-a}\int_a^b \sqrt{\det(\varphi^*(t))}dt\right) \leq l_{\varphi^*}(\mu_1,\ldots,\mu_n) \leq \frac{1}{b-a}\int_a^b f\left(\sqrt{\det(\varphi^*(t))}\right)dt \leq$$

$$\leq L_{\varphi^*}(\mu_1,\ldots,\mu_n) \leq \frac{f\left(\sqrt{\det(\varphi^*(a))}\right) + f\left(\sqrt{\det(\varphi^*(b))}\right)}{2}$$

holds true for arbitrary $n \in \mathbb{N}$, $\mu_0 = 0$, $\mu_{n+1} = 1$, $0 \leq \mu_1 \leq \ldots \leq \mu_n \leq 1$, where $l_{\varphi^*}(\mu_1,\ldots,\mu_n)$ and $L_{\varphi^*}(\mu_1,\ldots,\mu_n)$ are defined in Theorem 1.5.

Then taking the product of this with $\sqrt{\det(g(0))}$ and integrating we have obtain

$$\int_{M^n} f\left(\frac{1}{b-a}\int_a^b \sqrt{\det(\varphi^*(t))}dt\right)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n \leq$$

$$\leq \int_{M^n} l_{\varphi^*}(\mu_1,\ldots,\mu_n)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n \leq$$

$$\leq \int_{M^n}\left(\frac{1}{b-a}\int_a^b f\left(\sqrt{\det(\varphi^*(t))}\right)dt\right)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n \leq$$

$$\leq \int_{M^n} L_{\varphi^*}(\mu_1,\ldots,\mu_n)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n \leq$$

$$\leq \int_{M^n}\frac{f\left(\sqrt{\det(\varphi^*(a))}\right) + f\left(\sqrt{\det(\varphi^*(b))}\right)}{2}\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n. \tag{4.2}$$

Since it follows from the hypothesis that $f(\sqrt{\det(\varphi^*(t))})$ is a convex function and

$$\int_{M^n}\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n = 1, \tag{4.3}$$

by using Jensen's inequality for manifolds, we get

$$\int\limits_{M^n} f\left(\frac{1}{b-a}\int\limits_a^b \sqrt{\det(\varphi^*(t))}dt\right)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n \geq$$

$$\geq f\left(\int\limits_{M^n}\frac{1}{b-a}\int\limits_a^b \sqrt{\det(\varphi^*(t))}dt\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n\right) =$$

$$= f\left(\frac{1}{b-a}\int\limits_a^b\int\limits_{M^n} \sqrt{\det(\varphi^*(t))}\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n dt\right) =$$

$$= f\left(\frac{1}{b-a}\int\limits_a^b V(t)dt\right) \tag{4.4}$$

and

$$\int\limits_{M^n}\left(\frac{1}{b-a}\int\limits_a^b f\left(\sqrt{\det(\varphi^*(t))}\right)dt\right)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n =$$

$$= \frac{1}{b-a}\int\limits_a^b\int\limits_{M^n} f\left(\sqrt{\det(\varphi^*(t))}\right)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n dt \geq$$

$$\geq \frac{1}{b-a}\int\limits_a^b f\left(\int\limits_{M^n}\sqrt{\det(\varphi^*(t))}\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n\right)dt =$$

$$= \frac{1}{b-a}\int\limits_a^b f(V(t))dt. \tag{4.5}$$

On the other hand, by using (4.3) we have

$$\int\limits_{M^n} l_{\varphi^*}(\mu_1,\ldots,\mu_n)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n = l_{\varphi^*}(\mu_1,\ldots,\mu_n),$$

$$\int\limits_{M^n} L_{\varphi^*}(\mu_1,\ldots,\mu_n)\sqrt{\det(g(0))}dx^1\wedge\ldots\wedge dx^n = L_{\varphi^*}(\mu_1,\ldots,\mu_n).$$

Together with (4.2), (4.4) and (4.5) we obtain

$$f\left(\frac{1}{b-a}\int\limits_a^b V(t)dt\right) \leq l_{\varphi^*}(\mu_1,\ldots,\mu_n)$$

and

$$\frac{1}{b-a}\int_a^b f(V(t))dt \leq L_{\varphi^*}(\mu_1,\ldots,\mu_n).$$

Theorem 1.5 is proved.

1. *Hadamard J.* Etude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann // J. math. pures et appl. — 1893. — **58**. — P. 171–215.
2. *Mitrinović D. S., Lacković I. B.* Hermite and convexity // Aequat. math. — 1985. — **28**. — P. 229–232.
3. *Niculescu C., Persson L.-E.* Old and new on the Hermite–Hadamard inequality // Real Analysis Exchange. — 2004.
4. *Dragomir S. S., Pearce C. E. M.* Selected topics on Hermite–Hadamard inequalities (RGMIA Monographs http://rgmia.vu.edu.au /monographs/hermite hadamard.html).
5. *Ky Fan.* Convex sets and their applications // Lect. Notes. Appl. Math. Division. — Argonne Nat. Laboratory, Summer, 1959.
6. *Horn R. A., Johnson C. R.* Matrix analysis. — Cambridge Univ. Press, 1999.
7. *Morgan J., Tian G.* Ricci flow and the Poincaré conjecture // Clay Math. Monographs 3. Amer. Math. Soc. — 2007.
8. *Topping P.* Lectures on the Ricci flow // London Math. Soc. Lect. Note Ser. 325. — Cambridge Univ. Press, 2006.
9. *Hardy G., Littlewood J. E., Polya G.* Inequalities // Cambridge Math. Library. — Cambridge Univ. Press, 1999.

Received 25.06.12,
after revision — 17.05.14