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NONLINEAR MODAL EQUATIONS FOR A LEVITATING DROP

НЕЛІНІЙНІ МОДАЛЬНІ РІВНЯННЯ ЛЕВІТУЮЧОЇ КРАПЛІ

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Based on variational method, the paper derives nonlinear modal equations describing the dynamics of a levitating drop. Using these equations, we construct an asymptotic modal theory for axisymmetric drop oscillations. We consider nonlinear free oscillations of the drop with the frequency close to the lowest natural frequency, the results are compared with experimental data and numerical results obtained by other authors.

Базуючись на варіаційному методі, виведено нелінійні модальні рівняння динаміки левітуючої краплі. З допомогою цих рівнянь побудовано асимптотичну модальну теорію осесиметричних коливань. Розглянуто нелінійні вільні коливання з частотою, близькою до першої власної частоти. Результати порівнюються з експериментальними даними та чисельними результатами інших авторів.

1. Introduction. Modern chemical industry employs levitating drops in an ullage gas [5, 6, 15] for optimizing the reaction flow. The levitation is typically provided by acoustic and/or electromagnetic fields created in the gas [2–4, 12, 14, 16–18, 20]. Apart from experimental studies of the levitating drop dynamics, the literature contains theoretical works in which the problem is solved by numerical methods within a fully spatial-and-time discretization. One can also mention the Lord Rayleigh solution [10, 11] of the linearized problem as well as a few attempts to construct an asymptotic solution of the nonlinear free-surface problem [20].

In summary, the state-of-the-art is similar to that observed in the 50-60s for the liquid sloshing dynamics [7, 13] when the nonlinear analytical sloshing theory was not constructed, yet. The latter theory appeared only in the 70-80s after implementing the so-called nonlinear multimodal methods. The methods reduce the original free-surface problem to an approximate low-dimensional system of nonlinear ordinary differential equations with respect to generalized coordinates responsible for amplification of natural sloshing modes. A breakthrough in the multimodal methods was done by Lukovsky and Miles. In the present paper, we generalize Lukovsky-and-Miles' results for the nonlinear liquid drop dynamics.

2. Statement of the problem. We consider a levitating drop $Q(t)$ of an ideal incompressible liquid that performs oscillatory motions as illustrated in Fig. 1. Due to the surface tension, the drop takes spherical shape in its hydrostatic state. We choose the radius R_0 of the sphere to be the characteristic length and introduce the characteristic time $t_* = \sqrt{\rho R_0^3/T_s}$ (T_s is the surface tension coefficient). The nondimensional drop dynamics is considered in the spherical coordinate system $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$ ($r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$) so that the free surface $\Sigma(t)$ is described by the equation

$$r = \zeta(\theta,\varphi,t) = 1 + \xi(\theta,\varphi,t). \quad (1)$$

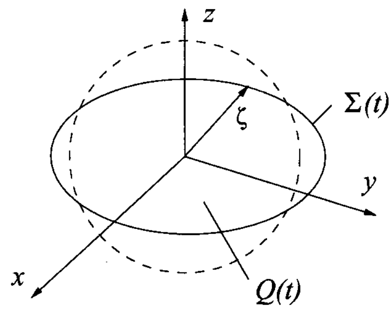

Fig. 1. Geometric notations.

According to (1), perturbations of $\Sigma(t)$ relative to the static spherical shape are subject of the volume conservation condition

$$\int\limits_{Q(t)} dQ = \frac{4}{3}\pi \Rightarrow \int\limits_0^{2\pi}\int\limits_0^{\pi}\left(\frac{1}{3}\xi^3 + \xi^2 + \xi\right)\sin\theta\, d\theta\, d\varphi = 0 \tag{2}$$

playing the role of a *holonomic constrain*.

Following [13], we employ the Bateman – Luke variational formulation which states that the original free-surface problem is the necessary extrema condition of the action

$$A(\Phi, \zeta) = \int\limits_{t_1}^{t_2} BL(\Phi, \zeta)\, dt \tag{3}$$

for arbitrary instants t_1 and t_2 ($t_1 < t_2$) and independent variables ζ and Φ (velocity potential) restricted to

$$\delta\Phi|_{t_1,t_2} = 0, \quad \delta\zeta|_{t_1,t_2} = 0 \tag{4}$$

within the Lagrangian

$$BL(\Phi, \zeta) = -\int\limits_{Q(t)}\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right) dQ - |\Sigma(t)| - \bar{p}_0\left(\int\limits_{Q(t)} dQ - \frac{4}{3}\pi\right). \tag{5}$$

Here, $|\cdot|$ defines the area, $V_l = \frac{4}{3}$ is the nondimensional liquid volume and $\bar{p}_0$ is the Lagrange multiplier (a time-dependent function) caused by the holonomic constrain (2).

The aforementioned free-surface problem takes the form

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0 \quad \text{in} \quad Q(t), \tag{6a}$$

$$\zeta\Phi_r - \Phi_\theta\zeta_\theta - \Phi_\varphi\zeta_\varphi/\sin\theta = \zeta\zeta_t \quad \text{on} \quad \Sigma(t), \tag{6b}$$

$$\left(\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left(\nabla\Phi\right)^2\right)+\left[\frac{2+(\zeta_\theta/\zeta)^2+(\zeta_\varphi/(\zeta\sin\theta))^2}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}-\frac{1}{\zeta^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\zeta\zeta_\theta\sin\theta}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\right)-\right.$$

$$\left.-\frac{1}{\zeta^2\sin^2\theta}\frac{\partial}{\partial\varphi}\left(\frac{\zeta\zeta_\varphi}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\right)\right]+\bar{p}_0=0\quad\text{on}\quad\Sigma(t),\qquad(6c)$$

subject to the volume conservation condition (2). Here, (6a) and (6b) constitute the *kinematic* sub-problem and (6c) is the so-called *dynamic* boundary condition in which the square bracket term is the sum of the principal curvatures $[k_1+k_2]$.

The problem (6) requires either the initial conditions $\zeta(\theta,\varphi,0)=\zeta_0(\theta,\varphi)$, $\Phi(r,\theta,\varphi,0)=$ $=\Phi_0(r,\theta,\varphi)$ defining the initial drop shape and velocity field or the periodicity condition $\zeta(\theta,\varphi,t)=\zeta(\theta,\varphi,t+T)$, $\Phi(r,\theta,\varphi,t)=\Phi(r,\theta,\varphi,t+T)$, where $T=2\pi/\sigma$ is a fixed period.

3. Linear eigensolution and natural modes. Consider small oscillations of a drop with respect to its spherical shape by linearizing the volume conservation as well as the kinematic (6b) and dynamic (6c) boundary conditions in terms of Φ and ξ. The linearized volume conservation condition (2) takes the form

$$\int\limits_0^{2\pi}\int\limits_0^{\pi}\xi\sin\theta\,d\theta\,d\varphi=0,\qquad(7)$$

but the linearized boundary conditions are

$$\frac{\partial\Phi}{\partial r}=\frac{\partial\xi}{\partial t},\quad\frac{\partial\Phi}{\partial t}+\left\{-2\xi\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\xi}{\partial\theta}\right]+\frac{1}{\sin^2\theta}\frac{\partial^2\xi}{\partial\varphi^2}\right)\right\}=0,\quad r=1,\qquad(8)$$

that can be combined to exclude ξ and derive the boundary condition

$$\frac{\partial^2\Phi}{\partial t^2}-\left\{2\frac{\partial\Phi}{\partial r}+\frac{\partial}{\partial r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi}\right]\right\}=0,\quad r=1,\qquad(9)$$

with respect to Φ.

Postulating $\Phi(r,\theta,\varphi,t)=\phi(r,\theta,\varphi)\exp(\mathrm{i}\sigma t)$ where σ is the so-called *linear eigenfrequency* leads to the spectral boundary problem

$$\nabla^2\phi=0,\quad r<1,\quad\int\limits_0^{2\pi}\int\limits_0^{\pi}\left.\frac{\partial\phi}{\partial r}\right|_{r=1}\sin\theta\,d\theta\,d\varphi=0,$$
$$(10)$$
$$-\sigma^2\phi=\left\{2\frac{\partial\phi}{\partial r}+\frac{\partial}{\partial r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right]\right\},\quad r=1,$$

with respect to the spectral parameter σ^2 and the eigenfunction ϕ.

The spectral boundary problem (10) can be solved by separating the spatial variables $\phi(r,\theta,\varphi) = \bar{Y}_{lm}(r,\theta,\varphi) = r^l Y_{lm}(\theta,\varphi)$, $l \geq 0$, that leads to the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\varphi} = -l(l+1)Y_{lm}. \tag{11}$$

The analytical *eigensolution* follows from (11) and consists of the eigenfrequencies

$$\sigma^2 = \sigma^2_{lm} = l(l-1)(l+2), \quad l = 0,1,\ldots, \quad m = 0,\ldots,l, \tag{12}$$

and the eigenfunctions

$$\phi_{lm} = \bar{Y}_{lm}(r,\theta,\varphi) = N_{lm} r^l P_l^{(m)}(\cos\theta)\left\{\begin{array}{l}\cos m\varphi,\\ \sin m\varphi,\end{array}\right. \quad N_{lm} = \left\{\begin{array}{ll}\sqrt{\dfrac{(2l+1)(l-m)!}{4\pi(l+m)!}}, & m=0,\\[2ex] \sqrt{\dfrac{(2l+1)(l-m)!}{2\pi(l+m)!}}, & m \geq 1,\end{array}\right. \tag{13}$$

where $P_l^{(m)}$ are the associated Legendre polynomials.

Four eigenfunctions with $l = 0$ and 1 correspond to zero-eigenfrequency and, therefore, do not imply, from a physical point of view, *natural linear modes*. The case $l = 1$ with $m = 0$ gives $\phi_{10} = z = r\cos\theta$ that describes a translatory drop motion (as a solid body) along Oz, but $l = 1$ and $m = 1$ yield $y = r\sin\theta\sin\varphi$ and $x = r\sin\theta\cos\varphi$ responsible the same translatory motions but along Oy and Ox, respectively. The case $l = 0$ corresponds to $\phi_{00} = 1/2\sqrt{\pi}$. Excluding these four exceptional functions makes the functional basis (13) incomplete from a mathematical point of view.

4. Nonlinear modal equations. The multimodal methods suggest the solution of (6),

$$\zeta(\theta,\varphi,t) = 1 + \sum_I \beta_I(t) f_I(\theta,\varphi), \quad \Phi(r,\theta,\varphi,t) = \sum_N F_N(t)\phi_N(r,\theta,\varphi), \tag{14}$$

where f_I and ϕ_N are complete sets of functions to define admissible shapes $Q(t)$ satisfying the volume conservation condition and approximating the velocity field. Dealing with the star-shaped domains $Q(t)$, the solid harmonics (13) provide the completeness and we have

$$\begin{aligned}
&\phi_l = N_{l0} r^l P_l(\cos\theta), \quad l \geq 0,\\
&\phi_{lm,c} = \phi_{lm}(r,\theta)\cos m\varphi = N_{lm} r^l P_l^{(m)}(\cos\theta)\cos m\varphi, \quad l \geq 1, \quad m = 1,\ldots,l, \qquad (15a)\\
&\phi_{lm,s} = \phi_{lm}(r,\theta)\sin m\varphi = N_{lm} r^l P_l^{(m)}(\cos\theta)\sin m\varphi, \quad l \geq 1, \quad m = 1,\ldots,l,
\end{aligned}$$

$$\begin{aligned}
&f_l = N_{l0} P_l(\cos\theta), \quad l \geq 0,\\
&f_{lm,c} = f_{lm}(\theta)\cos m\varphi = N_{lm} P_l^{(m)}(\cos\theta)\cos m\varphi, \quad l \geq 1, \quad m = 1,\ldots,l, \qquad (15b)\\
&f_{lm,s} = f_{lm}(\theta)\sin m\varphi = N_{lm} P_l^{(m)}(\cos\theta)\sin m\varphi, \quad l \geq 1, \quad m = 1,\ldots,l.
\end{aligned}$$

This transforms (14) to the form

$$\zeta(\theta,\varphi,t) = 1 + \sum_{l=0}^{\infty}\beta_l(t)f_l(\theta) + \sum_{l=1}^{\infty}\sum_{m=1}^{l}(\beta_{c,lm}(t)\cos m\varphi + \beta_{s,lm}(t)\sin m\varphi)f_{lm}(\theta), \tag{16}$$

$$\Phi(r,\theta,\varphi,t) = \sum_{l=0}^{\infty}F_l(t)\phi_l(r,\theta) + \sum_{l=1}^{\infty}\sum_{m=1}^{l}(F_{c,lm}(t)\cos m\varphi + F_{s,lm}(t)\sin m\varphi)\phi_{lm}(r,\theta). \tag{17}$$

Substituting (16) into (2) gives the holonomic constrain for the generalized coordinates,

$$2\sqrt{\pi}\beta_0 + \sum_{i=0}^{\infty}\beta_i^2 + \sum_{l=1}^{\infty}\sum_{m=1}^{l}(\beta_{c,lm}^2 + \beta_{s,lm}^2) + \tilde{G}_3(\beta_i,\beta_{c,lm},\beta_{s,lm}) = 0, \tag{18}$$

where $\tilde{G}_3$ implies the cubic and higher polynomial terms. Using the implicit function theorem we find β_0 as follows:

$$\beta_0 = G(\beta_i,\beta_{c,lm},\beta_{s,lm}, i \geq 1, l \geq 1) =$$

$$= -\frac{1}{2\sqrt{\pi}}\left[\sum_{i=1}\beta_i^2 + \sum_{l=1}^{\infty}\sum_{m=1}^{l}(\beta_{c,lm}^2 + \beta_{s,lm}^2)\right] - \frac{1}{2\sqrt{\pi}}G_3(\beta_i,\beta_{c,lm},\beta_{s,lm}, i \geq 1, l \geq 1) \tag{19}$$

(G_3 also denotes the cubic and higher polynomial terms in β_*) and transforms (16) to the form

$$\zeta(\theta,\varphi,t) = 1 - \frac{1}{4\pi}\left(\sum_{i=1}^{\infty}\beta_i^2 + \sum_{l=1}^{\infty}\sum_{m=1}^{l}(\beta_{c,lm}^2 + \beta_{s,lm}^2) + G_3\right) + \sum_{l=1}^{\infty}\beta_l(t)f_l(\theta) +$$

$$+ \sum_{l=1}^{\infty}\sum_{m=1}^{l}(\beta_{c,lm}(t)\cos m\varphi + \beta_{s,lm}(t)\sin m\varphi)f_{lm}(\theta) \tag{20}$$

defining the free surface as a function of β_i, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \geq 1$, $l \geq 1$, and providing the volume conservation. As a consequence, the Lagrange multiplier becomes zero, i.e., $\bar{p}_0 = 0$.

The generalized velocity F_0 can be excluded due to

$$\frac{2}{3}\sqrt{\pi}\int_{t_1}^{t_2}\delta\dot{F}_0\,dt = \frac{2}{3}\sqrt{\pi}\,[\delta F_0(t_2) - \delta F_0(t_1)] = 0,$$

provided by (4) with

$$\delta\beta_i(t_1) = \delta\beta_i(t_2) = \delta\beta_{c,lm}(t_1) = \delta\beta_{c,lm}(t_2) = \delta\beta_{s,lm}(t_1) = \delta\beta_{s,lm}(t_2) =$$

$$= \delta F_i(t_1) = \delta F(t_2) = \delta F_{c,lm}(t_1) = \delta F_{c,lm}(t_2) = \delta F_{s,lm}(t_1) = \delta F_{s,lm}(t_2) = 0. \tag{21}$$

Substituting (17) into (5) yields the Lagrangian as a function of the generalized coordinates and velocities

$$BL = -\sum_{i=1}^{\infty} A_i \dot{F}_i - \sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{c,lm}\dot{F}_{c,lm} - \sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{c,lm}\dot{F}_{c,lm} - \frac{1}{2}\sum_{n,k=1}^{\infty} A_{n,k}F_nF_k -$$

$$-\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2} A_{(c,l_1m_1),(c,l_2m_2)}F_{(c,l_1m_1)}F_{(c,l_2m_2)} -$$

$$-\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2} A_{(s,l_1m_1),(s,l_2m_2)}F_{(s,l_1m_1)}F_{(s,l_2m_2)} -$$

$$-\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2} A_{(c,l_1m_1),(s,l_2m_2)}F_{(c,l_1m_1)}F_{(s,l_2m_2)} -$$

$$-\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{n,(s,lm)}F_nF_{(s,lm)} - \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{l} A_{n,(c,lm)}F_nF_{(c,lm)} - TS = 0, \qquad (22)$$

where

$$A_n = \int_{Q(t)} \phi_n dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} \phi_n \, r^2 \sin\theta \, dr \, d\theta \, d\varphi, \qquad (23a)$$

$$A_{c,lm} = \int_{Q(t)} \phi_{lm}\cos(m\varphi)\, dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} \phi_{lm}\cos(m\varphi) r^2 \sin\theta \, dr \, d\theta \, d\varphi, \qquad (23b)$$

$$A_{s,lm} = \int_{Q(t)} \phi_{lm}\sin(m\varphi)\, dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} \phi_{lm}\sin(m\varphi) r^2 \sin\theta \, dr \, d\theta \, d\varphi, \qquad (23c)$$

$$A_{n,k} = A_{k,n} = \int_{Q(t)} (\nabla\phi_n \cdot \nabla\phi_k)\, dQ = \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} (\nabla\phi_n \cdot \nabla\phi_k)\, r^2 \sin\theta \, dr \, d\theta \, d\varphi, \qquad (24a)$$

$$A_{n,(c,lm)} = A_{(c,lm),n} = \int_{Q(t)} (\nabla\phi_n \cdot \nabla[\phi_{lm}\cos m\varphi])\, dQ =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta} (\nabla\phi_n \cdot \nabla[\phi_{lm}\cos m\varphi])\, r^2 \sin\theta \, dr \, d\theta \, d\varphi, \qquad (24b)$$

$$A_{n,(s,lm)} = A_{(s,lm),n} = \int\limits_{Q(t)} (\nabla\phi_n \cdot \nabla[\phi_{lm}\sin m\varphi])\, dQ =$$

$$= \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{\zeta} (\nabla\phi_n \cdot \nabla[\phi_{lm}\sin m\varphi]) r^2 \sin\theta\, dr\, d\theta\, d\varphi, \qquad (24c)$$

$$A_{(c,l_1m_1),(s,l_2m_2)} = A_{(s,l_2m_2),(c,l_1m_1)} = \int\limits_{Q(t)} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\sin m_2\varphi])\, dQ =$$

$$= \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{\zeta} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\sin m_2\varphi]) r^2 \sin\theta\, dr\, d\theta\, d\varphi, \qquad (24d)$$

$$A_{(c,l_1m_1),(c,l_2m_2)} = A_{(c,l_2m_2),(c,l_1m_1)} = \int\limits_{Q(t)} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\cos m_2\varphi])\, dQ =$$

$$= \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{\zeta} (\nabla[\phi_{l_1m_1}\cos m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\cos m_2\varphi]) r^2 \sin\theta\, dr\, d\theta\, d\varphi, \qquad (24e)$$

$$A_{(s,l_1m_1),(s,l_2m_2)} = A_{(s,l_2m_2),(s,l_1m_1)} = \int\limits_{Q(t)} (\nabla[\phi_{l_1m_1}\sin m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\sin m_2\varphi])\, dQ =$$

$$= \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{\zeta} (\nabla[\phi_{l_1m_1}\sin m_1\varphi] \cdot \nabla[\phi_{l_2m_2}\sin m_2\varphi]) r^2 \sin\theta\, dr\, d\theta\, d\varphi, \qquad (24f)$$

and

$$TS = \int\limits_{\Sigma(t)} dS = \int\limits_0^{2\pi}\int\limits_0^{\pi} \zeta\sqrt{\zeta^2 + \zeta_\theta^2 + \frac{\zeta_\varphi^2}{\sin^2\theta}}\, \sin\theta\, d\theta\, d\varphi. \qquad (25)$$

Performing a variation of independent generalized velocities F_i, $F_{c,lm}$, $F_{s,lm}$, $i \geq 1$, $l \geq 1$, in the action within the Lagrangian (22) leads to the equations

$$\frac{dA_n}{dt} = \sum_{k=1}^{\infty} A_{n,k}F_k + \sum_{k=1}^{\infty}\sum_{m=1}^{k} \left(A_{n,(c,km)}F_{c,km} + A_{n,(s,km)}F_{s,km}\right), \quad n \geq 1, \qquad (26a)$$

$$\frac{dA_{c,lm}}{dt} = \sum_{k=1}^{\infty} A_{(c,lm),k}F_k + \sum_{k=1}^{\infty}\sum_{n=1}^{k} \left(A_{(c,lm),(c,kn)}F_{c,kn} + A_{(c,lm),(s,kn)}F_{s,kn}\right), \qquad (26b)$$

$$\frac{dA_{s,lm}}{dt} = \sum_{k=1}^{\infty} A_{(s,lm),k} F_k + \sum_{k=1}^{\infty}\sum_{n=1}^{k} \left(A_{(s,lm),(c,kn)} F_{c,kn} + A_{(s,lm),(s,kn)} F_{s,kn} \right), \tag{26c}$$

$$l \geq 1, \quad m = 1, \ldots, l.$$

The differentiation rule

$$\frac{dA_n}{dt} = \sum_{i=1}^{\infty} \frac{\partial A_n}{\partial \beta_i}\, \dot\beta_i + \sum_{l=1}^{\infty}\sum_{m=1}^{l} \left(\frac{\partial A_n}{\partial \beta_{c,lm}}\, \dot\beta_{c,lm} + \frac{\partial A_n}{\partial \beta_{s,lm}}\, \dot\beta_{s,lm} \right), \tag{27a}$$

$$\frac{dA_{c,lm}}{dt} = \sum_{i=1}^{\infty} \frac{\partial A_{c,lm}}{\partial \beta_i} \dot\beta_i + \sum_{j=1}^{\infty}\sum_{n=1}^{j} \left(\frac{\partial A_{c,lm}}{\partial \beta_{c,jn}}\, \dot\beta_{c,jn} + \frac{\partial A_{c,lm}}{\partial \beta_{s,jn}}\, \dot\beta_{s,jn} \right), \tag{27b}$$

$$\frac{dA_{s,lm}}{dt} = \sum_{i=1}^{\infty} \frac{\partial A_{s,lm}}{\partial \beta_i}\, \dot\beta_i + \sum_{j=1}^{\infty}\sum_{n=1}^{j} \left(\frac{\partial A_{s,jn}}{\partial \beta_{c,jn}}\, \dot\beta_{c,jn} + \frac{\partial A_{s,lm}}{\partial \beta_{s,jn}}\, \dot\beta_{s,jn} \right) \tag{27c}$$

shows that (26) is a system of nonlinear ordinary differential equations with respect to generalized coordinates β_*. On the other hand, relations (26) can be considered as a system of algebraic equations with respect to generalized velocities F_i, $F_{c,lm}$, $F_{s,lm}$, $i \geq 1$, $l \geq 1$, where $A_{n,k}$ are nonlinear functions of generalized coordinates β_i, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \geq 1$, $l \geq 1$, but the left-hand side dA_n/dt, $dA_{c,lm}/dt$, $dA_{s,lm}/dt$, implies expressions with respect to generalized coordinates β_i, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \geq 1$, $l \geq 1$, and their first derivative.

Equations (26) can be interpreted as *kinematic equations* or a non-holonomic constrain. The the Euler – Lagrange equations follow from the extrema condition of the action with respect to the generalized coordinates β_i, $\beta_{c,lm}$, $\beta_{s,lm}$, $i \geq 1$, $l \geq 1$. These are *dynamic* equations taking the form

$$\sum_{n=1}^{\infty} \frac{\partial A_n}{\partial \beta_\mu}\, \dot F_n + \sum_{l=1}^{\infty}\sum_{m=1}^{l} \left(\frac{\partial A_{c,lm}}{\partial \beta_\mu}\, \dot F_{c,lm} + \frac{\partial A_{c,lm}}{\partial \beta_\mu}\, \dot F_{c,lm} \right) +$$

$$+ \frac{1}{2} \sum_{n,k=1}^{\infty} \frac{\partial A_{n,k}}{\partial \beta_\mu}\, F_n F_k + \sum_{n,l=1}^{\infty}\sum_{m=1}^{l} F_n \left(\frac{\partial A_{n,(c,lm)}}{\partial \beta_\mu}\, F_{c,lm} + \frac{\partial A_{n,(s,lm)}}{\partial \beta_\mu} F_{s,lm} \right) +$$

$$+ \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{c,l_1m_1} F_{s,l_2m_2} \frac{\partial A_{(c,l_1m_1),(s,l_2m_2)}}{\partial \beta_\mu} +$$

$$+ \frac{1}{2} \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{c,l_1m_1} F_{c,l_2m_2} \frac{\partial A_{(c,l_1m_1),(c,l_2m_2)}}{\partial \beta_\mu} +$$

$$+ \frac{1}{2} \sum_{l_1,l_2=1}^{\infty} \sum_{m_1,m_2=1}^{l_1,l_2} F_{s,l_1m_1} F_{s,l_2m_2} \frac{\partial A_{(s,l_1m_1),(s,l_2m_2)}}{\partial \beta_\mu} + \frac{\partial TS}{\partial \beta_\mu} = 0, \quad \mu \geq 1, \tag{28a}$$

$$\sum_{n=1}^{\infty}\frac{\partial A_n}{\partial\beta_{c,\mu\nu}}\,\dot F_n+\sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\frac{\partial A_{c,lm}}{\partial\beta_{c,\mu\nu}}\dot F_{c,lm}+\frac{\partial A_{c,lm}}{\partial\beta_{c,\mu\nu}}\,\dot F_{c,lm}\right)+$$

$$+\frac{1}{2}\sum_{n,k=1}^{\infty}\frac{\partial A_{n,k}}{\partial\beta_{c,\mu\nu}}\,F_nF_k+\sum_{n,l=1}^{\infty}\sum_{m=1}^{l}F_n\left(\frac{\partial A_{n,(c,lm)}}{\partial\beta_{c,\mu\nu}}\,F_{c,lm}+\frac{\partial A_{n,(s,lm)}}{\partial\beta_{c,\mu\nu}}F_{s,lm}\right)+$$

$$+\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{c,l_1m_1}F_{s,l_2m_2}\frac{\partial A_{(c,l_1m_1),(s,l_2m_2)}}{\partial\beta_{c,\mu\nu}}+$$

$$+\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{c,l_1m_1}F_{c,l_2m_2}\frac{\partial A_{(c,l_1m_1),(c,l_2m_2)}}{\partial\beta_{c,\mu\nu}}+$$

$$+\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{s,l_1m_1}F_{s,l_2m_2}\frac{\partial A_{(s,l_1m_1),(s,l_2m_2)}}{\partial\beta_{c,\mu\nu}}+\frac{\partial TS}{\partial\beta_{c,\mu\nu}}=0, \qquad (28b)$$

$$\mu\geq 1,\quad n=1,\ldots,\mu,$$

$$\sum_{n=1}^{\infty}\frac{\partial A_n}{\partial\beta_{s,\mu\nu}}\,\dot F_n+\sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\frac{\partial A_{c,lm}}{\partial\beta_{s,\mu\nu}}\,\dot F_{c,lm}+\frac{\partial A_{c,lm}}{\partial\beta_{s,\mu\nu}}\,\dot F_{c,lm}\right)+$$

$$+\frac{1}{2}\sum_{n,k=1}^{\infty}\frac{\partial A_{n,k}}{\partial\beta_{s,\mu\nu}}\,F_nF_k+\sum_{n,l=1}^{\infty}\sum_{m=1}^{l}F_n\left(\frac{\partial A_{n,(c,lm)}}{\partial\beta_{s,\mu\nu}}\,F_{c,lm}+\frac{\partial A_{n,(s,lm)}}{\partial\beta_{s,\mu\nu}}\,F_{s,lm}\right)+$$

$$+\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{c,l_1m_1}F_{s,l_2m_2}\frac{\partial A_{(c,l_1m_1),(s,l_2m_2)}}{\partial\beta_{s,\mu\nu}}+$$

$$+\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{c,l_1m_1}F_{c,l_2m_2}\frac{\partial A_{(c,l_1m_1),(c,l_2m_2)}}{\partial\beta_{s,\mu\nu}}+$$

$$+\frac{1}{2}\sum_{l_1,l_2=1}^{\infty}\sum_{m_1,m_2=1}^{l_1,l_2}F_{s,l_1m_1}F_{s,l_2m_2}\frac{\partial A_{(s,l_1m_1),(s,l_2m_2)}}{\partial\beta_{s,\mu\nu}}+\frac{\partial TS}{\partial\beta_{s,\mu\nu}}=0, \qquad (28c)$$

$$\mu\geq 1,\quad n=1,\ldots,\mu.$$

It is important that differentiation with respect to β_* is done assuming (20) that accounts for the volume conservation condition so that, e.g.,

$$\frac{\partial TS}{\partial \beta_\mu} = \int\limits_0^{2\pi}\int\limits_0^{\pi}(k_1+k_2)\zeta^2\left[f_\mu - \frac{1}{2\pi}\,\beta_\mu - \frac{1}{4\pi}\,\frac{\partial G_3}{\partial \beta_\mu}\right]\sin\theta\,d\theta\,d\varphi$$

$$= \int\limits_{\Sigma(t)} \frac{\zeta(k_1+k_2)}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\left[f_\mu - \frac{1}{2\pi}\,\beta_\mu - \frac{1}{4\pi}\,\frac{\partial G_3}{\partial \beta_\mu}\right]dS, \tag{29a}$$

$$\frac{\partial TS}{\partial \beta_{c,\mu\nu}} = \int\limits_0^{2\pi}\int\limits_0^{\pi}(k_1+k_2)\zeta^2\left[f_{\mu\nu}\cos(\nu\varphi) - \frac{1}{2\pi}\,\beta_{c,\mu\nu} - \frac{1}{4\pi}\,\frac{\partial G_3}{\partial \beta_{c,\mu\nu}}\right]\sin\theta\,d\theta\,d\varphi =$$

$$= \int\limits_{\Sigma(t)} \frac{\zeta(k_1+k_2)}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\left[f_{\mu\nu}\cos(\nu\varphi) - \frac{1}{2\pi}\,\beta_{c,\mu\nu} - \frac{1}{4\pi}\,\frac{\partial G_3}{\partial \beta_{c,\mu\nu}}\right]dS, \tag{29b}$$

$$\frac{\partial TS}{\partial \beta_{s,\mu\nu}} = \int\limits_0^{2\pi}\int\limits_0^{\pi}(k_1+k_2)\zeta^2\left[f_{\mu\nu}\sin(\nu\varphi) - \frac{1}{2\pi}\,\beta_{s,\mu\nu} - \frac{1}{4\pi}\,\frac{\partial G_3}{\partial \beta_{s,\mu\nu}}\right]\sin\theta\,d\theta\,d\varphi =$$

$$= \int\limits_{\Sigma(t)} \frac{\zeta(k_1+k_2)}{\sqrt{\zeta^2+\zeta_\theta^2+(\zeta_\varphi/\sin\theta)^2}}\left[f_{\mu\nu}\sin(\nu\varphi) - \frac{1}{2\pi}\,\beta_{s,\mu\nu} - \frac{1}{4\pi}\,\frac{\partial G_3}{\partial \beta_{s,\mu\nu}}\right]dS. \tag{29c}$$

5. Asymptotic modal theory for axisymmetric dynamics In the case of axisymmetric drops, the velocity potential takes the form

$$\Phi(r,\theta,\varphi,t) = \sum_{l=1}^{\infty} F_l(t)\phi_l(r,\theta) \tag{30}$$

and the free-surface equation is as follows:

$$\zeta(\theta,\varphi,t) = 1 - \frac{1}{4\pi}\left(\sum_{i=1}^{\infty}\beta_i^2 + \frac{1}{3}\sum_{i,j,k=1}^{\infty}\Lambda_{ijk}^{(3)}\beta_i\beta_j\beta_k + G_5\right) + \sum_{l=1}^{\infty}\beta_l(t)f_l(\theta), \tag{31}$$

where, using [21],

$$\Lambda_{ijm}^{(3)} = 2\pi\int\limits_0^{\pi} f_i f_j f_m \sin\theta d\theta = \frac{1}{2}\sqrt{\frac{(2i+1)(2j+1)}{\pi(2m+1)}}\,\left(C_{i0,j0}^{m0}\right)^2 \tag{32}$$

($C_{i0,j0}^{m0}$ are the Clebsch – Gordan coefficients).

Henceforth, we postulate

$$\beta_l \sim F_l = O(\epsilon^{1/3}), \quad \epsilon \ll 1, \tag{33}$$

and neglect the $o(\epsilon)$-terms in kinematic (26) and dynamic (28) equations. Accounting for (27a), (26) reads as

$$\sum_{i=1}^{\infty} \frac{\partial A_n}{\partial \beta_i} \dot{\beta}_i = \sum_{k=1}^{\infty} A_{nk} F_k, \quad n \geq 1, \tag{34}$$

where neglecting the $o(\epsilon)$-terms implies that $\partial A_n/\partial \beta_i$ and A_{nk} should keep only the second-order polynomial terms:

$$\frac{\partial A_n}{\partial \beta_i} = \delta_{ni} + (2+n) \sum_{j=1}^{\infty} \Lambda^{(3)}_{nij} \beta_j + \frac{(n+1)(n+2)}{2} \sum_{j,k=1}^{\infty} \Lambda^{(4)}_{nijk} \beta_j \beta_k -$$
$$- \frac{2+n}{4\pi} \left[\delta_{ni} \sum_{j=1}^{\infty} \beta_j^2 + 2\beta_i \beta_n \right] = \delta_{ni} + \sum_{j=1}^{\infty} \chi^{(1)}_{n,i,j} \beta_j + \sum_{j,k=1}^{\infty} \chi^{(2)}_{n,i,jk} \beta_j \beta_k, \tag{35a}$$

$$A_{nk} = n\delta_{nk} + \sum_{j=1}^{\infty} \left[nk \Lambda^{(3)}_{knj} + \Lambda^{(-3)}_{nk,j} \right] \beta_j + \frac{n+k}{2} \sum_{i,j=1}^{\infty} \left[nk \Lambda^{(4)}_{knij} + \Lambda^{(-4)}_{nk,ij} \right] \beta_i \beta_j -$$
$$- \frac{n(n+k+1)}{4\pi} \delta_{nk} \sum_{j=1}^{\infty} \beta_j^2 = n\delta_{nk} + \sum_{j=1}^{\infty} \Pi^{(1)}_{nk,j} \beta_j + \sum_{i,j=1}^{\infty} \Pi^{(2)}_{nk,ij} \beta_i \beta_j. \tag{35b}$$

Here, $\Lambda^{(3)}_{ijk}$ is defined by (32) and $\Lambda^{(4)}_{ijkm}$ is expressed via the Clebsch – Gordan coefficients as

$$\Lambda^{(4)}_{ijkm} = 2\pi \int_0^{\pi} f_i f_j f_k f_m \sin\theta d\theta = \frac{\sqrt{(2i+1)(2j+1)(2k+1)(2m+1)}}{4\pi} \times$$
$$\times \sum_{n=\max(|i-j|,|k-m|)}^{\min(i+j,k+m)} \frac{1}{2n+1} \left(C^{n0}_{i0,j0} \, C^{n0}_{k0,m0} \right)^2 .$$

Furthermore,

$$\Lambda^{(-2)}_{nk} = 2\pi \int_0^{\pi} \frac{\partial f_n}{\partial \theta} \frac{\partial f_k}{\partial \theta} \sin\theta \, d\theta = n(n+1)\delta_{nk}$$

and

$$\Lambda^{(-3)}_{in,k} = 2\pi \int_0^{\pi} \frac{\partial f_i}{\partial \theta} \frac{\partial f_n}{\partial \theta} f_k \sin\theta \, d\theta = -\frac{1}{2} \sqrt{\frac{i(i+1)(2i+1)n(n+1)(2n+1)}{\pi(2k+1)}} \, C^{k0}_{i0,n0} \, C^{k0}_{i(-1),n1},$$

$$\Lambda^{(-4)}_{in,kj} = 2\pi \int\limits_0^\pi \frac{\partial f_i}{\partial \theta}\frac{\partial f_n}{\partial \theta} f_k f_j \sin\theta\, d\theta = \frac{1}{4\pi}\sqrt{i(i+1)(2i+1)n(n+1)(2n+1)}\times$$
$$\times\sqrt{k(k+1)(2k+1)j(j+1)(2j+1)}\times$$
$$\times\sum_{m=\max(|i-n|,|k-j|)}^{\min(i+n,k+j)} \frac{1}{2m+1} C^{m0}_{i0,n0}\, C^{m0}_{i(-1),n1}\, C^{m0}_{k0,j0}\, C^{m0}_{k(-1),j1}.$$

We have also introduced the coefficients

$$\chi^{(1)}_{n,i,j} = (n+2)\Lambda^{(3)}_{nij}, \quad \chi^{(2)}_{n,i,jk} = \frac{n+2}{2}\left[(n+1)\Lambda^{(4)}_{nijk} - \frac{1}{2\pi}\left(\delta_{in}\delta_{jk} + 2\delta_{ij}\delta_{kn}\right)\right],$$

$$\Pi^{(1)}_{nk,j} = nk\Lambda^{(3)}_{nkj} + \Lambda^{(-3)}_{nk,j}, \quad \Pi^{(2)}_{nk,ij} = \frac{n+k}{2}\left(nk\Lambda^{(4)}_{nkij} + \Lambda^{(-4)}_{nk,ij}\right) - \frac{(n+k+1)n\delta_{nk}\delta_{ij}}{4\pi}.$$

Based on (35), kinematic equations (34) can be considered as linear algebraic equations with respect to F_k whose asymptotic solution should keep the quadratic terms,

$$F_l = \frac{\dot\beta_l}{l} + \sum_{i,j=1}^{\infty} V^{(2)}_{l,i,j}\dot\beta_i\beta_j + \sum_{i,j,k=1}^{\infty} V^{(3)}_{l,i,j,k}\dot\beta_i\beta_j\beta_k, \quad l \ge 1, \tag{36}$$

$$\dot F_l = \frac{\ddot\beta_l}{l} + \sum_{i,j=1}^{\infty} V^{(2)}_{l,i,j}\ddot\beta_i\beta_j + \sum_{i,j,k=1}^{\infty} V^{(3)}_{l,i,j,k}\ddot\beta_i\beta_j\beta_k + \sum_{i,j=1}^{\infty} V^{(2)}_{l,i,j}\dot\beta_i\dot\beta_j + \sum_{i,j,k=1}^{\infty} \bar V^{(3)}_{l,i,j,k}\dot\beta_i\dot\beta_j\beta_k,$$
$$\bar V^{(3)}_{l,i,j,k} = V^{(3)}_{l,i,j,k} + V^{(3)}_{l,i,k,j}, \quad l \ge 1. \tag{37}$$

Substituting (36) into (34) and gathering all the similar polynomial terms gives

$$V^{(2)}_{n,i,j} = \frac{\chi^{(1)}_{n,i,j} - \Pi^{(1)}_{ni,j}/i}{n}, \quad V^{(3)}_{n,i,j,k} = \frac{\chi^{(2)}_{n,i,j,k} - \Pi^{(2)}_{ni,jk}/i - \sum\limits_{l=1}^{\infty} V^{(2)}_{l,i,j}\Pi^{(1)}_{nl,k}}{n}.$$

For axisymmetric drops, the dynamic modal equations (28) take the form

$$\sum_{n=1}^{\infty}\frac{\partial A_n}{\partial\beta_\mu}\dot F_n + \frac{1}{2}\sum_{n,k=1}^{\infty}\frac{\partial A_{n,k}}{\partial\beta_\mu}F_n F_k + \frac{\partial TS}{\partial\beta_\mu} = 0, \quad \mu \ge 1, \tag{38}$$

where

$$\frac{\partial A_{n,k}}{\partial\beta_\mu} = \Pi^{(1)}_{nk,\mu} + 2\sum_{i=1}^{\infty}\Pi^{(2)}_{nk,i\mu}\beta_i, \tag{39}$$

and formula (29a) leads to

$$\frac{\partial TS}{\partial \beta_\mu} = 2\pi \int\limits_0^\pi \zeta^2 \sin\theta \left(\frac{2+(\zeta_\theta/\zeta)^2}{\sqrt{\zeta^2\zeta_\theta^2}} - \frac{1}{\zeta^2 \sin\theta}\,\frac{\partial}{\partial\theta}\left(\frac{\zeta\zeta_\theta\sin\theta}{\sqrt{\zeta^2\zeta_\theta^2}}\right)\right) \times$$

$$\times \left[f_\mu - \frac{1}{2\pi}\,\beta_\mu - \frac{1}{4\pi}\sum_{i,j=1}^{\infty}\Lambda_{ij\mu}\beta_i\beta_j \right] d\theta =$$

$$= 2\pi \int\limits_0^\pi (2+2\xi)\left[f_\mu - \frac{1}{2\pi}\,\beta_\mu - \frac{1}{4\pi}\sum_{i,j=1}^{\infty}\Lambda_{ij\mu}\beta_i\beta_j \right]\sin\theta\, d\theta +$$

$$+ 2\pi \int\limits_0^\pi \left[\xi_\theta - \frac{1}{2}\,\xi_\theta^3\right]\frac{\partial f_\mu}{\partial\theta}\sin\theta\, d\theta =$$

$$= (\mu+2)(\mu-1)\beta_\mu + \sum_{i,j=1}^{\infty} T_{ij}^{(2\mu)}\beta_i\beta_j + \sum_{i,j,k=1}^{\infty} T_{i,j,k}^{(3\mu)}\beta_i\beta_j\beta_k, \tag{40}$$

where

$$T_{ij}^{(2\mu)} = -2\Lambda_{ij\mu}^{(3)}, \quad T_{i,j,k}^{(3\mu)} = -\frac{1}{2}\,\Lambda_{ijk\mu}^{(-4\theta)} + \frac{1}{\pi}\,\delta_{i\mu}\delta_{jk}, \quad \Lambda_{ijk\mu}^{(-4\theta)} = 2\pi\int\limits_0^\pi \frac{\partial f_i}{\partial\theta}\,\frac{\partial f_j}{\partial\theta}\,\frac{\partial f_k}{\partial\theta}\,\frac{\partial f_\mu}{\partial\theta}\,\sin\theta d\theta.$$

Substituting (35a), (36), (37), (39), and (40) into (38) and neglecting the $o(\epsilon)$-terms gives the required asymptotic nonlinear modal equations,

$$\sum_{i=1}^{\infty}\left[\delta_{\mu i} + \sum_{j=1}^{\infty} d_{i,j}^{1,\mu}\beta_j + \sum_{j,k=1}^{\infty} d_{i,j,k}^{2,\mu}\beta_j\beta_k\right]\ddot{\beta}_i + \sum_{n,k=1}^{\infty}\left[t_{n,k}^{0,\mu} + \sum_{m=1}^{\infty} t_{n,k}^{1,\mu}\beta_m\right]\dot{\beta}_n\dot{\beta}_k + \sigma_\mu^2\beta_\mu +$$

$$+ \sum_{i,j=1}^{\infty}\left[\mu T_{ij}^{2\mu}\right]\beta_i\beta_j + \sum_{i,j,k=1}^{\infty}\left[\mu T_{i,j,k}^{3\mu}\right]\beta_i\beta_j\beta_k = 0, \quad \mu \geq 1, \tag{41}$$

where $\sigma_\mu = \sigma_{\mu 0}$ are the nondimensional frequencies by (12) and

$$d_{i,j}^{1,\mu} = \mu\left[\frac{\chi_{i,\mu,j}^{(1)}}{i} + V_{\mu,i,j}^{(2)}\right], \quad d_{i,j,k}^{2,\mu} = \mu\left[\frac{\chi_{i,\mu,j,k}^{(2)}}{i} + \sum_{\alpha=1}^{\infty}\chi_{\alpha,\mu,j}^{(1)}V_{\alpha,i,k}^{(2)} + V_{\mu,i,j,k}^{(3)}\right],$$

$$t_{n,k}^{0,\mu} = \mu\left[V_{\mu,n,k}^{(2)} + \frac{\Pi_{nk,\mu}^{(1)}}{nk}\right],$$

$$t_{n,k,m}^{1,\mu} = \mu\left[\bar{V}_{\mu,n,k,m}^{(3)} + \frac{\Pi_{nk,\mu m}^{(2)}}{nk} + \sum_{\alpha=1}^{\infty}\left(\chi_{\alpha,\mu,m}^{(1)}V_{\alpha,n,k}^{(2)} + \frac{\Pi_{\alpha k,\mu}^{(1)}V_{\alpha,n,m}^{(2)}}{k} + \frac{\Pi_{\alpha n,\mu}^{(1)}V_{\alpha,k,m}^{(2)}}{n}\right)\right].$$

The nonlinear asymptotic modal equations (41) constitute an infinite-dimensional system of ordinary differential equations with respect to generalized coordinates β_n that are not resolved relative to the higher derivative.

6. Nonlinear axisymmetric eigenoscillations. We consider almost periodic (free) oscillations of an axisymmetric drop with the frequency σ close to the lowest linear eigenfrequency σ_{20} subject to the Duffing-type third order asymptotics implying the dominant generalized coordinate $\beta_2 = O(\epsilon^{1/3})$. Analyzing the nonzero coefficients in (41) shows that

$$\beta_2 = O(\epsilon^{1/3}), \quad \beta_4 = O(\epsilon^{2/3}), \quad \beta_6 = O(\epsilon), \quad \beta_l = o(\epsilon), \quad l \neq 2,4,6, \tag{42}$$

so that neglecting the $o(\epsilon)$-terms in (41) leads to the only nonlinear modal equations

$$\ddot\beta_2 + \sigma_2^2\beta_2 + d_1\ddot\beta_2\beta_4 + d_2\ddot\beta_4\beta_2 + d_3\dot\beta_2\dot\beta_4 + d_4\ddot\beta_2\beta_2^2 + \\ + d_5\dot\beta_2^2\beta_2 + t_1\beta_2^2 + t_2\beta_2\beta_4 + t_3\beta_2^3 + c_1\ddot\beta_2\beta_2 + c_2\dot\beta_2^2 = 0, \qquad (43a)$$

$$\ddot\beta_4 + \sigma_4^2\beta_4 + d_6\ddot\beta_2\beta_2 + d_7\dot\beta_2^2 + t_4\beta_2^2 + t_5\beta_2\beta_4 + c_3\ddot\beta_4\beta_2 + c_4\dot\beta_4\dot\beta_2 = 0, \qquad (43b)$$

$$\ddot\beta_6 + \sigma_6^2\beta_6 + d_8\ddot\beta_2\beta_4 + d_9\ddot\beta_4\beta_2 + d_{10}\dot\beta_4\dot\beta_2 + d_{11}\ddot\beta_2\beta_2^2 + d_{12}\dot\beta_2^2\beta_2 + t_6\beta_2\beta_4 + t_7\beta_2^3 = 0, \qquad (43c)$$

where

$$d_1 = \frac{24}{4\sqrt{\pi}},\ d_2 = \frac{15}{14\sqrt{\pi}},\ d_3 = \frac{75}{14\sqrt{\pi}},\ d_4 = -\frac{67}{98\pi},\ d_5 = -\frac{585}{196\pi},\ d_6 = \frac{15}{7\sqrt{\pi}},$$

$$d_7 = -\frac{9}{7\sqrt{\pi}},\ d_8 = \frac{105\sqrt{65}}{286\sqrt{\pi}},\ d_9 = \frac{30\sqrt{65}}{143\sqrt{\pi}},\ d_{10} = -\frac{75\sqrt{65}}{143\sqrt{\pi}},\ d_{11} = \frac{135\sqrt{65}}{1001\pi},\ d_{12} = \frac{135\sqrt{65}}{2002\pi},$$

$$t_1 = -\frac{4\sqrt{5}}{7\sqrt{\pi}},\ t_2 = -\frac{24}{7\sqrt{\pi}},\ t_3 = -\frac{76}{7\pi},\ t_4 = -\frac{24}{7\sqrt{\pi}},$$

$$t_5 = -\frac{160\sqrt{5}}{77\sqrt{\pi}},\ t_6 = -\frac{180\sqrt{65}}{143\sqrt{\pi}},\ t_7 = \frac{540\sqrt{65}}{143\pi},$$

$$c_1 = \frac{9\sqrt{5}}{14\sqrt{\pi}},\ c_2 = \frac{4\sqrt{5}}{7\sqrt{\pi}},\ c_3 = \frac{75\sqrt{5}}{154\sqrt{\pi}},\ c_4 = \frac{185\sqrt{5}}{154\sqrt{\pi}}.$$

Other modal equations of (41) do not include nonlinear terms.

To construct a periodic asymptotic solution of (43), we assume, as usually [9], the closeness condition between σ and σ_2,

$$\frac{\sigma - \sigma_2}{\sigma_2} = O(\epsilon^{2/3}) \tag{44}$$

(the nonlinear eigenfrequency σ is unknown). The wanted periodic solution takes the form

$$\beta_2 = A\cos(\sigma t) + A^2(E_1 + E_2\cos(2\sigma t)) + O(A^3),$$

$$\beta_4 = A^2(E_3 + E_4\cos(2\sigma t)) + O(A^3), \quad \beta_6 = O(A^3), \quad A = O(\epsilon^{1/3}), \tag{45}$$

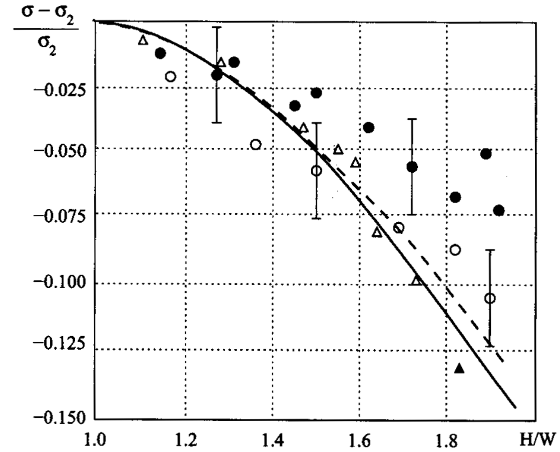

Fig. 2. Theoretical (Eq. (48), solid line) and experimental dependency between $(\sigma - \sigma_2)/\sigma_2$ (σ_2 is the first linear eigenfrequency, σ is the corresponding nonlinear eigenfrequency) and the maximum ratio between the drop heights and and width (H/W).

where A is the unknown dominant amplitude. Substituting (45) into ($43a$) and ($43b$) and accounting for (44) gives

$$E_1 = \frac{-t_1 + (c_1 - c_2)\sigma^2}{2\sigma_2^2} = \frac{-t_1 + (c_1 - c_2)\sigma_2^2}{2\sigma_2^2} + O(A^2), \tag{46a}$$

$$E_2 = \frac{-t_1 + (c_1 + c_2)\sigma^2}{2(\sigma_2^2 - 4\sigma^2)} = \frac{t_1 - (c_1 + c_2)\sigma_2^2}{6\sigma_2^2} + O(A^2), \tag{46b}$$

$$E_3 = \frac{-t_4 + (d_6 - d_7)\sigma^2}{2\sigma_4^2} = \frac{-t_4 + (d_6 - d_7)\sigma_2^2}{2\sigma_4^2} + O(A^2), \tag{46c}$$

$$E_4 = \frac{-t_4 + (d_6 + d_7)\sigma^2}{2(\sigma_4^2 - 4\sigma_2^2)} = \frac{-t_4 + (d_6 + d_7)\sigma_2^2}{2(\sigma_4^2 - 4\sigma_2^2)} + O(A^2). \tag{46d}$$

Gathering the A^3-order terms at the first harmonics in ($43a$) and using (44) leads to the secular equation for the non-zero A,

$$\frac{\sigma - \sigma_2}{\sigma_2} = m_1^{(0)} A^2, \quad m_1^{(0)} = -\frac{6347}{7840\pi}, \tag{47}$$

implying a dependence between the normalized nonlinear eigenfrequency $(\sigma - \sigma_2)/\sigma_2 = O(\epsilon^{2/3})$ and the nondimensional amplitude parameter $A^2 = O(\epsilon^{2/3})$. Since $m_1^{(0)} < 0$, (47) determines the so-called 'soft-type' response suggesting that amplitude A increases with decreasing frequency σ.

Fig. 2 compares our asymptotic result with experimental data in [19] (see also [20]) where an experimental dependence between $(\sigma - \sigma_2)/\sigma_2$ and the maximum ratio (H/W) between the instant drop height and width were reported. The experimental data for $R_0 = 0.49$ cm are denoted by $\bullet$, but $\circ$ marks measurements made for $R_0 = 0.62$ cm. In the lowest-order approximation, the Legendre polynomials properties give

$$\frac{\mathrm{H}}{\mathrm{W}} = \frac{1 + \sqrt{5/(4\pi)}A}{1 - \frac{1}{2}\sqrt{5/(4\pi)}A}, \tag{48}$$

where A is defined by (47). Fig. 2 shows that (48) provides a good agreement with the numerical values in [20] (dashed line), [8] ($\triangle$) and [1] ($\blacktriangle$). Our theoretical values are in an qualitative agreement with experimental measurements for the lower drop radius. A quantitative discrepancy can be related to viscous effects discussed in [20].

7. Conclusions. Employing the Lukovsky – Miles nonlinear multimodal method developed for nonlinear sloshing problems, we derived modal equations describing nonlinear dynamics of a levitating drop. The nonlinear modal equations simplify for axisymmetric drop motions by implementing a perturbation technique. These equations become finite-dimensional for the case of almost periodic drop motions with the nonlinear eigenfrequency close to the lowest linear eigenfrequency. This case was studied by other authors, experimentally and numerically. To compare ou analytical asymptotic results with earlier experimental [19, 18] and numerical [20, 8, 1] data, we constructed an asymptotic periodic solution of the last modal equations. The results are in a good agreement.

1. *Alonso C. T.* The dynamics of colliding and oscillating drops // Jet Propulsion Laboratory Proc. Int. Colloq. on Drops and Bubbles / Eds D. J. Collins, M. S. Plesset, M. M. Saffren. — 1974. — P. 14.
2. *Basaran O. A.* Nonlinear oscillations of viscous liquid drops // J. Fluid Mech. — 1992. — **241**. — P. 169 – 198.
3. *Becker E., Hiller W. J., Kowalewski T. A.* Experimental and theoretical investigation of large-amplitude oscillations of liquid droplets // J. Fluid Mech. — 1991. — **231**. — P. 189 – 210.
4. *Becker E., Hiller W. J., Kowalewski T. A.* Nonlinear dynamics of viscous droplets // J. Fluid Mech. — 1994. — **258**. — P. 191 – 216.
5. *Brandt E. H.* Suspended by sound // Nature. — 2001. — **413**. — P. 474 – 475.
6. *Eberhardt R., Neidhart B.* Acoustic levitation device for sample pretreatment in microanalysis and trace analysis // Fresenius J. Anal. Chem. — 1999. — **365**. — P. 475 – 479.
7. *Faltinsen O. M., Timokha A. N.* Sloshing. — Cambridge: Cambridge Univ. Press, 2009. — 608 p.
8. *Foote G. B.* A numerical method for studying simple drop behavior: simple oscillation // J. Comput. Phys. — 1973. — **11**. — P. 507 – 530.
9. *Hermann M., Timokha A.* Modal modelling of the nonlinear resonant sloshing in a rectangular tank I: A single-dominant model // Math. Models and Methods Appl. Sci. — 2005. — **15**. — P. 1431 – 1458.
10. *Lord Rayleigh.* On the capillary phenomena of jets // Proc. Roy. Soc. London. — 1879. — **29**. — P. 71 – 97.
11. *Landau L. D., Lifschitz E. M.* Hydrodynamics. — Berlin: Verlag Harri Deutsch, 1991. — 496 p.
12. *Liu Y., Zhu Da-M., Strayer D. M., Israelsson U. E.* Magnetic levitation of large water droplets and mice // Adv. Space Res. — 2010. — **45**. — P. 208 – 213.
13. *Lukovsky I. A.* Introduction to the nonlinear dynamics of solid bodies with cavities filled by a liquid. — Kiev: Naukova dumka, 1990. — 296 p. (in Russian).
14. *Natarajan R., Brown R. A.* Third-order effects and the nonlinear stability of drop oscillations // J. Fluid Mech. — 1987. — **183**. — P. 95 – 121.
15. *Priego-Capote F., de Castro L.* Ultrasound-assisted levitation: Lab-on-a-drop // Trends in Anal. Chemistry. — 2006. — **25**. — P. 856 – 867.
16. *Prosperetti A.* Free oscillations of drops and bubbles: the initial-value problem // J. Fluid Mech. — 1980. — **100**. — P. 333 – 347.

17. *Shen C. L., Xie W. J., Wei B.* Parametrically excited sectorial oscillation of liquid drops floating in ultrasound // Phys. Rev. E. — 2010. — **81**. — Art. No. 046305.

18. *Trinh E., Wang T. G.* Large-amplitude free and driven drop-shape oscillations: experimental observations // J. Fluid Mech. — 1982. — **122**. — P. 315 – 338.

19. *Trinh E., Wang T. G.* Large amplitude drop shape oscillations // Pasadena Proc. 2nd Int. Colloq. Drops and Bubbles. — 1982. — P. 82 – 87.

20. *Tsamopoulos J. A., Brown R. A.* Nonlinear oscillations of inviscid drops and bubbles // J. Fluid Mech. — 1983. — **127**. — P. 519 – 537.

21. *Zharov F. N., Grigoriev F. I., Shiryaev S. O.* Some properties of expansion by derivatives of the Legendre polynomials appearing for nonlinear oscillations of a viscous drop // J. Techn. Phys. — 2005. — **75**. — P. 131 – 140.

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