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EXPONENTIAL STABILIZATION OF LONGITUDINAL VIBRATIONS OF AN INHOMOGENEOUS BEAM

ЕКСПОНЕНЦІАЛЬНА СТАБІЛІЗАЦІЯ ПОЗДОВЖНІХ ВІБРАЦІЙ НЕОДНОРІДНОЇ БАЛКИ

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In this paper, we consider the longitudinal vibrations of an inhomogeneous beam with Kelvin – Voigt damping distributed along the length of the beam. We establish the uniform exponential decay of solution with an explicit form of exponential decay of energy. The result is achieved directly by considering an energy like Lyapunov functional without using the frequency domain approach in the literature of the semigroup theory.

Розглянуто поздовжні вібрації неоднорідної балки з затуханням типу Кельвіна – Войта, розподіленим вздовж балки. Встановлено, що розв'язки мають експоненціальне затухання, і знайдено явний вигляд експоненціального зменшення енергії. Результат отримано безпосереднім розглядом енергії у формі функціонала Ляпунова без використання частотної області з теорії напівгруп.

1. Introduction. In this paper, we consider an inhomogeneous beam of length L which is clamped at both ends. Suppose that it is made of a viscoelastic material with Kelvin – Voigt constitutive relation (cf. Fung [3, p. 22 – 26]). Consequently, the longitudinal vibrations of the beam satisfies the differential equation [11]:

$$\rho(x)\,\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\,\frac{\partial y}{\partial x} + 2\delta(x)\frac{\partial^2 y}{\partial t \partial x}\,x\right) = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \tag{1}$$

where $\mathbb{R}^+ := (0, \infty)$, $\rho \in C([0, L])$ and $p, \delta \in C^1([0, L])$. The three coefficients ρ, p, δ are evidently continuous functions on $[0, L]$ and moreover, these are essentially real valued positive for such an inhomogeneous beam.

The mathematical theory of stabilization of distributed parameter system is currently of interest in view of application to vibration control of various structural elements. The question of energy decay estimates, in the context of boundary stabilization of a wave equation has earlier been studied by several authors [1, 9, 10] and a list of references cited therein. A particular problem that was initiated by K. Liu and Z. Liu in [11] wherein, it is not possible to prove the uniform exponential stability of the beam equation (1) by applying the frequency domain approach in the literature of the semigroup theory. In our discussion, herein we consider the same mathematical equation (1), keeping in view both the general inhomogeneity and the viscous damping of Kelvin – Voigt type. This type of material damping mechanism is always present, however small it may be, in all real materials as long as the system vibrates (cf. Christensen [2, p. 16 – 20]). The aim of this paper is to study the uniform exponentially stability result for the solution of the mathematical problem (1) by means of an explicit form of the exponentially energy decay estimate. To achieve the result, we adopt here a direct method by constructing suitable Lyapunov functional related to the energy functional without going through the literature of semigroup theory. Such exponential result has earlier been obtained directly, by Gorain [4] for internally damped wave equation in a bounded domain in $\mathbb{R}^n$ and by Gorain and Bose [7] for torsional modes of vibrations. The similar result, for the quasilinear vibrations of a beam or a string can be found in Gorain [5, 6]. For a clamped beam, the boundary conditions are (cf. K. Liu and Z. Liu [11])

$$y(0,t) = y(L,t) = 0 \quad \text{on} \quad \mathbb{R}^+ \tag{2}$$

and set the initial conditions as

$$y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = y_1(x) \quad \text{on} \quad (0,L). \tag{3}$$

The functions $y_0(x)$ and $y_1(x)$ are assumed to be continuous over $[0,L]$ so that the solution $y(x,t)$ is continuously differentiable on the closer half-strip $[0,L] \times [0,\infty)$.

For the above system (1) – (3), our objective is here to investigate the behaviour of the real and smooth solutions $y(x,t)$ at time $t \to +\infty$ by means of the total energy of the system.

2. Total energy of the system. For every real and smooth solution $y = y(x,t)$ of the system (1) – (3), we define the energy of y at instant t by the functional

$$E(y,t) := \frac{1}{2}\int\limits_0^L \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 + p\left(\frac{\partial y}{\partial x}\right)^2\right] dx \quad \text{for} \quad t \geq 0. \tag{4}$$

A differentiation with respect to t and then replace of $\dfrac{\partial^2 y}{\partial t^2}$ by the governing equation (1) yields

$$\frac{dE}{dt} = \int\limits_0^L \left[\frac{\partial y}{\partial t}\frac{\partial}{\partial x}\left(p\frac{\partial y}{\partial x} + 2\delta\frac{\partial^2 y}{\partial t\partial x}\right) + p\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t\partial x}\right] dx. \tag{5}$$

On integration by parts in classical sense and application of the boundary conditions in (2), the above becomes

$$\frac{dE}{dt} = -2\int\limits_0^L \delta\left(\frac{\partial^2 y}{\partial t\partial x}\right)^2 dx \leq 0 \quad \text{for} \quad t \geq 0. \tag{6}$$

We choose the solution y of the system so smooth that the partial derivative $\dfrac{\partial^2 y}{\partial t\partial x}$ involved in (6) is to be a continuous function on the closer half-strip $[0,L] \times [0,\infty)$. The result (6) implies that energy E of the system (1) – (3) is a decreasing function of time. The negativity of the integral in (6) shows that some amount of energy of the system is dissipating due to consideration of small Kelvin – Voigt damping of the structure. Hence, every regular and smooth solution of the system satisfies the energy estimate $E(y,t) \le E(y,0)$, where

$$E(y,0) = \frac{1}{2}\int_0^L \left[\rho(x)y_1^2 + p(x){y_0'}^2\right] dx \quad \text{for} \quad t \ge 0. \tag{7}$$

As the system (1) – (3) is a nonconserving and also energy dissipating, naturally, the question arises as to whether the solution of this system decays with time uniformly or not. To obtain an affirmative answer, we are focussing directly a method that will explicitly establish the uniform exponential energy decay estimate of the system. In other words, our aim is to establish the following result:

$$E(y,t) \le Me^{-\mu t}E(y,0) \quad \text{for} \quad t \ge 0 \tag{8}$$

explicitly, for some reals $\mu > 0$ and $M > 1$. Here the constant μ and M depend on the interval $[0, L]$ and eventually on the initial values $\{y_0, y_1\}$.

It can be easily verified that the above system must have nontrivial solution, unless both y_0, y_1 are identically zero on $[0, L]$. As a simple example, we observe that

$$y(x,t) = e^{-\frac{p}{\delta}t}\sin\frac{\pi}{L}x \tag{9}$$

is a nontrivial solution of the system (1) – (3), in the case of constants ρ, p, δ with $L^2\rho p = \pi^2\delta^2$ for the initial values

$$y_0(x) = \sin\frac{\pi}{L}x \quad \text{and} \quad y_1(x) = -\frac{p}{\delta}\sin\frac{\pi}{L}x. \tag{10}$$

3. Main result. As the system evolves from its initial state (y_0, y_1) to the state $(y, dy/dt)$ at instant $t \in \mathbb{R}^+$, the energy $E(y,t)$ diminishes from its initial value $E(y,0)$ driven by the work done due to Kelvin – Voigt damping. The result of the uniform exponential energy decay estimate for the solution of system (1) – (3) can be found in the ensuing theorem.

Theorem 1. *Let $y(x,t)$ be a smooth solution of the initial boundary-value problem (1) – (3). Then the solution tends to zero exponentially as time $t \to +\infty$. In other words, the energy functional E as defined by (4) satisfies the result (8).*

The theorem will be proved after some preliminary steps. We need firstly, the following two inequalities.

For any real constant $c > 0$, we have a trivial inequality

$$|uv| \le \frac{1}{2}\left(c|u|^2 + \frac{|v|^2}{c}\right). \tag{11}$$

Also, we have Poincaré type Scheeffer's inequality (cf. Mitrinović et. al. [12, p. 67])

$$\int_0^L y^2\,dx \le \frac{L^2}{\pi^2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx, \tag{12}$$

as y satisfies the boundary condition (2).

Since $\rho(x)$, $\delta(x)$, $p(x)$ are continuous functions of x over $[0, L]$, by the application of the mean value theorem of integral calculus, we have real numbers $\xi_1, \xi_2, \eta_1, \eta_2, \zeta \in [0, L]$ satisfying

$$\int_0^L \delta \left(\frac{\partial y}{\partial x}\right)^2 dx = \delta(\xi_1) \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx, \tag{13}$$

$$\int_0^L \delta \left(\frac{\partial^2 y}{\partial t \partial x}\right)^2 dx = \delta(\xi_2) \int_0^L \left(\frac{\partial^2 y}{\partial t \partial x}\right)^2 dx, \tag{14}$$

$$\int_0^L \rho y^2 \, dx = \rho(\eta_1) \int_0^L y^2 \, dx, \tag{15}$$

$$\int_0^L \rho \left(\frac{\partial y}{\partial t}\right)^2 dx = \rho(\eta_2) \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx, \tag{16}$$

$$\int_0^L p \left(\frac{\partial y}{\partial x}\right)^2 dx = p(\zeta) \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx, \tag{17}$$

and define

$$\alpha := \sup \left\{ \frac{L}{\pi} \sqrt{\frac{\rho(\eta_1)}{p(\zeta)}} \right\}, \quad \beta := \sup \left\{ \frac{2\delta(\xi_1)}{p(\zeta)} \right\}, \quad \gamma := \sup \left\{ \frac{L^2}{\pi^2} \frac{\rho(\eta_2)}{\delta(\xi_2)} \right\} \tag{18}$$

for all $\xi_1, \xi_2, \eta_1, \eta_2, \zeta \in [0, L]$.

Next, we require the following lemmas.

Lemma 1. *Let $y(x,t)$ be a smooth solution of the initial boundary-value problem (1) – (3). Then the time derivative of the functional G defined by*

$$G(y,t) := \int_0^L \rho y \frac{\partial y}{\partial t} \, dx + \int_0^L \delta \left(\frac{\partial y}{\partial x}\right)^2 dx \quad \textit{for} \quad t \geq 0, \tag{19}$$

yields

$$\frac{dG}{dt} = \int_0^L \left[\rho \left(\frac{\partial y}{\partial t}\right)^2 - p \left(\frac{\partial y}{\partial x}\right)^2 \right] dx. \tag{20}$$

Proof. See the Appendix.

Lemma 2. *Let $y(x,t)$ be a smooth solution of the initial boundary-value problem (1)–(3). Then the functional G defined by (19) satisfies*

$$-\alpha E(y,t) \leq G(y,t) \leq (\alpha+\beta)E(y,t) \quad \textit{for every} \quad t \geq 0. \tag{21}$$

Proof. See the Appendix.

To establish the main theorem, we proceed as in [6, 8, 9] and introduce an energy like Lyapunov functional V defined by

$$V(y,t) := E(y,t) + \varepsilon G(y,t) \quad \text{for} \quad t \geq 0, \tag{22}$$

where $\varepsilon > 0$ is a small but fixed real number. The Lemma 2 yields for the functional V that estimates

$$(1-\alpha\varepsilon)E(y,t) \leq V(y,t) \leq [1+(\alpha+\beta)\varepsilon]E(y,t) \quad \text{for} \quad t \geq 0, \tag{23}$$

where we choose $\varepsilon < 1/\alpha$, so that $V(y,t) \geq 0$ for $t \geq 0$.

Now, taking the time derivative of (22) and applying the result (6) and (20), we obtain

$$\begin{aligned}
\frac{dV}{dt} &= -2\int_0^L \delta\left(\frac{\partial^2 y}{\partial t\partial x}\right)^2 dx + \varepsilon\int_0^L \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 - p\left(\frac{\partial y}{\partial x}\right)^2\right] dx = \\
&= -2\varepsilon E(y,t) - 2\int_0^L \delta\left(\frac{\partial^2 y}{\partial t\partial x}\right)^2 dx + 2\varepsilon\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx \quad (\text{by} \quad (4)) = \\
&= -2\varepsilon E(y,t) - 2\int_0^L \delta\left(\frac{\partial^2 y}{\partial t\partial x}\right)^2 dx + 2\varepsilon\rho(\eta_2)\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx \quad (\text{by} \quad (16)) \leq \\
&\leq -2\varepsilon E(y,t) - 2\int_0^L \delta\left(\frac{\partial^2 y}{\partial t\partial x}\right)^2 dx + 2\varepsilon\rho(\eta_2)\frac{L^2}{\pi^2}\int_0^L \left(\frac{\partial^2 y}{\partial t\partial x}\right)^2 dx \quad (\text{by} \quad (12) \leq \\
&\leq -2\varepsilon E(y,t) - 2(1-\gamma\varepsilon)\int_0^L \delta\left(\frac{\partial^2 y}{\partial t\partial x}\right)^2 dx
\end{aligned} \tag{24}$$

by the relations (14) and (18). Since $\varepsilon > 0$ is small, we assume that

$$0 < \varepsilon < \varepsilon_0 = \min\left\{\frac{1}{\alpha}, \frac{1}{\gamma}\right\}. \tag{25}$$

Hence (24) leads to the differential inequality

$$\frac{dV}{dt} + \mu V \leq 0 \tag{26}$$

in view of (23), where

$$\mu := \frac{2\varepsilon}{1 + (\alpha + \beta)\varepsilon} > 0. \tag{27}$$

Multiplying (26) by $e^{\mu t}$ and integrating over the time interval $[0, t]$ for $t \geq 0$, we obtain

$$V(y, t) \leq e^{-\mu t} V(y, 0). \tag{28}$$

Invoking the inequality (23) again in (28), we finally obtain the result

$$E(y, t) \leq M e^{-\mu t} E(y, 0) \quad \text{for} \quad t \geq 0, \tag{29}$$

where

$$M := \frac{1 + (\alpha + \beta)\varepsilon}{1 - \alpha\varepsilon} > 1,$$

μ is in (27) and $E(y, 0)$ is in (7). Hence the theorem.

The result of the theorem shows that the smooth solution of the system $y(x, t) \to 0$ uniformly exponentially as time $t \to \infty$. Again, it follows from (27) that

$$\frac{d\mu}{d\varepsilon} = \frac{2}{[1 + (\alpha + \beta)\varepsilon]^2} > 0.$$

Hence, the exponential energy decay rate μ as function of ε will be maximum for largest admissible value of ε. In view of (25), an upper bound of which is given by ε_0 that depends explicitly on α and γ. As defined in (18), it signifies that the decay of energy will be slower for a longer beam.

4. Conclusions. This mathematical study deals with a unform exponential stability result of an inhomogeneous beam, modeled by the differential equation (1) for longitudinal modes of vibrations. We have achieved the uniform decay of solution following an explicit form of exponential decay of energy. By the application of frequency domain approach in the literature of semigroup theorem, establishment of this result has not been possible (cf. [11]). As the system is uniformly stable, it is controllable in particular, from an arbitrary initial state to a desired final state. Our discussion here, has significantly covered the case of uniform stability of the other structural vibrations, like the vibrations of strings, rods etc. satisfying (1), from mathematical point of view.

Appendix. ***Proof of Lemma 1.*** If we differentiate (19) with respect to t and use of the governing equation (1), then we have

$$\frac{dG}{dt} = \int_0^L y \left[\frac{\partial}{\partial x} \left(p(x) \frac{\partial y}{\partial x} + 2\delta(x) \frac{\partial^2 y}{\partial t \partial x} \right) \right] dx + \int_0^L \rho(x) \left(\frac{\partial y}{\partial t} \right)^2 dx + 2 \int_0^L \delta(x) \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x} \, dx.$$

Integrating by parts and applying the boundary conditions in (2), the lemma follows immediately.

Proof of Lemma 2. It follows from (13) and (17) that

$$0 \le \int_0^L \delta \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{\delta(\xi_1)}{p(\zeta)} \int_0^L p \left(\frac{\partial y}{\partial x}\right)^2 dx \le \beta E(y,t) \quad \text{for} \quad t \ge 0, \tag{30}$$

with the help of (4) and (18). Again by the inequality (11), we can write for every $t \ge 0$,

$$\left|\int_0^L \rho y \frac{\partial y}{\partial t} dx\right| = \int_0^L \left|\sqrt{\rho}\frac{\partial y}{\partial t}\right| |\sqrt{\rho} y| \, dx \le$$

$$\le \frac{L}{2\pi}\sqrt{\frac{\rho(\eta_1)}{p(\zeta)}} \int_0^L \rho \left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{\pi}{2L}\sqrt{\frac{p(\zeta)}{\rho(\eta_1)}} \int_0^L \rho y^2 \, dx \le$$

$$\le \frac{L}{2\pi}\sqrt{\frac{\rho(\eta_1)}{p(\zeta)}} \int_0^L \rho \left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{\pi}{2L}\sqrt{\frac{p(\zeta)}{\rho(\eta_1)}} \int_0^L y^2 \, dx \le$$

$$\le \frac{L}{2\pi}\sqrt{\frac{\rho(\eta_1)}{p(\zeta)}} \int_0^L \rho \left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{L}{2\pi}\sqrt{\frac{p(\zeta)}{\rho(\eta_1)}} \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx =$$

$$= \frac{L}{2\pi}\sqrt{\frac{\rho(\eta_1)}{p(\zeta)}} \left[\int_0^L \rho \left(\frac{\partial y}{\partial t}\right)^2 dx + p\left(\frac{\partial y}{\partial x}\right)^2 dx\right] \le \alpha E(y,t),$$

by the use of the relations (15), (12), (17) and (18) successively. The above leads to satisfy the inequalities

$$-\alpha E(y(t) \le \int_0^L \rho y \frac{\partial y}{\partial t} dx \le \alpha E(y(t) \quad \text{for} \quad t \ge 0. \tag{31}$$

Thus the inequalities (30) and (31) yield for $G(y,t)$ defined by (19) that estimates (21).

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