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CONVERGENCE OF THE POSITIVE SOLUTIONS OF A NONLINEAR NEUTRAL DIFFERENCE EQUATION

ЗБІЖНІСТЬ ДОДАТНИХ РОЗВ'ЯЗКІВ НЕЛІНІЙНОГО РІЗНИЦЕВОГО РІВНЯННЯ НЕЙТРАЛЬНОГО ТИПУ

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Sufficient conditions are established which guarantee the convergence of the positive solutions of the neutral type difference equation of the form $\Delta[x(n) - q(n)x(\sigma(n))] + p(n)f(x(\tau_1(n)), \ldots, x(\tau_k(n))) = 0$, *where* $\sigma(n)$, $n = 1, 2, \ldots$, *are retarded arguments and* $\tau_j(n)$, $j = 1, \ldots, k$, *are general deviated arguments.*

Знайдено достатні умови збіжності додатних розв'язків різницевого рівняння нейтрального типу вигляду $\Delta[x(n) - q(n)x(\sigma(n))] + p(n)f(x(\tau_1(n)), \ldots, x(\tau_k(n))) = 0$, *де* $\sigma(n)$, $n = 1, 2, \ldots$, — *запізнілі аргументи і* $\tau_j(n)$, $j = 1, \ldots, k$, — *загальні відхилені аргументи.*

1. Introduction. Neutral difference and differential equations arise in many areas of applied mathematics, such as circuit theory [1, 3], bifurcation analysis [2], population dynamics [8], stability theory [16, 17], dynamical behavior of delayed network systems [22], and so on. This is the reason that during the last few decades these equations are in the main interest of the literature.

In the present paper, we are interested in the first order neutral type difference equation

$$\Delta\left[x(n) - q(n)x(\sigma(n))\right] + p(n)f(x(\tau_1(n)), \ldots, x(\tau_k(n))) = 0. \tag{1.1}$$

Here $\sigma(n)$, $n = 1, 2, \ldots$, is a retarded argument and $(\tau_j(n))$, $n = 1, 2, \ldots$, are retarded or advanced arguments, for all $j = 1, 2, \ldots, k$. We assume that among others the coefficients $p(n)$ are positive real numbers and the function f satisfies some rather mild conditions.

The search for the asymptotic behavior and, especially, for oscillation criteria and stability of difference equations has received a great attention in the last few years. The purpose of this paper is to derive sufficient conditions for the convergence of the positive solutions of equation (1.1), when the coefficient $q(n)$ is either positive or negative for all n. Our criteria are new and, due to the presence of the nonlinearity f, they do not use the approaches like those used elsewhere, see e.g. [4–7, 9–15, 18–21] and the references therein. In most of these works the algebraic characteristic equation gives useful information about oscillation and stability. Our results given here are new, even for the known cases, when all the arguments are of the form $n - \tau$.

In Section 2, we present some preliminaries required in the proofs. Sufficient conditions which ensure that the solutions converge to $+\infty$ are given in Section 3 and in the final Section 4 we investigate the convergence of the solutions to zero.

2. Some preliminaries. We shall denote by $\mathbb{N}$ the set of all positive integers. Moreover we assume the following conditions:

(C_1) The sequence $(\sigma(n))$ satisfies

$$\sigma(n) \leq n - 1. \tag{2.1}$$

Also, given any large integer N, assume that there exists an integer ζ such that

$$[N, +\infty) \cap \mathbb{N} \subseteq \mathcal{R}(\sigma|_{[\zeta,+\infty)\cap\mathbb{N}}). \tag{2.2}$$

We denote by $\zeta(N)$ the smallest integer with this property and assume that $\lim \zeta(N) = +\infty$.

The following lemma provides some tools which are useful for the main results:

Lemma 2.1. *Let*

$$M(n) := \min\{r : \sigma(r) \geq n\}.$$

If the condition (2.1) is satisfied, then for each $\hat{n}$ there is a certain β such that, for all $n \geq M(\hat{n})$, there is a positive integer $m(n)$ satisfying the double inequality

$$\hat{n} \leq \sigma^{(m(n))}(n) \leq \beta \tag{2.3}$$

and

$$\lim m(n) = +\infty. \tag{2.4}$$

Proof. Assume that (2.3) is not true. Then there is some $\hat{n}$ such that, for each k there is some $n(k) \geq M(\hat{n})$ having the property that, for all positive integers m,

$$\sigma^{(m)}(n(k)) > k.$$

Therefore we have

$$k < \sigma^{(m)}(n(k)) \leq \sigma^{(m-1)}(n(k)) - 1 \leq \sigma^{(m-2)}(n(k)) - 2 \leq \ldots \leq \sigma(n(k)) - (m-1) \leq n(k) - m.$$

Thus we get the inequality

$$k \leq \lim_m [n(k) - m] = -\infty,$$

which is impossible. This proves (2.3).

To prove (2.4) assume that it is not true. Then there is some $\gamma > 0$ satisfying

$$m(n) \leq \gamma, \quad n = 1, 2, \ldots.$$

Since the sequence $(m(n))$ consists of nonnegative integers, there exists a (strictly increasing) sequence of positive integers n_λ such that all the terms of the sequence $(m(n_\lambda))$ are equal to a constant positive integer, say, A. Thus we have

$$m(n_\lambda) =: A, \quad \lambda = 1, 2, \ldots .$$

From statement (2.3) we have

$$\hat{n} \le \sigma^A(n_\lambda) \le \beta =: B.$$

Therefore it follows that

$$\sigma(\sigma^{A-1}(n_\lambda)) \le \beta =: B_1.$$

Again, we have

$$\sigma(\sigma^{A-2}(n_\lambda)) \le \beta =: B_2$$

and so, finally, we obtain

$$n_\lambda \le \beta =: B_A.$$

Hence the sequence (n_λ) is bounded, a contradiction. This proves statement (2.4) and the proof of the lemma is complete.

Given $\hat{n}$ and any $n \ge \hat{n}$ the smallest positive integer β corresponding to n will be denoted by $\beta(n)$. Also, the biggest $m(n)$ corresponding to $\hat{n}$ and n via the previous lemma will be denoted by $m(\hat{n}, n)$.

Here we present an example which illustrates the results of the previous lemma.

Example 2.1. For each positive integer consider the greatest odd integer $\sigma(n)$ which is (strictly) smaller than n. This means that, if n is an odd integer, say, $2\lambda + 1$, then we have

$$\sigma(n) = 2\lambda - 1 \quad \text{and} \quad \sigma(n+1) = \sigma(2\lambda + 2) = 2\lambda + 1$$

and if n is an even integer, say, 2λ, then

$$\sigma(n) = 2\lambda - 1 \quad \text{and} \quad \sigma(n+1) = \sigma(2\lambda + 1) = 2\lambda - 1.$$

These facts show that for each n the quantity

$$M(\hat{n}) := \beta(n) = 2\lfloor n/2 \rfloor + 1$$

satisfies the requirements of Lemma 2.1, where $\lfloor a \rfloor$ denotes the integer part of a. Also it is not hard to show that

$$\sigma^{(k)} := 2 \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2k$$

and the sequence $(m(\hat{n}, n))$ (guaranteed from Lemma 2.1) has terms given by

$$m(\hat{n}, n) := \left\lfloor \frac{n - \hat{n}}{2} \right\rfloor .$$

3. The case $q(n) \geq 0$. Before giving our main results it is convenient to assume that there is a *starting* point $n_0 \in \mathbb{N}$ such that

$$\min\{\sigma(n), \tau_1(n), \tau_2(n), \ldots, \tau_k(n)\} \geq 1, \quad n \geq n_0,$$

and a *starting* set

$$J(n_0) := [n_0, \beta(n_0)] \cap \mathbb{N}.$$

The following conditions are assumed throughout this section:

(C_2) The sequence $(p(n))$ consists of nonnegative real numbers and satisfies the property: There exists a certain $\alpha > 0$ and for each $n \in \mathbb{N}$ there is $N(n) \in \mathbb{N}$ such that $N(n) > n$ and

$$\sum_{j=n}^{N(n)} p(j) \geq \alpha. \tag{3.1}$$

(C_3) The sequence $(q(n))$ consists of nonnegative real numbers and it is bounded.

(C_4) For each $\varepsilon > 0$,

$$\inf\{f(u^1, u^2, \ldots, u^k) : u^j \geq \varepsilon, \quad j = 1, 2, \ldots, k\} > 0.$$

Remark 3.1. It is easy to see that the next two conditions are equivalent to condition (C_4):

(C_{4_a}) For each $\varepsilon > 0$ there is some $\delta > 0$ such that the relations $u^j \geq \varepsilon$, $j = 1, 2, \ldots, k$, imply that

$$f(u^1, u^2, \ldots, u^k) \geq \delta.$$

(C_{4_b}) Given sequences $y^j(n)$, $j = 1, 2, \ldots, k$, the relation

$$\lim f(y^1(n), y^2(n), \ldots, y^k(n)) = 0$$

implies that there is some index $j \in \{1, 2, \ldots, k\}$ and a subsequence $(\mu(n))$ such that

$$\lim y^j(\mu(n)) = 0.$$

It is easy to see that from condition (C_4) it follows that the function $f : \mathbb{R}^k \to \mathbb{R}$ satisfies the sign property $f(u^1, u^2, \ldots, u^k) > 0$ for all reals $u^j > 0$.

Theorem 3.1. *Assume that conditions $(C_1)-(C_4)$ hold and moreover let*

$$\lim \sigma(n) = \lim \tau_1(n) = \lim \tau_2(n) = \ldots = \lim \tau_k(n) = +\infty.$$

If $m(n_0, n)$ is the quantity defined in Lemma 2.1 and the condition

$$\lim \prod_{j=0}^{m(n_0,n)-1} q(\sigma^{(j)}(n)) = +\infty \tag{3.2}$$

is satisfied, then any eventually non-negative solution $(x(n))$ *of the difference equation (1.1) with*

$$S(n_0) := \min\{x(n) : n \in J(n_0)\} > 0 \tag{3.3}$$

tends to $+\infty$.

Proof. Let $\beta(n)$ be the quantity guaranteed by (and having the properties as described in) Lemma 2.1. For each $n \geq n_0$ we let

$$z(n) := x(n) - q(n)x(\sigma(n))$$

and then observe that

$$\Delta[z(n)] = -p(n)f(x(\tau_1(n)), x(\tau_2(n)), \ldots, x(\tau_k(n))) \leq 0.$$

This means that the sequence $(z(n))$ is decreasing and so the limit

$$l := \lim z(n)$$

exists in $[-\infty, +\infty)$.

We shall examine the following cases:

Case 1: $l \geq 0$. Then we have

$$x(n) - q(n)x(\sigma(n)) = z(n) \geq 0$$

and therefore

$$x(n) \geq q(n)x(\sigma(n)). \tag{3.4}$$

Let $m(n_0, n)$ be the quantity given in Lemma 2.1. Applying repeatedly inequality (3.4), finally, we obtain

$$x(n) \geq q(n)x(\sigma(n)) \geq q(n)q(\sigma(n))x(\sigma^{(2)}(n)) \geq \ldots \geq \prod_{j=0}^{m(n_0,n)-1} q(\sigma^{(j)}(n))S(n_0), \tag{3.5}$$

which tends to $+\infty$ because of (3.2). Thus $\lim x(n) = +\infty$ and the result is true in this case.

Case 2: $l < 0$. We claim that l cannot be finite. Indeed, assume that this is true. From (1.1) we get

$$z(n+1) - z(n) + p(n)f(x(\tau_1(n)), x(\tau_2(n)), \ldots, x(\tau_k(n))) = 0. \tag{3.6}$$

If $N(n)$ is the quantity defined in condition (C_2), summing up both sides of relation (3.6) from n to $N(n)$, we get

$$z(N(n)+1) - z(n) + \sum_{j=n}^{N(n)} p(j)f(x(\tau_1(j)), x(\tau_2(j)), \ldots, x(\tau_k(j))) = 0.$$

Thus it follows that

$$-[z(N(n)+1)-z(n)] = \sum_{j=n}^{N(n)} p(j) f(x(\tau_1(j)), x(\tau_2(j)), \ldots, x(\tau_k(j))) \geq$$

$$\geq \alpha \min\{f(x(\tau_1(j)), x(\tau_2(j)), \ldots, x(\tau_k(j))) : j = n, \ldots N(n)\} =$$

$$= \alpha f(x(\tau_1(t(n))), x(\tau_2(t(n))), \ldots, x(\tau_k(t(n))))$$

for some $t(n) \in [n, N(n)] \cap \mathbb{N}$.

By using the three facts:

$\alpha > 0$,

the solution is nonnegative, and

the limit l is a finite real number,

we conclude that

$$\lim f(x(\tau_1(t(n))), x(\tau_2(t(n))), \ldots, x(\tau_k(t(n)))) = 0$$

and, due to condition (C_{4_b}), there is some index $j \in \{1, 2, \ldots, k\}$ and a subsequence $(\mu(n))$ such that

$$\lim x(\tau_j(\mu(n))) = 0.$$

Take any large n which guarantees the existence of the number $\zeta(\tau_j(\mu(n)))$ satisfying condition (2.2). Hence, it follows that there is a certain integer $r \geq \zeta(\tau_j(\mu(n)))$ such that

$$\sigma(r) = \tau_j(\mu(n)).$$

We let $r(n)$ be the smallest of all such r which correspond to n. Then, clearly, we have

$$\lim r(n) = +\infty$$

and moreover

$$z(r(n)) - x(r(n)) = -q(r(n))x(\sigma(r(n))) = -q(r(n))x(\tau_j(\mu(n))),$$

which tends to zero. Thus we get

$$0 = \lim(z(r(n)) - x(r(n))) = l - \lim x(r(n)),$$

which is impossible, since we have $l < 0$ and the solution x is nonnegative. This proves our claim. Thus it follows that $l = -\infty$, namely,

$$\lim z(n) = -\infty.$$

This fact, together with the inequality

$$z(n) = x(n) - q(n)x(\sigma(n)) \geq -q(n)x(\sigma(n)),$$

imply that

$$\lim x(\sigma(n)) = +\infty, \tag{3.7}$$

because the sequence $(q(n))$ is bounded.

We shall show that the whole sequence $(x(n))$ converges to $+\infty$. Indeed, otherwise, there is a subsequence $(x(s(n)))$ of $(x(n))$ having a finite limit. Then this sequence is bounded from above; let b be an upper bound of it, i.e.,

$$0 \le x(s(n)) \le b. \tag{3.8}$$

From (2.2), it follows that for any large n, there is some $\nu(n) \ge \zeta(\sigma(n))$ such that

$$s(n) = \sigma(\nu(n)).$$

Hence from (3.8) we get

$$x(\sigma(\nu(n))) \le b,$$

which contradicts (3.7).

Theorem 3.1 is proved.

In the following example it is shown that, if conditions (2.2) and (3.3) are not satisfied, then the result may not hold.

Example 3.1. Consider the sequence $(\sigma(n))$ as in Example 2.1. Also, consider the set E of all positive integers which are of the form 2^k for some positive integer k. If χ_E denotes the characteristic function of E, we formulate the neutral difference equation

$$\Delta[x(n) - 2x(\sigma(n))] + \chi_E(n)x(\sigma(n) - 1) = 0. \tag{3.9}$$

It is clear that the constant sequence $q(n) := 2$, $n = 1, 2, \ldots$, satisfies (3.2). Indeed, first of all we observe that for each $\hat{n}$ there exists the number $\beta(\hat{n}) = 2\left\lfloor \frac{\hat{n}}{2} \right\rfloor + 1$ satisfying the requirements of Lemma 2.1. We observe that

$$\prod_{j=0}^{m(n_0,n)-1} q(\sigma^{(j)}(n)) = 2^{\left\lfloor \frac{n-n_0}{2} \right\rfloor},$$

which obviously tends to $+\infty$. This proves that (3.2) is satisfied.

Moreover, we can see that the sequence $p(n) := \chi_E(n)$ satisfies condition (3.1). Indeed, for each positive integer n take as $N(n)$ the smallest number of the form 2^k which is greater than n. Then condition (C_2) is satisfied with $\alpha = 1$.

On the other hand the range of the function σ consists only of odd integers. Thus (2.2) is not true. Conditions (C_3), (C_4) obviously are satisfied.

Now write equation (3.9) in the form

$$x(n+1) = 2x(\sigma(n+1)) + x(n) - 2x(\sigma(n)) - \chi_E(n)x(\sigma(n) - 1) \tag{3.10}$$

and observe that any four consecutive values x_1, x_2, x_3, x_4 of the sequence $x(n)$ are sufficient to define all the next terms of the sequence. Also, due to the linearity of equation (3.10), if these values are equal to 0, then all the next terms of the sequence following them must be equal to 0. For instance, consider the solution $(x(n))$ having initial values

$$x(0) := 0, \quad x(1) = -\frac{1}{2}, \quad x(2) := 1.$$

Then observe that $J(0) = \{0, 1\}$ and condition (3.3) is not satisfied with $n_0 = 0$. The solution $(x(n))$ with these initial values satisfies

$$x(3) = 0, \quad x(4) = 1, \quad x(5) = x(6) = x(7) = x(8) = 0$$

and therefore all the terms $x(j)$ for $j \geq 9$ are equal to 0. Thus the solution has limit zero and not $+\infty$. This proves our original claim.

4. The case $q(n) < 0$. First we shall assume the following condition:

(C_5) The sequence $(q(n))$ consists of negative real numbers.

We are going to investigate the convergence to zero of all positive solutions of equation (1.1). We give the following result:

Theorem 4.1. *Assume that relation (2.2) as well as the conditions (C_2), (C_4) and (C_5) hold. Then any eventually nonnegative solution satisfies*

$$\liminf x(n) = 0. \tag{4.1}$$

Moreover, if the condition

$$-1 < \liminf q(n) \leq 0 \tag{4.2}$$

holds, then $\lim x(n) = 0$.

Proof. Consider equation (1.1), where $q(n) < 0$ for all n. Assuming that $(x(n))$ is a nonnegative solution, the function

$$z(n) := x(n) - q(n)x(\sigma(n)) \tag{4.3}$$

is nonnegative and, due to (1.1) and (C_4), it satisfies $\Delta(z(n)) \leq 0$. This implies that $(z(n))$ is a nonnegative decreasing sequence and, thus, it converges to some $l \geq 0$.

If $l = 0$, then due to the fact that $z(n) \geq x(n) \geq 0$, we have $\lim x(n) = 0$ and therefore (4.1) is true.

We shall discuss the case $l > 0$. Assume that (4.1) does not hold. Then for some $\varepsilon > 0$ and n_0 we have $x(\tau_j(n)) \geq \varepsilon$ for all $j = 1, 2, \ldots, k$ and $n \geq n_0$. Hence, from (C_4) there is a $\delta > 0$ such that

$$f(x(\tau_1(n)), x(\tau_2(n)), \ldots, x(\tau_k(n))) \geq \delta \tag{4.4}$$

for all $n \geq n_0$. Hence we have

$$0 = \Delta z(n) + p(n)f(x(\tau_1(n)), x(\tau_2(n)), \ldots, x(\tau_k(n))) \geq \Delta z(n) + p(n)\delta,$$

or

$$\Delta z(n) \le -p(n)\delta.$$

Summing up from n to $N(n)$ (given in condition (C_2)) we obtain

$$z(N(n)+1) - z(n) \le -\delta\alpha.$$

Taking the limits as $n \to +\infty$ we get $0 = l - l \le -\delta\alpha$, a contradiction. Therefore (4.1) is true.

Now assume that (4.2) holds. Again, if $l = 0$, then it follows that $\lim x(n) = 0$.

Assume that $l > 0$. Then, from the previous arguments, the existence of a sequence $(\mu(n))$ is guaranteed such that

$$\lim x(\mu(n)) = 0.$$

From relation (4.3) on the one hand we get

$$l = \lim z(n) = \lim[x(n) - q(n)x(\sigma(n))] \ge \limsup x(n) \ge 0$$

and, on the other hand,

$$z(\mu(n)) = x(\mu(n)) - q(\mu(n))x(\sigma(\mu(n))).$$

Passing to the limits, from (4.2) we obtain

$$l = \limsup[-q(\mu(n))x(\sigma(\mu(n)))] \le \limsup(-q(n)) \limsup x(\sigma(\mu(n))) < l,$$

a contradiction. Hence $l = 0$, a fact which implies that $(x(n))$ converges to zero.

Theorem 4.1 is proved.

Remark 4.1. In the proof of the previous result the condition (4.2) plays a significant role. Indeed, we claim that if (4.2) is replaced with $-1 \le \liminf q(n) \le 0$, then the result $\lim x(n) = 0$ might not be true.

To show this fact consider the neutral difference equation

$$\Delta[x(n) - q(n-1)x(n-1)] + \min\{x(n), x(n-1)\} = 0, \tag{4.5}$$

where the coefficient $q(n)$ is defined by

$$q(n) := \frac{1}{2}[(-1)^n - 1], \quad n = 1, 2, \ldots.$$

Observe that (4.5) can be written in the difference form

$$x(n+1) = (1 + q(n))x(n) - q(n-1)x(n-1) - \min\{x(n), x(n-1)\},$$

while the coefficient $q(n)$ satisfies $\liminf q(n) = -1$. It is easy to see that the function

$$f(u_1, u_2) := \min\{u_1, u_2\}$$

satisfies the conditions of the theorem. Seeking for the solution $(x(n))$ with the initial values $x(0) = 0$ and $x(1) = 1$ we obtain that $x(n) = 0$, if n is even and $x(n) = 1$, if n is odd, which, obviously, proves our claim.

Next we assume that

$$\tau_j(n) = \tau(n), \quad j = 1, 2, \ldots, k,$$

and moreover

(C_6) Given any large integer N, there is an integer ρ such that

$$[N, +\infty) \cap \mathbb{N} \subseteq \mathcal{R}(\tau|_{[\rho, +\infty) \cap \mathbb{N}}).$$

We denote by $\rho(N)$ the smallest integer with this property and assume that $\lim \rho(N) = +\infty$.

(C_7) It holds $\liminf p(n) > 0$.

Theorem 4.2. *Assume that condition (2.2) as well as* (C_4), (C_5), (C_6) *and* (C_7) *hold. Then any nonnegative solution of the equation*

$$\Delta[x(n) - q(n)x(\sigma(n))] + p(n)f(x(\tau(n))) = 0 \tag{4.6}$$

converges to zero.

Proof. As previously, assuming that $(x(n))$ is a nonnegative solution, the function

$$z(n) := x(n) - q(n)x(\sigma(n))$$

decreases and it converges to some $l \geq 0$.

Let $\varepsilon > 0$. It follows that there is some n_1 such that $|\Delta z(n)| < \varepsilon$ for all $n \geq n_1$. Therefore we have

$$p(n)f(x(\tau(n))) < \varepsilon \tag{4.7}$$

for all $n \geq n_1$.

Assume that there exists an increasing sequence of positive integers $\mu(n)$ such that

$$\lim x(\mu(n)) =: l_1 > 0. \tag{4.8}$$

It is clear that $l > 0$, because, otherwise, $l = 0$ and so it holds

$$\lim q(\mu(n))x(\sigma(\mu(n))) = l_1.$$

Thus the sequence $(q(n))$ consists of positive terms, which is not true due to (C_5). Hence $l > 0$.

We claim that there exists a sequence of positive integers $\xi(n)$ converging to $+\infty$ such that for some $\delta > 0$ and an index $n_2 \geq n_1$ it holds

$$n \geq n_2 \Longrightarrow x(\tau(\xi(n))) \geq \delta. \tag{4.9}$$

Observe, also, that

$$n \geq n_2 \Longrightarrow x(\mu(n)) + [-q(\mu(n))x(\sigma(\mu(n)))] = x(\mu(n)) - q(\mu(n))x(\sigma(\mu(n))) \geq l.$$

This implies that for any fixed $n \geq n_2$ at least one of the following cases holds:

$$(a) \quad x(\mu(n)) \geq \frac{l}{2}, \qquad (b) \quad -q(\mu(n))x(\sigma(\mu(n))) \geq \frac{l}{2}.$$

Let us assume that case (a) holds for infinitely many indices, i.e., a sequence $\eta(n)$ exists satisfying

$$n \geq n_2 \Longrightarrow x(\mu(\eta(n))) \geq \frac{l}{2}.$$

Then, because of (C_6), there is a $n_3 \geq n_2$ and an increasing sequence $(\xi(n))$ with $\xi(n) \geq \geq \rho(\mu(\eta(n)))$ and such that $\tau(\xi(n)) = \mu(\eta(n))$. Clearly we have $\lim \xi(n) = +\infty$. Hence it holds

$$x(\tau(\xi(n))) \geq \frac{l}{2}$$

for all $n \geq n_3$ and (4.9) is proved with $\delta = \dfrac{l}{2}$.

Next, assume that case (b) holds for all terms of some sequence $\eta(n)$. Consider an upper bound B of the sequence $(-q(n))$. Then we obtain

$$\frac{l}{2} \leq -q(\eta(n))x(\sigma(\eta(n))) \leq Bx(\sigma(\eta(n)))$$

for all $n \geq n_2$. This and condition (2.2) imply the existence of a increasing unbounded sequence $(\xi(n))$ with $\xi(n) \geq \rho(\sigma(\eta(n)))$ and some $n_3 \geq n_2$, such that $\tau(\xi(n)) = \sigma(\eta(n))$. Hence we have

$$x(\tau(\xi(n))) \geq \frac{l}{2B},$$

for all $n \geq n_3$, which shows that our claim in (4.9) is true with $\delta = \dfrac{l}{2B}$.

Now, from condition (C_4), it follows that there exists a $\theta > 0$ satisfying

$$f(x(\tau(\xi(n)))) \geq \theta$$

for all $n \geq n_3$. Taking into account (C_4) and (4.7) we get

$$\varepsilon > p(\xi(n))f(x(\tau(\xi(n)))) > p(\xi(n))\theta$$

for all $n \geq n_3$. The last inequality, in view of (C_7) and the arbitrariness of ε, leads to a contradiction. Thus there is no sequence $(\mu(n))$ satisfying (4.8), which means that the solution $(x(n))$ converges to zero. The proof is complete.

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