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EXISTENCE RESULTS FOR THIRD ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH MULTIPLIER $p(t)$*

ПРО ІСНУВАННЯ РОЗВ'ЯЗКІВ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ ТРЕТЬОГО ПОРЯДКУ З МУЛЬТИПЛІКАТОРОМ $p(t)$

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In this paper, we study the existence of solutions for third order impulsive functional differential inclusions with multiplier $p(t)$. Two new results are obtained by suitable fixed point theorem combined with multi-valued analysis theory.

Вивчається питання існування розв'язків для функціонально-диференціальних включень третього порядку з імпульсною дією та мультиплікатором $p(t)$. Отримано два нових результати за допомогою придатної теореми про нерухому точку та результатів з аналізу багатозначних функцій.

1. Introduction. This paper is concerned with the existence of solutions for third order impulsive functional differential inclusions

$$
\begin{aligned}
&(p(t)u')''(t) \in F(t,u_t), \quad t \in [0,T], \quad t \neq t_k, \quad k = 1,\ldots,m,\\
&\Delta u^{(i)}(t_k) = I_{ik}(u(t_k)), \quad i = 0,1,2, \quad k = 1,\ldots,m, \qquad (1.1)\\
&u(t) = \phi(t), \quad t \in [-r,0], \quad u^{(i)}(0) = \eta_i, \quad i = 1,2,
\end{aligned}
$$

where $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map, $D = \{\psi : [-r,0] \to \mathbb{R}^n; \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t}^-)$ and $\psi(\bar{t}^+)$ exist with $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $\mathcal{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$, $I_{ik} \in C(\mathbb{R}^n,\mathbb{R}^n)$, $i = 0,1,2$, $k = 1,2,\ldots,m$, $\phi \in D$, $p \in C^1([0,T],\mathbb{R}_+)$ and p'' exists.

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$u^{(i)} : [0,T] \to \mathbb{R}^n$ which is piecewise continuous in $[0,T]$ with points of discontinuity of the first kind at the points $t_k \in [0,T]$, i.e., there exist the limits $u^{(i)}(t_k^+) < \infty$ and $u^{(i)}(t_k^-) = u^{(i)}(t_k) < < \infty$, $u''' : [0,T] \to \mathbb{R}^n$, and $\Delta u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k)$, $i = 0,1,2$, $k = 1,2,\ldots,m$.

For any continuous function u defined on $[-r,T]\setminus\{t_1,\ldots,t_m\}$ and any $t \in [0,T]$, we denote by u_t the element of D defined by $u_t(\theta) = u(t+\theta), \theta \in [-r,0]$. Here $u_t(\cdot)$ represents the history of the state from $t-r$, up to the present time t.

The theory of impulsive functional differential equations and inclusions has become more important in recent years in some mathematical models of real phenomena, especially in control, biological or medical domains.

The reason for this applicability arises from the fact that impulsive differential problems are an appropriate model for describing processes which at certain moments change their state rapidly (in a mathematical simulation it is opportune to assume that the changes of the state are instantaneous and given by jumps) and which cannot be described using classical differential problems. And functional differential inclusions are well known as differential inclusions with memory, expressing the fact that the velocity of the system depends not only on the state of the system at given instant but also on the history of the trajectory up to that instant. In addition, this theory is interesting in itself since it exhibits several new phenomena such as rhythmical beating, merging of solutions and non-continuity of solutions.

In recent years, M. Benchohra et al. [1] and Y. K Chang et al. [2] have investigated the existence of solutions for impulsive functional differential inclusions

$$\begin{aligned}
&(p(t)y'(t))' \in F(t,y_t), \quad t \in [0,T], \quad t \neq t_k, \quad k = 1,\ldots,m,\\
&\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\ldots,m,\\
&\Delta y'|_{t=t_k} = J_k(y(t_k^-)), \quad k = 1,\ldots,m,\\
&y(t) = \phi(t), \quad t \in [-r,0], \quad y'(0) = \eta,
\end{aligned} \tag{1.2}$$

with $p(t) = 1$ and (1.2), respectively. Motivated by the above mentioned work, here we want to derive the existence of solutions of (1.1).

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts from [1 – 10] which are used throughout this paper.

Let (X,d) be a metric space and $N : X \to \mathcal{P}(X)$ be a multivalued map. We use the notations $P(X) = \{Y \in \mathcal{P}(X) : Y \neq \varnothing\}$, $P_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_c(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, and $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$.

Definition 2.1. *A multivalued map $N : [0,T] \to P_{\mathrm{cl}}(X)$ is said to be measurable if for each $x \in X$ the function $g : [0,T] \to \mathbb{R}_+$, defined by $g(t) = E_d(x,N(t)) = \inf\{d(x,z) : z \in N(t)\}$, belongs to $L^1([0,T],\mathbb{R})$.*

Lemma 2.1 [1]. *Let $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ by $H_d(A,B) = \max\{\sup_{a\in A} E_d(a,B), \sup_{b\in B} E_d(b,A)\}$. Then $(P_{b,\mathrm{cl}}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a complete metric space.*

Definition 2.2. *Let X be a nonempty closed subset of $\mathbb{R}^n$, and $N : X \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty closed values. N is lower semicontinuous (l.s.c.) on X if the set $\{x \in X : N(x) \cap C \neq \varnothing\}$ is open for each open set C in $\mathbb{R}^n$.*

Definition 2.3. *Let W be a subset of $[0,T] \times D$. W is $\mathcal{L} \otimes \mathcal{B}$ measurable if W belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$ where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in D.*

Definition 2.4. *A subset $\mathcal{U}$ of $L^1([0,T],\mathbb{R}^n)$ is decomposable if for each $u, v \in \mathcal{U}$ and $J \subset$ $\subset [0,T]$ measurable the function $u\chi_J + v\chi_{[0,T]\setminus J} \in \mathcal{U}$, where χ stands for the characteristic function.*

Definition 2.5. *Let X be a separable metric space and $\mathcal{N} : X \to \mathcal{P}(L^1([0,T],\mathbb{R}^n))$ be a multivalued operator. We say $\mathcal{N}$ has property* (BC) *if*

(1) $\mathcal{N}$ is (l.s.c.);

(2) $\mathcal{N}$ has nonempty closed and decomposable values.

In order to define the solution of (1.1), we consider the following spaces: $PC = \{u : [0,T] \to \mathbb{R}^n | u_k \in C((t_k, t_{k+1}], \mathbb{R}^n), k = 0, \ldots, m$, and there exist $u(t_k^-)$ and $u(t_k^+)$ with $u(t_k^-) =$ $= u(t_k), k = 0, \ldots, m\}$, which is a Banach space with the norm $\|u\|_{PC} = \max\{\|u_k\|_{(t_k,t_{k+1}]}, k =$ $= 0, \ldots, m\}$, where u_k is the restriction of u to $(t_k, t_{k+1}]$, $k = 0, \ldots, m$.

Lemma 2.2 [1]. *Let $\Omega = D \cup PC$. Then Ω is a Banach space with norm $\|u\|_\Omega = \max\{\|u\|_D, \|u\|_{PC}\}$.*

Definition 2.6. *Let $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator $\mathcal{F} : \Omega \to \mathcal{P}(L^1([0,T],\mathbb{R}^n))$ by letting $\mathcal{F}(u) =$ $= \{v \in L^1([0,T],\mathbb{R}^n) : v(t) \in F(t, u_t)$ for a.e. $t \in [0,T]\}$. The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.*

Definition 2.7. *A function $u \in \Omega$ is said to be a solution of (1.1) if u satisfies (1.1).*

Definition 2.8. *A multivalued operator $N : X \to P_{cl}(X)$ is called*

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma\, d(x,y) \quad \textit{for each} \quad x, y \in X,$$

(2) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Definition 2.9. *Let $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values, where for D and $\mathcal{P}(\mathbb{R}^n)$ we refer to (1.1). We say F is of lower semicontinuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is l.s.c. and has nonempty closed and decomposable values.*

Definition 2.10. *The multivalued map N has a fixed point if there exists $x \in X$ such that $x \in N(x)$. The set of fixed points of the multivalued map N will be denoted by* Fix N.

Definition 2.11. *For a function $u : [-r,T] \to \mathbb{R}^n$, the set $S_{F,u} = \{v \in L^1([0,T],\mathbb{R}^n) : v(t) \in F(t,u_t)\}$ is known as the set of selection functions.*

Definition 2.12. *F has a measurable selection if there exists a measurable function(single - valued) $h : [0,T] \to \mathbb{R}^n$ such that $h(t) \in S_{F,u}$ for each $t \in [0,T]$.*

Lemma 2.3 [5]. *Let X be a separable metric space and $N : X \to \mathcal{P}(L^1([0,T],\mathbb{R}^n))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e., there exists a continuous function $f : X \to L^1([0,T],\mathbb{R}^n)$ such that $f(x) \in N(x)$ for each $x \in X$.*

Lemma 2.4 [6]. *Let X be a normed linear space with $S \subset X$ convex and $0 \in S$. Assume $H : S \to S$ is a completely continuous operator. If the set $\varepsilon(H) = \{x \in S : x = \lambda H(x)$ for some $\lambda \in (0,1)\}$ is bounded, then H has at least one fixed point in S.*

Lemma 2.5 [7]. *Let (X, d) be a complete metric space. If $N : X \to P_{\text{cl}}(X)$ is a contraction, then* Fix $N \neq \varnothing$.

Lemma 2.6 [8]. *$E \subseteq \Omega$ is a relatively compact set if and only if $E \subseteq \Omega$ is uniformly bounded and equicontinuous on each J_k, $k = 0, \ldots, p$, where $J_0 = [-r, 0]$, $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, p$.*

3. Main result. Let us introduce the following conditions for later use:

(H1) $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ has the property that $F(\cdot, \psi) : [0,T] \to P_{cp}(\mathbb{R}^n)$ is measurable for each $\psi \in D$, where for D and $\mathcal{P}(\mathbb{R}^n)$ we refer to (1.1).

(H2) There exist nonnegative constants c_{ik}, $i = 0,1,2$, $k = 1, \ldots, m$, such that $|I_{ik}(u(t_k)) - I_{ik}(v(t_k))| \leq c_{ik}|u(t_k) - v(t_k)|$, $I_{ik}(0) = 0$, $i = 0,1,2$, $k = 1, \ldots, m$, and for all $u, v \in \Omega$.

(H3) There exists a function $l \in L^1([0,T], \mathbb{R}_+)$ such that $H_d(F(t,\psi), F(t,\varphi)) \leq l(t)\|\psi - \varphi\|_D$, for a.e. $t \in [0,T]$ and any $\psi, \varphi \in D$, and $E_d(0, F(t,0)) \leq l(t)$ for a.e. $t \in [0,T]$, where for F and D we refer to (1.1).

(H4) Let $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a nonempty and compact valued multivalued map, where for D and $\mathcal{P}(\mathbb{R}^n)$ we refer to (1.1), such that $(t,\psi) \mapsto F(t,\psi)$ is $\mathcal{L} \times \mathcal{B}$ measurable, and $\psi \mapsto F(t,\psi)$ is l.s.c. for a.e. $t \in [0,T]$.

(H5) There exists a function $M \in L^1([0,T], \mathbb{R}_+)$ such that $\|F(t,\psi)\| = \sup\{|v(t)| : v(t) \in F(t,\psi)\} \leq M(t)$ for each $t \in [0,T]$, where for F we refer to (1.1).

Lemma 3.1 [11]. *Let $F : [0,T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty and compact values, where for D and $\mathcal{P}(\mathbb{R}^n)$ we refer to (1.1). Assume (H4) and (H5) hold. Then F is of l.s.c. type.*

Theorem 3.1. *Assume that (H1), (H2) and (H3) are satisfied. Then (1.1) has at least one solution on $[-r, T]$, provided*

$$\gamma = \frac{T^2}{p_0}\|l\|_{L^1} + \sum_{k=1}^{m}\left\{c_{0k} + \frac{T - t_k}{p_0}\, p(t_k)c_{1k} + \frac{(T - t_k)^2}{p_0}\, \left[p'(t_k)c_{1k} + p(t_k)c_{2k}\right]\right\} < 1,$$

where $p_0 = \min\{p(t) : t \in [0,T]\}$.

Proof. We transform the problem (1.1) into a fixed point problem. Consider the multivalued map $G : \Omega \to \mathcal{P}(\Omega)$, defined by $G(u) = \{g \in \Omega\}$, where

$$g(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + p(0)\eta_1 \displaystyle\int_0^t \frac{ds}{p(s)} + [p'(0)\eta_1 + p(0)\eta_2] \int_0^t \frac{s\,ds}{p(s)} + & \\ + \displaystyle\int_0^t \frac{ds}{p(s)} \int_0^s (s - \tau)h(\tau)d\tau + \sum_{0 < t_k < t} \Bigg\{ I_{0k}(u(t_k)) + & \\ + p(t_k)I_{1k}(u(t_k)) \displaystyle\int_{t_k}^t \frac{ds}{p(s)} + \big[p'(t_k)I_{1k}(u(t_k)) + & \\ + p(t_k)I_{2k}(u(t_k))\big] \displaystyle\int_{t_k}^t \frac{(s - t_k)ds}{p(s)} \Bigg\}, & t \in [0,T] \quad \text{and} \quad h \in S_{F,u}. \end{cases}$$

It is clear that the fixed points of G are solutions of (1.1). For each $u \in \Omega$, the set $S_{F,u}$ is nonempty since by (H1), F has a measurable selection [16].

We shall show that G satisfies the assumptions of Lemma 2.5. The proof will be given in two steps.

Step 1. $G(u) \subseteq P_{cl}(\Omega)$ for each $u \in \Omega$.

Indeed, let $\{u_n\} \subseteq G(u)$ such that $u_n \to u_*$. Then there exists $h_n \in S_{F,u}$ such that for each $t \in [0,T]$,

$$u_n(t) = \phi(0) + p(0)\eta_1 \int_0^t \frac{ds}{p(s)} + [p'(0)\eta_1 + p(0)\eta_2] \int_0^t \frac{sds}{p(s)} + \int_0^t \frac{ds}{p(s)} \int_0^s (s-\tau)h_n(\tau)d\tau +$$

$$+ \sum_{0<t_k<t} \left\{ I_{0k}(u(t_k)) + p(t_k)I_{1k}(u(t_k)) \int_{t_k}^t \frac{ds}{p(s)} + \right.$$

$$\left. + \left[p'(t_k)I_{1k}(u(t_k)) + p(t_k)I_{2k}(u(t_k))\right] \int_{t_k}^t \frac{(s-t_k)ds}{p(s)} \right\}.$$

Since $F(0,\psi)$ has compact values and (H3) holds, we may pass to a subsequence if necessary to get that h_n converges to h in $L^1([0,T],\mathbb{R}^n)$ and hence $h \in S_{F,u}$. Then for each $t \in [0,T]$,

$$u_n(t) \to u_*(t) = \phi(0) + p(0)\eta_1 \int_0^t \frac{ds}{p(s)} + \left[p'(0)\eta_1 + p(0)\eta_2\right] \int_0^t \frac{sds}{p(s)} +$$

$$+ \int_0^t \frac{ds}{p(s)} \int_0^s (s-\tau)h(\tau)d\tau + \sum_{0<t_k<t} \left\{ I_{0k}(u(t_k)) + p(t_k)I_{1k}(u(t_k)) \int_{t_k}^t \frac{ds}{p(s)} + \right.$$

$$\left. + \left[p'(t_k)I_{1k}(u(t_k)) + p(t_k)I_{2k}(u(t_k))\right] \int_{t_k}^t \frac{(s-t_k)ds}{p(s)} \right\}.$$

So $u_* \in G(u)$, and in particular, $G(u) \subseteq P_{\mathrm{cl}}(\Omega)$.

Step 2. It can be shown that there exists $\gamma < 1$ such that $H_d(G(u), G(\overline{u})) \le \gamma \|u - \overline{u}\|_\Omega$ for all $u, \overline{u} \in \Omega$.

Let $u, \overline{u} \in \Omega$ and $g \in G(u)$. Then there exists $h(t) \in F(t, u_t)$ such that for each $t \in [0,T]$,

$$g(t) = \phi(0) + p(0)\eta_1 \int_0^t \frac{ds}{p(s)} + [p'(0)\eta_1 + p(0)\eta_2] \int_0^t \frac{sds}{p(s)} + \int_0^t \frac{ds}{p(s)} \int_0^s (s-\tau)h(\tau)d\tau +$$

$$+ \sum_{0<t_k<t} \left\{ I_{0k}(u(t_k)) + p(t_k)I_{1k}(u(t_k)) \int_{t_k}^t \frac{ds}{p(s)} + \right.$$

$$+\ \left[p'(t_k)I_{1k}(u(t_k))+p(t_k)I_{2k}(u(t_k))\right]\int\limits_{t_k}^{t}\frac{(s-t_k)ds}{p(s)}\Bigg\}.$$

From (H3) it follows that, for each $t\in[0,T]$,

$$H_d\big(F(t,u_t),F(t,\overline{u}_t)\big)\le l(t)\|u_t-\overline{u}_t\|_D.$$

Hence there exists $\omega(t)\in F(t,\overline{u}_t)$ such that

$$|h(t)-\omega(t)|\le l(t)\|u_t-\overline{u}_t\|_D,\quad t\in[0,T].$$

Consider $U:[0,T]\to\mathcal{P}(\mathbb{R}^n)$, given by $U(t)=\{\omega(t):|h(t)-\omega(t)|\le l(t)\|u_t-\overline{u}_t\|_D\}$. Since the multivalued operator $V(t)=U(t)\cap F(t,\overline{u}_t)$ is measurable [16], there exists a function $\overline{h}(t)$, which is a measurable selection for V. So, $\overline{h}(t)\in F(t,\overline{u}_t)$ and $|h(t)-\overline{h}(t)|\le l(t)\|u_t-\overline{u}_t\|_D$, for each $t\in[0,T]$.

We define, for each $t\in[0,T]$,

$$\overline{g}(t)=\phi(0)+p(0)\eta_1\int\limits_0^t\frac{ds}{p(s)}+\left[p'(0)\eta_1+p(0)\eta_2\right]\int\limits_0^t\frac{sds}{p(s)}+\int\limits_0^t\frac{ds}{p(s)}\int\limits_0^s(s-\tau)\overline{h}(\tau)d\tau+$$

$$+\sum_{0<t_k<t}\Bigg\{I_{0k}(\overline{u}(t_k))+p(t_k)I_{1k}(\overline{u}(t_k))\int\limits_{t_k}^{t}\frac{ds}{p(s)}+$$

$$+\ \left[p'(t_k)I_{1k}(\overline{u}(t_k))+p(t_k)I_{2k}(\overline{u}(t_k))\right]\int\limits_{t_k}^{t}\frac{(s-t_k)ds}{p(s)}\Bigg\}.$$

Then we have

$$|g(t)-\overline{g}(t)|\le\int\limits_0^t\frac{ds}{p(s)}\int\limits_0^s(s-\tau)|h(\tau)-\overline{h}(\tau)|d\tau+\sum_{0<t_k<t}\Bigg\{|I_{0k}(u(t_k))-I_{0k}(\overline{u}(t_k))|+$$

$$+p(t_k)|I_{1k}(u(t_k))-I_{1k}(\overline{u}(t_k))|\int\limits_{t_k}^{t}\frac{ds}{p(s)}+$$

$$+\left[p'(t_k)|I_{1k}(u(t_k))-I_{1k}(\overline{u}(t_k))|+p(t_k)|I_{2k}(u(t_k))-I_{2k}(\overline{u}(t_k))|\right]\times$$

$$\times\int\limits_{t_k}^{t}\frac{(s-t_k)ds}{p(s)}\Bigg\}\le\frac{T}{p_0}\int\limits_0^t(t-s)|h(s)-\overline{h}(s)|ds+\sum_{0<t_k<t}\Bigg\{c_{0k}|u(t_k)-\overline{u}(t_k)|+$$

$$+\frac{T-t_k}{p_0}p(t_k)c_{1k}|u(t_k)-\overline{u}(t_k)|+$$

$$+\frac{(T-t_k)^2}{p_0}\left[p'(t_k)c_{1k}|u(t_k)-\overline{u}(t_k)|+p(t_k)c_{2k}|u(t_k)-\overline{u}(t_k)|\right]\Bigg\}\leq$$

$$\leq\frac{T^2}{p_0}\int\limits_0^t|h(s)-\overline{h}(s)|ds+\sum_{0<t_k<t}\Bigg\{c_{0k}+\frac{T-t_k}{p_0}p(t_k)c_{1k}+$$

$$+\frac{(T-t_k)^2}{p_0}\left[p'(t_k)c_{1k}+p(t_k)c_{2k}\right]\Bigg\}|u(t_k)-\overline{u}(t_k)|\leq$$

$$\leq\frac{T^2}{p_0}\int\limits_0^t l(s)\|u_s-\overline{u}_s\|_D ds+\sum_{k=1}^m\Bigg\{c_{0k}+\frac{T-t_k}{p_0}p(t_k)c_{1k}+$$

$$+\frac{(T-t_k)^2}{p_0}\left[p'(t_k)c_{1k}+p(t_k)c_{2k}\right]\Bigg\}|u(t_k)-\overline{u}(t_k)|\leq$$

$$\leq\frac{T^2}{p_0}\int\limits_0^t l(s)\,ds\|u-\overline{u}\|_\Omega+\sum_{k=1}^m\Bigg\{c_{0k}+\frac{T-t_k}{p_0}p(t_k)c_{1k}+$$

$$+\frac{(T-t_k)^2}{p_0}\left[p'(t_k)c_{1k}+p(t_k)c_{2k}\right]\Bigg\}\|u-\overline{u}\|_\Omega\leq$$

$$\leq\left\{\frac{T^2}{p_0}\|l\|_{L^1}+\sum_{k=1}^m\left[c_{0k}+\frac{T-t_k}{p_0}p(t_k)c_{1k}+\frac{(T-t_k)^2}{p_0}\left(p'(t_k)c_{1k}+p(t_k)c_{2k}\right)\right]\right\}\|u-\overline{u}\|_\Omega.$$

So $\|g(t)-\overline{g}(t)\|_\Omega\leq\gamma\|u-\overline{u}\|_\Omega$. By an analogous reasoning, obtained by interchanging the roles of u and $\bar{u}$, it follows that $H_d(G(u),G(\bar{u}))\leq\gamma\|u-\bar{u}\|_\Omega$. Therefore, G is a contraction. By Lemma 2.5, G has a fixed point which is a solution of (1.1).

Theorem 3.2. *In addition to (H4) and (H5), assume that the following condition holds:*

(H6). There exist constants d_{ik}, $i=0,1,2$, $k=1,\ldots,m$, such that $|I_{ik}(u(t_k))|\leq d_{ik}|u(t_k)|$ for each $u\in\Omega$. Then (1.1) has at least one solution on $[-r,T]$, provided

$$\gamma=\sum_{k=1}^m\left\{d_{0k}+\frac{T-t_k}{p_0}p(t_k)d_{1k}+\frac{(T-t_k)^2}{p_0}\left[p'(t_k)d_{1k}+p(t_k)d_{2k}\right]\right\}<1.$$

Proof. Note that (H4), (H5), and Lemma 3.1 imply that F is of l.s.c. type. Then, from Lemma 2.3, there exists a continuous function $f:\Omega\to L^1([0,T],\mathbb{R}^n)$ such that $f(u)\in\mathcal{F}(u)$ for each $u\in\Omega$.

We consider the equation

$$
\begin{aligned}
&(p(t)u')''(t) = f(u)(t), \quad t \in [0,T], \quad t \neq t_k, \quad k = 1,\ldots,m,\\
&\Delta u^{(i)}(t_k) = I_{ik}(u(t_k)), \quad i = 0,1,2, \quad k = 1,\ldots,m, \\
&u(t) = \phi(t), \quad t \in [-r,0], \quad u^{(i)}(0) = \eta_i, \quad i = 1,2,
\end{aligned}
\tag{3.1}
$$

It is clear that if $u \in \Omega$ is a solution of (3.1), then u is a solution of (1.1). Transform the problem (3.1) into a fixed point problem. Consider the operator $J : \Omega \to \Omega$, defined by

$$
J(u)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) + p(0)\eta_1 \displaystyle\int_0^t \frac{ds}{p(s)} + [p'(0)\eta_1 + p(0)\eta_2] \int_0^t \frac{sds}{p(s)} + & \\ + \displaystyle\int_0^t \frac{ds}{p(s)} \int_0^s (s-\tau) f(u)(\tau) d\tau + \sum_{0<t_k<t} \Bigg\{ I_{0k}(u(t_k)) + & \\ + p(t_k) I_{1k}(u(t_k)) \displaystyle\int_{t_k}^t \frac{ds}{p(s)} + \Bigg[p'(t_k) I_{1k}(u(t_k)) + & \\ + p(t_k) I_{2k}(u(t_k)) \Bigg] \displaystyle\int_{t_k}^t \frac{(s-t_k)ds}{p(s)} \Bigg\}, & t \in [0,T]. \end{cases}
$$

We shall show that J satisfies all assumptions of Lemma 2.4. The proof will be given in four steps.

Step 1. J is continuous.

Since the functions f and I_{ik} are continuous, this conclusion can be easily obtained.

Step 2. J maps arbitrary bounded subset of Ω into one bounded set in Ω.

Let $B_a = \{u \in \Omega : \|u\|_\Omega \leq a\}$ be arbitrary bounded subset of Ω and $u \in B_a$, there exists $f \in \mathcal{F}(u)$ such that for $t \in [0,T]$,

$$
\begin{aligned}
J(u)(t) = {} & \phi(0) + p(0)\eta_1 \int_0^t \frac{ds}{p(s)} + [p'(0)\eta_1 + p(0)\eta_2] \int_0^t \frac{sds}{p(s)} + \int_0^t \frac{ds}{p(s)} \int_0^s (s-\tau) f(u)(\tau) d\tau + \\
& + \sum_{0<t_k<t} \Bigg\{ I_{0k}(u(t_k)) + p(t_k) I_{1k}(u(t_k)) \int_{t_k}^t \frac{ds}{p(s)} + \\
& + \left[p'(t_k) I_{1k}(u(t_k)) + p(t_k) I_{2k}(u(t_k)) \right] \int_{t_k}^t \frac{(s-t_k)ds}{p(s)} \Bigg\}.
\end{aligned}
\tag{3.2}
$$

From (H5) and (H6), we get for each $t \in [0, T]$,

$$|J(u)(t)| \leq |\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + \left[p'(0)|\eta_1| + p(0)|\eta_2|\right]\frac{T^2}{p_0} + \frac{T^2}{p_0}\int\limits_0^T |f(u)(s)|\,ds +$$

$$+ \sum_{k=1}^{m}\Bigg\{|I_{0k}(u(t_k))| + \frac{T-t_k}{p_0}p(t_k)|I_{1k}(u(t_k))| +$$

$$+ \frac{(T-t_k)^2}{p_0}\left[p'(t_k)|I_{1k}(u(t_k))| + p(t_k)|I_{2k}(u(t_k))|\right]\Bigg\} \leq$$

$$\leq |\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + \left[p'(0)|\eta_1| + p(0)|\eta_2|\right]\frac{T^2}{p_0} + \frac{T^2}{p_0}\int\limits_0^T M(s)\,ds +$$

$$+ \sum_{k=1}^{m}\left\{d_{0k} + \frac{T-t_k}{p_0}p(t_k)d_{1k} + \frac{(T-t_k)^2}{p_0}\left[p'(t_k)d_{1k} + p(t_k)d_{2k}\right]\right\}|u(t_k)| \leq$$

$$\leq |\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + \left[p'(0)|\eta_1| + p(0)|\eta_2|\right]\frac{T^2}{p_0} + \frac{T^2}{p_0}\|M\|_{L^1} +$$

$$+ \sum_{k=1}^{m}\left\{d_{0k} + \frac{T-t_k}{p_0}p(t_k)d_{1k} + \frac{(T-t_k)^2}{p_0}\left[p'(t_k)d_{1k} + p(t_k)d_{2k}\right]\right\}\|u\|_{\Omega}.$$

Then, for each $u \in B_a$, we have

$$\|J(u)\|_{\Omega} \leq |\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + \left[p'(0)|\eta_1| + p(0)|\eta_2| + \|M\|_{L^1}\right]\frac{T^2}{p_0} + \gamma\|u\|_{\Omega} \leq$$

$$\leq |\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + \left[p'(0)|\eta_1| + p(0)|\eta_2| + \|M\|_{L^1}\right]\frac{T^2}{p_0} + \gamma a. \tag{3.3}$$

Therefore, $J(B_a)$ is bounded.

Step 3. J maps arbitrary bounded set into one equicontinuous set in Ω.

Let $\rho_1, \rho_2 \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, $\rho_1 < \rho_2$, and $u \in B_a$ be arbitrary bounded subset of Ω. By (3.2), we get

$$|J(u)(\rho_2) - J(u)(\rho_1)| \leq p(0)|\eta_1|\int\limits_{\rho_1}^{\rho_2}\frac{ds}{p(s)} + \left[p'(0)|\eta_1| + p(0)|\eta_2|\right]\int\limits_{\rho_1}^{\rho_2}\frac{s\,ds}{p(s)} +$$

$$+ \int\limits_{\rho_1}^{\rho_2}\frac{ds}{p(s)}\int\limits_0^s (s-\tau)|f(u)(\tau)|\,d\tau + \sum_{0<t_k<\rho_2}\Bigg\{p(t_k)|I_{1k}(u(t_k))|\int\limits_{\rho_1}^{\rho_2}\frac{ds}{p(s)} +$$

$$+\ \left[p'(t_k)|I_{1k}(u(t_k))| + p(t_k)|I_{2k}(u(t_k))|\right] \int\limits_{\rho_1}^{\rho_2} \frac{(s-t_k)ds}{p(s)} \Bigg\} \leq$$

$$\leq \left\{p(0)|\eta_1| + T\left[p'(0)|\eta_1| + p(0)|\eta_2|\right]\right\} \int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)} + T\int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)} \int\limits_{0}^{T} M(\tau)d\tau +$$

$$+ \sum_{0<t_k<\rho_2} \left\{p(t_k)d_{1k}|u(t_k)| + T\left[p'(t_k)d_{1k}|u(t_k)| + p(t_k)d_{2k}|u(t_k)|\right]\right\} \int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)} \leq$$

$$\leq \left\{p(0)|\eta_1| + T\left[p'(0)|\eta_1| + p(0)|\eta_2|\right]\right\} \int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)} + T\|M\|_{L^1} \int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)} +$$

$$+ \|u\|_\Omega \sum_{k=1}^{m} \left\{p(t_k)d_{1k} + T\left[p'(t_k)d_{1k} + p(t_k)d_{2k}\right]\right\} \int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)} \leq$$

$$\leq \left\{p(0)|\eta_1| + T\left[p'(0)|\eta_1| + p(0)|\eta_2| + \|M\|_{L^1}\right]\right\} \int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)} +$$

$$+ a\sum_{k=1}^{m} \left\{p(t_k)d_{1k} + T\left[p'(t_k)d_{1k} + p(t_k)d_{2k}\right]\right\} \int\limits_{\rho_1}^{\rho_2} \frac{ds}{p(s)}.$$

According to the complete continuity of integrable function M, the right-hand side of the above inequality tends to zero as $\rho_2 \to \rho_1$. The consequence for the cases $\rho_1, \rho_2 \in (0, t_1]$ and $[-r, 0]$ is obvious. Then $J(B_a)$ is one equicontinuous set in Ω.

As a consequence of Step 1 to Step 3 together with Lemma 2.6 and the Ascoli – Arzela theorem, we conclude that $J : \Omega \to \Omega$ is completely continuous.

Step 4. The set $\varepsilon(J) = \{u \in \Omega : u = \lambda J(u), \text{for some } 0 < \lambda < 1\}$ is bounded.

For each $u \in \varepsilon(J)$, by (3.3), we have

$$\|u\|_\Omega = \lambda\|J(u)\|_\Omega \leq \|J(u)\|_\Omega \leq |\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + \left[p'(0)|\eta_1| + p(0)|\eta_2| + \|M\|_{L^1}\right]\frac{T^2}{p_0} +$$

$$+ \sum_{k=1}^{m} \left\{d_{0k} + \frac{T-t_k}{p_0}p(t_k)d_{1k} + \frac{(T-t_k)^2}{p_0}\left[p'(t_k)d_{1k} + p(t_k)d_{2k}\right]\right\} \|u\|_\Omega =$$

$$= |\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + \left[p'(0)|\eta_1| + p(0)|\eta_2| + \|M\|_{L^1}\right]\frac{T^2}{p_0} + \gamma\|u\|_\Omega.$$

Then

$$\|u\|_\Omega \leq \frac{|\phi(0)| + p(0)|\eta_1|\frac{T}{p_0} + [p'(0)|\eta_1| + p(0)|\eta_2| + \|M\|_{L^1}]\frac{T^2}{p_0}}{1-\gamma},$$

i.e., $\varepsilon(J)$ is bounded.

In view of Lemma 2.4, we deduce that J has a fixed point which in turn is a solution of (1.1).

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