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EXISTENCE OF LOCAL AND GLOBAL SOLUTIONS OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

ІСНУВАННЯ ЛОКАЛЬНИХ ТА ГЛОБАЛЬНИХ РОЗВ'ЯЗКІВ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРОБОВОГО ПОРЯДКУ

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In this paper we shall study the existence of local and global mild solutions of the fractional order differential equations in an arbitrary Banach space by using the semigroup theory and Schauder's fixed point theorem. We also give some examples to illustrate the applications of the abstract results.

Вивчено питання існування локальних та глобальних м'яких розв'язків диференціальних рівнянь дробового порядку в довільному банаховому просторі з використанням теорії півгруп та теореми Шаудера про нерухому точку. Також наведено кілька прикладів, які ілюструють застосування абстрактного результату.

1. Introduction. We consider the following fractional order differential equation in a Banach space $(H, \|.\|)$:

$$\frac{d^\beta u(t)}{dt^\beta} + Au(t) = f(t, u(t)), \quad t \in (0, T], \qquad u(0) = u_0, \tag{1.1}$$

where A is a closed linear operator defined on a dense set, $0 < \beta \leq 1$, $0 < T < \infty$ and $\dfrac{d^\beta u(t)}{dt^\beta}$ denotes the derivative of u in the Caputo sense. We assume $-A$ is the infinitesimal generator of a compact analytic semigroup $\{S(t) : t \geq 0\}$ in H and the nonlinear map f is defined from $[0, T] \times H$ into H satisfying certain conditions to be specified later.

For the initial works on existence and uniqueness of solutions to different type of differential equations we refer to [1–9] and references cited in these papers.

Jardat et al. [3] have considered the following fractional order differential equation in a Banach space of the form

$$\begin{aligned} &\frac{d^\beta u(t)}{dt^\beta} = Au(t) + f(t, u(t), Gu(t), Su(t)), \quad t > t_0, \quad \beta \in (0,1], \\ &u(t_0) = u_0, \end{aligned} \tag{1.2}$$

where A generates a strongly continuous semigroup. They have used the semigroup and fixed point method to prove the existence and uniqueness of solutions.

In this paper, we use the Schauder's fixed point theorem and semigroup theory to prove the existence of local and global mild solutions to the given problem (1.1). With some extra assumptions, we can use all the results of this paper to the problem considered by Jardat [3].

The plan of the paper is as follows. Introduction and preliminaries are given respectively, in the first two sections. In Section 3, we prove the existence of local mild solutions and in Section 4, the existence of global mild solutions for the problem (1.1) is given. In the last section, we have given some examples

2. Preliminaries. We note that if $-A$ is the infinitesimal generator of an analytic semigroup then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence without loss of generality we suppose that

$$\|S(t)\| \le M \quad \text{for} \quad t \ge 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \le \alpha \le 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ being dense in H. We have $H_\kappa \hookrightarrow H_\alpha$ for $0 < \alpha < \kappa$ and the embedding is continuous. For more details on the fractional powers of closed linear operators we refer to Pazy [10]. It can be proved easily that $H_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ and it is equivalent to the graph norm of A^α.

We notice that $\mathcal{C}_T = C([0,T], H)$, the set of all continuous functions from $[0,T]$ into H, is a Banach space under the supremum norm given by

$$\|\psi\|_T := \sup_{0 \le \eta \le T} \|\psi(\eta)\|, \quad \psi \in \mathcal{C}_T.$$

We consider the following assumptions:

(H$_1$) $-A$ is the infinitesimal generator of a compact analytic semigroup $S(t)$.

(H$_2$) The nonlinear map $f : [0,T] \times H \to H$ is continuous in the first variable and satisfies the following condition:

$$\|f(t,x) - f(s,y)\| \le L_f(r)[|t-s| + \|x-y\|],$$

for all $x, y \in B_r(H, u_0)$, $t, s \in [0, T]$. Here, $L_f : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function and for $r > 0$

$$B_r(Z, z_1) = \{z \in Z : \|z - z_1\|_Z \leq r\},$$

where $(Z, \|.\|_Z)$ is a Banach space.

We need some basic definitions and properties of the fractional calculus theory.

Definition 2.1. *A real function $g(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$ such that $g(x) = x^p g_1(x)$, where $g_1 \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $g^{(m)} \in C_\mu$, $m \in \mathbb{N}$.*

Definition 2.2. *The Riemann–Liouville fractional integral operator of order $\beta \geq 0$ of a fucntion $g \in C_\mu$, $\mu \geq -1$ is defined as*

$$I^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \theta)^{\beta - 1} g(\theta) d\theta, \quad t > 0.$$

Definition 2.3. *If the function $g \in C_{-1}^m$ and m is a positive integer then we can define the fractional derivative of $g(t)$ in the Caputo sense as given below*

$$\frac{d^\beta g(t)}{dt^\beta} = \frac{1}{\Gamma(m - \beta)} \int_0^t (t - \theta)^{m - \beta - 1} g^m(\theta) d\theta, \quad m - 1 < \beta \leq m, \quad t > 0.$$

Definition 2.4. *By a mild solution of the differential equation (1.1), we mean a continuous solution u of the following integral equation given below,*

$$u(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} S(t - s) f(s, u(s))\, ds.$$

For more details on mild solution, we refer to [3].

3. Existence of local solutions. To prove the existence of mild solution of the evolution problem (1.1), we need the following lemma.

Lemma 3.1. *The differential equation (1.1) is equivalent to the following integral equation:*

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} (-Au(s)) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s, u(s)) ds,$$

where $0 < t \leq T$.

Proof. For the details, we refer to Lemma 1.1 in [3].

Now, we state the following theorem.

Theorem 3.1. *Assume the conditions (H_1), (H_2) are satisfied and* $u_0 \in D(A)$*. Then, there exists a* t_0, $0 < t_0 < T$*, such that the equation (1.1) has a local mild solution on* $[0, t_0]$.

Proof. Let $R > 0$ be such that $M\|u_0\| \leq \dfrac{R}{2}$ and $A_1 = \|A^{-\alpha}\|$.

Choose t_0, $0 < t_0 \leq T$ such that

$$t_0 < \left[\frac{R}{2}\left\{\frac{M}{\beta\Gamma(\beta)}\{L_f(R)[T+R] + \|f(0,u_0)\|\}\right\}^{-1}\right]^{\frac{1}{\beta}}.$$

We set

$$Y = \{u \in \mathcal{C}_{t_0} : u(0) = u_0, \|u(t) - u_0\| \leq R, \quad \text{for} \quad 0 \leq t \leq t_0\}.$$

Clearly, Y is a bounded, closed and convex subset of $\mathcal{C}_{t_0}$.

For any $0 < \tilde{T} \leq T$, we define a mapping F from $\mathcal{C}_{\tilde{T}}$ into $\mathcal{C}_{\tilde{T}}$ given by

$$(Fu)(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} S(t-s) f(s, u(s))\, ds.$$

Clearly, F is well defined.

We need to show that $F : Y \to Y$. For any $u \in Y$, we have $(Fu)(0) = u_0$. If $t \in [0, t_0]$, then we have,

$$\|(Fu)(t) - u_0\| \leq \|S(t)u_0 - u_0\| + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|S(t-s)\|\|f(s,u(s))\|ds \leq$$

$$\leq \frac{R}{2} + \frac{M}{\beta\Gamma(\beta)}\{L_f(R)[T+R] + \|f(0,u_0)\|\}t_0^{\beta} \leq R.$$

Hence, $F : Y \to Y$.

Now we will show that F maps Y into a precompact subset $F(Y)$ of Y. For this we will show that for fixed $t \in [0, t_0]$, $Y(t) = \{(Fu)(t) : u \in Y\}$ is precompact in H and $F(Y)$ is an uniformly equicontinuous family of functions. Here, for $t = 0$, $Y(0) = \{u_0\}$ is precompact in H.

Let $t > 0$ be fixed. For an arbitrary $\epsilon \in (0, t)$, define a mapping F_ϵ on Y by the formula

$$(F_\epsilon u)(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)}\int_0^{t-\epsilon} (t-s)^{\beta-1} S(t-s) f(s,u(s))ds =$$

$$= S(t)u_0 + \frac{S(\epsilon)}{\Gamma(\beta)}\int_0^{t-\epsilon} (t-s)^{\beta-1} S(t-s-\epsilon) f(s,u(s))\, ds.$$

Since $S(\epsilon)$ is compact for every $\epsilon > 0$, the set $Y_\epsilon(t) = \{(F_\epsilon u)(t) : u \in Y\}$ is precompact in H for every $\epsilon \in (0, t)$, where $t \in (0, t_0]$.

Also, we have

$$\|(Fu)(t) - (F_\epsilon u)(t)\| = \left\| \frac{1}{\Gamma(\beta)} \int\limits_{t-\epsilon}^{t} (t-s)^{\beta-1} S(t-s) f(s, u(s))\, ds \right\| \leq \epsilon^\beta R_1$$

for all $t \in (0, t_0]$, $u \in Y$ and $R_1 = \dfrac{M}{\beta\Gamma(\beta)} \{L_f(R)[T + R] + \|f(0, u_0)\|\}$. Consequently, the set $Y(t)$, where $t \geq 0$, is precompact in H.

For any $t_1, t_2 \in (0, t_0]$ with $t_1 < t_2$ and $u \in Y$, we have

$$(Fu)(t_2) - (Fu)(t_1) = [S(t_2) - S(t_1)]u_0 + \frac{1}{\Gamma(\beta)} \int\limits_{t_1}^{t_2} (t_2 - s)^{\beta-1} S(t_2 - s) f(s, u(s))\, ds +$$

$$+ \frac{-1}{\Gamma(\beta)} \int\limits_{0}^{t_1} [(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}] S(t_2 - s) f(s, u(s))\, ds +$$

$$+ \frac{1}{\Gamma(\beta)} \int\limits_{0}^{t_1} (t_1 - s)^{\beta-1} [S(t_2 - s) - S(t_1 - s)]\, f(s, u(s))\, ds =$$

$$= I_1 + I_2 + I_3 + I_4. \tag{3.1}$$

Hence,

$$\|(Fu)(t_2) - (Fu)(t_1)\| \leq \|I_1\| + \|I_2\| + \|I_3\| + \mathcal{I}_\triangle\|. \tag{3.2}$$

We have

$$I_1 = [S(t_2) - S(t_1)]u_0.$$

From Theorem 2.6.13 in Pazy [10], it follows that for every $0 < \eta < 1 - \alpha$, $t_2 > t_1 > 0$, we have

$$\|I_1\| \leq A_1 \|(S(t_2) - S(t_1)) A^\alpha u_0\| \leq A_1 C_\eta C_{\alpha+\eta} t_1^{-(\alpha+\eta)} (t_2 - t_1)^\eta \|u_0\| \leq M_1 (t_2 - t_1)^\eta,$$

where C_η is some positive constant satisfying $\|A^\eta S(t)\| \leq C_\eta t^{-\eta}$ for all $t > 0$. Also, M_1 depends on t_1 and blows up as t_1 decreases to zero.

From equation (3.1), we have

$$\|I_2\| \leq \frac{1}{\Gamma(\beta)} \int\limits_{t_1}^{t_2} (t_2 - s)^{\beta-1} \|S(t_2 - s)\| \|f(s, u(s))\| ds \leq \frac{M A_2}{\beta\Gamma(\beta)} (t_2 - t_1)^\beta,$$

where $A_2 = \{L_f(R)[T + R] + \|f(0, u_0)\|\}$. We have

$$I_3 = \frac{-1}{\Gamma(\beta)} \int\limits_{0}^{t_1} [(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}] S(t_2 - s) f(s, u(s))\, ds.$$

Hence,

$$\|I_3\| \le \frac{A_2\, A_1\, C_\alpha}{\Gamma(\beta)} \int\limits_0^{t_1} (t_1 - s)^{\lambda - 1} [(t_1 - s)^{-\lambda\mu} - (t_2 - s)^{-\lambda\mu}]\, ds,$$

where $\lambda = 1 - \alpha$, $\mu = \dfrac{1-\beta}{1-\alpha}$ and $\alpha \ne 1$.

Hence, after some calculation, we get

$$\|I_3\| \le \frac{A_2\, A_1\, C_\alpha}{\Gamma(\beta)} \mu {\delta_1}^{\mu-1} (1-c)^{-\lambda(1-\mu)-1} (t_2 - t_1)^{\lambda(1-\mu)},$$

where $c = \left(1 - \left(\dfrac{\mu}{\lambda}\right)^1 \lambda\mu\right)$ and $0 < \delta_1 \le 1$.

Similarly, we get

$$\|I_4\| \le \frac{A_1\, A_2\, C_{1+\alpha}}{\Gamma(\beta)} \int\limits_0^{t_1} (t_1 - s)^{\beta-1} [(t_1 - s)^{-1} - (t_2 - s)^{-1}]\, ds \le$$

$$\le \frac{A_1\, A_2\, C_{1+\alpha}}{\alpha\Gamma(\beta)} {\delta_2}^{(\frac{1}{\beta}-1)} (1 - c_1)^{-\beta} (t_2 - t_1)^{\beta(1-\frac{1}{\beta})},$$

where $C_{1+\alpha} = \left(1 - \dfrac{1}{\beta^2}\right)$, $0 < \delta_2 \le 1$ and $C_{1+\alpha}$ is some positive constant satisfying $\|A^{1+\alpha} S(t)\| \le C_{1+\alpha} t^{-1-\alpha}$ for all $t > 0$.

Thus from the above calculations we observe that the right hand side of the inequality (3.2) tends to zero when $t_2 - t_1 \to 0$. Hence, $F(Y)$ is a family of equicontinuous functions. Also, $F(Y)$ is bounded. Thus from the Arzela–Ascoli theorem (cf. see Dieudonne [11]), $F(Y)$ is precompact. The existence of a fixed point of F in Y is the consequence of *Schauder's* fixed point theorem.

Hence, there exists $u \in Y$, such that for all $t \in [0, t_0]$, we have

$$u(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} S(t-s) f(s, u(s)) ds, \tag{3.3}$$

where $u(0) = u_0$.

Applying the similar arguments as above, we see that the function u given by equation (3.3) is uniformly Hölder continuous on $[0, t_0]$. With the help of the condition (H_2), we can show that the map $t \longmapsto f_1(t, u(t))$ is Hölder continuous on $[0, t_0]$. This completesthe proof of the theorem.

4. Existence of global solutions.

Theorem 4.1. *Suppose that* $0 \in \rho(-A)$ *and* $-A$ *generates a compact analytic semigroup* $S(t)$ *with* $\|S(t)\| \le M$, *for* $t \ge 0$, $u_0 \in D(A)$, *and the function* $f_1 : [0, \infty) \times H \to H$ *satisfies the condition* (H_2). *If there is a continuous nondecreasing real valued function* $k(t)$ *such that*

$$\|f_1(t, \psi)\| \le k(t)(1 + \|\psi\|) \quad for \quad t \ge 0, \quad \psi \in H,$$

then the equation (1.1) has a unique mild solution u which exists for all $t \geq 0$.

Proof. By Theorem 3.1, we can continue the solution of equation (1.1) as long as $\|u(t)\|$ stays bounded. Therefore, we need to show that if u exists on $[0, T)$, then $\|u(t)\|$ is bounded as $t \uparrow T$.

For $t \in [0, T)$, we have

$$u(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} S(t-s) f(s, u(s)) ds.$$

From the above equation, we get

$$\|u(t)\| \leq M\|u_0\| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|S(t-s)\| \|f(s, u(s))\| ds.$$

Hence

$$\|u(t)\| \leq C_2 + C_3 \int_0^t (t-s)^{(\beta-1)} \|u(s)\| \, ds,$$

where

$$C_2 = M\|u_0\| + \frac{1}{\beta\Gamma(\beta)} Mk(T)T^{\beta}$$

and

$$C_3 = \frac{1}{\Gamma(\beta)} Mk(T).$$

Hence from Lemma 6.7 (Chapter 5 in Pazy [10]), u is a global solution.

To complete the proof of the theorem we only need to show that u is unique on the whole interval.

Let u_1 and u_2 be two solutions of the given fractional integral equation (1.1). Then, by a similar argument as above, we see that

$$\|u_1(t) - u_2(t)\| \leq$$

$$\leq \frac{1}{\Gamma(\beta)} ML_f(R) \int_0^t (t-s)^{(\beta-1)} \|u_1(s) - u_2(s)\| \, ds.$$

Hence from Lemma 6.7 (Chapter 5, Pazy [10]), the solution u is unique. This completes the proof of the theorem.

5. Examples. Let $H = L^2((0,1);\mathbb{R})$. Consider the following fractional partial differential equations,

$$\frac{\partial^\beta}{\partial t^\beta} w(t,x) - \partial_x^2 w(t,x) = F(t, w(t,x)) \quad x \in (0,1), \quad t > 0,$$
$$w(0,x) = u_0, \quad w(t,0) = w(t,1) = 0, \quad t \in [0,T], \quad 0 < T < \infty, \tag{5.1}$$

where F is a given functions and $0 < \beta < 1$.

We define an operator A,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = H_0^1(0,1).$$

Here clearly the operator A is self-adjoint, with compact resolvent and is the infinitesimal generator of a compact analytic semigroup $S(t)$. We take $\alpha = 1/2$, $D(A^{1/2})$ is a Banach space with norm

$$\|x\|_{1/2} := \|A^{1/2}x\|, \quad x \in D(A^{1/2}),$$

and we denote this space by $H_{1/2}$.

The equation (5.1) can be reformulated as the following abstract equation in $H = L^2((0,1);\mathbb{R})$:

$$\frac{d^\beta u(t)}{dt^\beta} + Au(t) = f(t, u(t)), \quad t > 0,$$

$$u(0) = u_0,$$

where $u(t) = w(t,.)$ that is $u(t)(x) = w(t,x)$, $t \in [0,T]$, $x \in (0,1)$, and the function $f : [0,T] \times H \to H$ is given by

$$f(t, u(t))(x) = F(t, w(t,x)).$$

We can take $f(t,u) = h(t)g(u^{'})$, where h is Lipschitz continuous and $g : H \to H$ is Lipschitz continuous on H. In particular, we can take $g(u) = \sin u$, $g(u) = \xi u$, $g(u) = \arctan(u)$, where ξ is constant.

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