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SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH AN INFINITE SET OF PERIODIC SOLUTIONS*

НЕЛІНІЙНІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ ДРУГОГО ПОРЯДКУ З НЕСКІНЧЕННОЮ МНОЖИНОЮ ПЕРІОДИЧНИХ РОЗВ'ЯЗКІВ

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For the differential equation $u'' = f(t, u, u')$, where the function $f : R \times R^2 \to R$ is periodic in the first argument and $f(t, x, 0) \equiv 0$, sufficient conditions for the existence of a continuum of nonconstant periodic solutions are found.

Для диференціального рівняння $u'' = f(t, u, u')$, де функція $f : R \times R^2 \to R$ є періодичною за першим аргументом і $f(t, x, 0) \equiv 0$, знайдено необхідні умови для існування континууму періодичних розв'язків, що не є сталими.

The problems on the existence, uniqueness and non-uniqueness of periodic solutions of nonlinear differential equations and systems attract attention of many mathematicians and are the subject of numerous investigations (see, e.g, [1 – 16] and the references therein). Nevertheless, the description of classes of equations having a continuum of periodic solutions is far from being complete. The goal of the present paper is to fill this gap to a certain extent.

Below we consider the differential equation

$$u'' = f(t, u, u'), \tag{1}$$

where the function $f : R \times R^2 \to R$ satisfies the local Carathéodory conditions, i.e., $f(t, \cdot, \cdot) : R^2 \to R$ is continuous for almost all $t \in R$, $f(\cdot, x, y) : R \to R$ is measurable for all $(x, y) \in R^2$, and for an arbitrary $\rho > 0$ the function f_ρ, given by

$$f_\rho(t) = \max\{|f(t, x, y)| : |x| + |y| \le \rho\} \quad \text{for} \quad t \in R,$$

is Lebesgue integrable on every finite interval.

We are interested in the case, where the equalities

$$f(t + \omega, x, y) = f(t, x, y), \quad f(-t, x, -y) = f(t, x, y),$$
$$f(t, -x, -y) = -f(t, x, y), \tag{2}$$

$$f(t, x, 0) = 0 \tag{3}$$

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are fulfilled on $R \times R^2$; here ω is a positive constant.

In view of (3), equation (1) has a continuum of constant solutions. There naturally arises the question whether equation (1) under the conditions (2) and (3) may have nonconstant periodic solutions. As is stated in the proven below Theorem 1, the answer is positive.

Let $R_+ = [0, +\infty)$, L_ω be the space of ω-periodic and Lebesgue integrable on $[0, \omega]$ real functions, and M_ω the set of functions $\varphi : R \times R_+ \to R_+$ such that $\varphi(\cdot, x) \in L_\omega$ for arbitrary $x \in R_+$, $\varphi(t, \cdot) : R_+ \to R$ a continuous nondecreasing function for almost all $t \in R$, $\varphi(t, 0) \equiv \equiv 0$ and

$$\int_0^\omega \varphi(t, x)\, dt > 0 \quad \text{for} \quad x > 0. \tag{4}$$

Theorem 1. *Let conditions (2), (3) be fulfilled and*

$$f(t, x, y) \le -\varphi(t, x)\psi(y) \quad \textit{for} \quad t \in R_+, \quad x \in R_+, \quad 0 \le y \le r, \tag{5}$$

where $r > 0$, $\varphi \in M_\omega$, and $\psi : [0, r] \to R_+$ is a continuous function such that

$$\psi(0) = 0, \quad \psi(y) > 0 \quad \textit{for} \quad 0 < y \le r, \quad \int_0^r \frac{dy}{\psi(y)} < +\infty. \tag{6}$$

Then equation (1) has a continuum of nonconstant periodic solutions.

To prove the theorem, we will need the following lemma.

Lemma 1. *Let inequality (5) be fulfilled, where $\varphi \in M_\omega$, and $\psi : [0, r] \to R_+$ is a continuous function satisfying condition (6). Then for an arbitrary $c \in (0, r)$, there exists $t_c \in (0, +\infty)$ such that equation (1) on the interval $[0, t_c]$ has a solution u_c satisfying the conditions*

$$u_c(0) = 0, \quad u_c'(0) = c, \tag{7}$$

$$u_c(t) > 0, \quad 0 < u_c'(t) < r \quad \textit{for} \quad 0 < t < t_c, \quad u_c'(t_c) = 0. \tag{8}$$

Proof. Let u_c be a maximally extended to the right solution of problem (1), (7). Then either u_c is defined on R_+, and

$$u_c(t) > 0, \quad 0 < u_c'(t) < r \quad \text{for} \quad t \in R_+, \tag{9}$$

or there exists $t_c \in (0, +\infty)$ such that

$$u_c(t) > 0, \quad 0 < u_c'(t) < r \quad \text{for} \quad 0 < t < t_c \tag{10}$$

and

$$u_c'(t_c) \in \{0, r\}. \tag{11}$$

First we assume that condition (9) is fulfilled. Then in view of (5), for an arbitrarily fixed $a > 0$ almost everywhere on $[a, +\infty)$ the inequality

$$\varphi(t, x) \le -\frac{u_c''(t)}{\psi(u_c'(t))}$$

is fulfilled, where $x = u_c(a) > 0$. Integrating this inequality from a to $a + k\omega$, where k is an arbitrary natural number, due to the ω-periodicity of $\varphi(\cdot, x)$ and condition (6) we find

$$k \int\limits_a^{a+\omega} \varphi(t, x)\, dt \le \int\limits_{u_c'(a+k\omega)}^{u_c'(a)} \frac{dy}{\psi(y)} < \rho,$$

where

$$\rho = \int\limits_0^r \frac{dy}{\psi(y)} < +\infty.$$

Consequently,

$$\int\limits_0^\omega \varphi(t, x)\, dt = \int\limits_a^{a+\omega} \varphi(t, x)\, dt \le \frac{\rho}{k} \to 0 \quad \text{as} \quad k \to +\infty,$$

which contradicts condition (4). The obtained contradiction proves that the function u_c does not satisfy inequalities (9). Hence for some $t_c \in (0, +\infty)$, conditions (10) and (11) are fulfilled.

According to (5) and (10), almost everywhere on $(0, t_c)$ the inequality

$$u_c''(t) \le 0$$

is satisfied. Therefore $u_c'(t_c) \le c < r$, whence by virtue of (11) it follows that $u_c(t_c) = 0$. Thus condition (8) holds.

The lemma is proved.

Lemma 2. *Let on $R \times R^2$ equalities (2) be fulfilled and let the function u be a solution of equation (1) on some interval $[0, t_0] \subset R_+$. Then for an arbitrary natural k the function v, given by the equality*

$$v(t) = u(k\omega - t) \quad \textit{for} \quad k\omega - t_0 \le t \le k\omega,$$

is a solution of equation (1) on $[k\omega - t_0, k\omega]$.

Proof. Indeed,

$$v''(t) = u''(k\omega - t) = f\left(k\omega - t, u(k\omega - t), u'(k\omega - t)\right) =$$

$$= f\left(k\omega - t, v(t), -v'(t)\right) \quad \text{almost everywhere on} \quad [k\omega - t, k\omega].$$

Thus according to (2), we find

$$v''(t) = f\left(-t, v(t), -v'(t)\right) =$$

$$= f\left(t, v(t), v'(t)\right) \quad \text{almost everywhere on} \quad [k\omega - t, k\omega].$$

The lemma is proved.

Proof of the theorem. Owing to Lemma 1, for an arbitrary $c \in (0, r)$ there exists $t_c \in \in (0, +\infty)$ such that equation (1) on $[0, t_c]$ has a solution u satisfying conditions (7) and (8). We choose a natural number k so that

$$k\omega \geq 2t_c$$

and extend u_c on R in the following manner:

$$u_c(t) = \begin{cases} u_c(t_c) & \text{for} \quad t_c \leq t \leq k\omega - t_c, \\ u_c(k\omega - t) & \text{for} \quad k\omega - t_c \leq t \leq k\omega, \end{cases}$$

$$u_c(t + k\omega) = -u_c(t) \quad \text{for} \quad t \in R.$$

By conditions (2), (3) and Lemma 2, the function u_c is a $2k\omega$-periodic solution of equation (1). On the other hand, it is evident that

$$u_{c_1}(t) \not\equiv u_{c_2}(t) \not\equiv \text{const} \quad \text{for} \quad 0 < c_1 < c_2 < r.$$

Consequently, if c runs through the interval $(0, r)$, we obtain a continuum of periodic nonconstant solutions of equation (1).

The theorem is proved.

As an example, we consider the generalized Emden – Fowler equation

$$u'' = \sum_{k=1}^{m} p_k(t)|u'|^{\mu_k}|u|^{\lambda_k} \operatorname{sgn} u, \tag{12}$$

where

$$\begin{gathered} \lambda_k > 0, \quad \mu_k > 0 \quad p_k \in L_\omega, \\ p_k(-t) = p_k(t) \leq 0 \quad \text{for} \quad t \in R, \end{gathered} \tag{13}$$

$$\int_0^\omega p_k(t)\,dt < 0, \quad k = 1, \ldots, m. \tag{14}$$

The following proposition holds.

Corollary. *Let conditions (13) and (14) be fulfilled. Then for the existence of a continuum of periodic solutions of equation (12) it is necessary and sufficient that*

$$\min\{\mu_1, \ldots, \mu_n\} < 1. \tag{15}$$

Proof. Assume first that along with (13) and (14) condition (15) is fulfilled. Then without loss of generality we can assume that $\mu_1 < 1$. Due to condition (13), the function f, given by the equality

$$f(t, x, y) = \sum_{k=1}^{m} p_k(t)|y|^{\mu_k}|x|^{\lambda_k} \operatorname{sgn} x,$$

satisfies conditions (2) and (3). On the other hand, for an arbitrary $r > 0$ inequality (5) is fulfilled, where

$$\varphi(t,x) = |p_1(t)|x^{\lambda_1}, \quad \psi(y) = y^{\mu_1}.$$

Moreover, $\varphi \in M_\omega$, and ψ satisfies condition (6) since

$$\int_0^\omega |p_1(t)|\,dt > 0 \quad \text{and} \quad 0 < \mu_1 < 1.$$

Consequently, all the conditions of the above-given theorem are fulfilled which guarantees the existence of a continuum of nonconstant ω-periodic solutions of equation (12).

It remains to state that if

$$\mu_k \geq 1, \quad k = 1, \ldots, m, \tag{16}$$

then an arbitrary periodic solution u of equation (12) is constant. Indeed, almost everywhere on R the equality

$$u''(t) = p(t)u'(t) \tag{17}$$

is fulfilled, where

$$p(t) = \sum_{k=1}^{m} p_k(t)|u'(t)|^{\mu_k - 1}|u(t)|^{\lambda_k}\mathrm{sgn}\,(u(t)u'(t));$$

in addition, in view of (16), we have

$$p \in L_\omega. \tag{18}$$

On the other hand, owing to the ω-periodicity of u, there exists $t_0 \in R$ such that

$$u'(t_0) = 0.$$

Thus it follows from (17) and (18) that $u'(t) \equiv 0$, i.e., $u(t) \equiv \mathrm{const}$.

The corollary is proved.

Remark. If $p_k(t) \equiv 0$, $k = 1, \ldots, m$, then equation (12) has no nonconstant periodic solution. Consequently, condition (14) in the above-given corollary is essential and it cannot be weakened.

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