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ON FREE VIBRATIONS OF A THICK PERIODIC JUNCTION WITH CONCENTRATED MASSES ON THE FINE RODS

ВЛАСНІ КОЛИВАННЯ ГУСТОГО ПЕРІОДИЧНОГО З'ЄДНАННЯ З КОНЦЕНТРОВАНОЮ МАСОЮ НА ТОНКИХ СТЕРЖНЯХ

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Convergence theorems and asymptotic estimates (as $\varepsilon \to 0$) are proved for eigenvalues and eigenfunctions of a boundary value problem for the Laplace operator in a plane thick periodic junction with concentrated masses. This junction consists of the junction's body and a large number $N = O(\varepsilon^{-1})$ of the fine rods. The density of the junction is order $O(\varepsilon^{-\alpha})$, $\alpha \geq 0$, on the rods (the concentrated masses if $\alpha > 0$), and $O(1)$ outside of them. There are three qualitatively different cases in the asymptotic behavior of the eigenvalues and eigenfunctions: $0 \leq \alpha < 2$, $\alpha = 2$, $\alpha > 2$. The main attention is payed to the case $0 \leq \alpha < 2$.

Доведені теореми про збіжність та асимптотичні оцінки (коли $\varepsilon \to 0$) для власних значень та власних функцій крайової задачі для оператора Лапласа в плоскому густому періодичному з'єднанні з концентрованою масою. Це з'єднання складається з деякої області і великої кількості $N = O(\varepsilon^{-1})$ тонких стержнів. Густина з'єднання є величиною порядку $O(\varepsilon^{-\alpha})$ на стержнях (концентрація маси при $\alpha > 0$) та $O(1)$ поза стержнями. Можливі три якісно різні випадки в асимптотичній поведінці власних значень та власних функцій: $0 \leq \alpha < 2$, $\alpha = 2$, $\alpha > 2$. Головна увага приділяється першому випадку.

1. Introduction and statement of the problem. Vibration systems with a concentration of mass on a small set of diameter $O(\varepsilon)$ have been studied for a long time. It is experimentally established that such concentration leads to the big reduction of the main frequency and to the big localization of vibrations. The new impulse in these research was given by E. Sanchez-Palencia in the paper [1] in which the effect of local vibrations was mathematically described. Then many articles appeared (see [2 – 9] and other) that deal with the asymptotic behavior of vibrations of a body containing a small region (many small regions) where the density is very much higher than elsewhere.

In this paper we investigate free vibrations of a plane thick periodic junction Ω_ε with concentrated masses on the fine rods. The asymptotic method developed in [10 – 13] for periodic thick junctions is used. Some results have already been announced in [14].

The junction Ω_ε consists of the junction's body

$$\Omega_0 = \{x \in \mathbf{R}^2 : 0 < x_1 < a,\ 0 < x_2 < \gamma(x_1)\},$$

and a large number N of the fine rods $G_\varepsilon = \bigcup_{j=0}^{N-1} G_\varepsilon^j$,

$$G_\varepsilon^j = \{x \in \mathbf{R}^2 :\ |x_1 - \varepsilon\,(j + 1/2)| < \varepsilon h/2,\ x_2 \in (-1, 0]\}, \quad j = 0, 1, \ldots , N - 1,$$

i.e., $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon$. Here $\gamma \in C^\infty([0,a])$, $0 < \gamma_0 = \min_{x_1 \in [0,a]} \gamma(x_1)$; h is a fix number from the interval $(0,1)$; N is a large positive integer, therefore $\varepsilon = a/N$ is a small discrete parameter which characterizes the distance between the rods and their thickness.

We consider the spectral boundary value problem

$$\begin{aligned} -\Delta_x\, u(\varepsilon,x) &= \lambda(\varepsilon)\, \rho_1(\varepsilon,x) u(\varepsilon,x), && x \in \Omega_\varepsilon, \\ \partial_\nu\, u(\varepsilon,x) &= 0, && x \in \partial\Omega_\varepsilon \cap \{x : x_2 \ge 0\}, \\ u(\varepsilon,x) &= 0, && x \in \Gamma_\varepsilon = \partial\Omega_\varepsilon \cap \{x : x_2 < 0\}, \end{aligned} \tag{1}$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; and $\rho_1(\varepsilon,x) = 1$ if $x \in \Omega_0$, and $\rho_1(\varepsilon,x) = \varepsilon^{-\alpha}$ if $x \in G_\varepsilon$; α is a nonnegative parameter.

For each $\varepsilon > 0$ there is a sequence of eigenvalues of problem (1)

$$0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \le \ldots \le \lambda_n(\varepsilon) \le \cdots \to +\infty \quad \text{as} \quad n \to \infty, \tag{2}$$

and a sequence of the corresponding eigenfunctions $\{u_n(\varepsilon,\cdot) : n \in \mathbf{N}\}$, that are orthonormalized by the following way

$$(u_n, u_m)_{\Omega_0} + \varepsilon^{-\alpha}(u_n, u_m)_{G_\varepsilon} = \delta_{n,m}, \quad \{n,\, m\} \in \mathbf{N}, \tag{3}$$

where $(\cdot,\cdot)_\Upsilon$ is the scalar product in $L_2(\Upsilon)$, and $\delta_{n,m}$ is the Kronecker delta.

Our aim is to describe the asymptotic behavior of eigenvalues $\{\lambda_n(\varepsilon) : n \in \mathbf{N}\}$ and eigenfunctions $\{u_n(\varepsilon,\cdot) : n \in \mathbf{N}\}$ as $\varepsilon \to 0$ ($N \to +\infty$). If $\alpha > 0$, then the passage to the limit is accompanied by the concentrated masses on the joined thin domains $G_\varepsilon^0, \ldots, G_\varepsilon^{N-1}$.

As we see in section 2, there are three qualitatively different cases in the asymptotic behavior of the eigenvalues and eigenfunctions: $0 \le \alpha < 2$, $\alpha = 2$, $\alpha > 2$. Here, we consider the case $0 \le \alpha < 2$. Some remarks for the other cases are given in Remark 2 and in the conclusion.

2. Auxiliary inequalities. The case $0 \le \alpha \le 2$. Consider the space $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ formed by functions of the Sobolev space $H^1(\Omega_\varepsilon)$ whose traces vanish on Γ_ε. In this subspace we introduce along with the norm $\|u\|_1 = (\int_{\Omega_\varepsilon} (|\nabla u|^2 + \rho_1 u^2)\, dx)^{1/2}$ a new norm $\|\cdot\|_\varepsilon$ that is generated by the scalar product

$$\langle u, v \rangle_\varepsilon = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx.$$

Denote the space $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ with this scalar product by H_ε.

Lemma 1. *For ε small enough, the norms $\|\cdot\|_1$ and $\|\cdot\|_\varepsilon$ are equivalent, i.e., there exist positive constants c_1, ε_0, such that for all $\varepsilon \in (0,\varepsilon_0)$ the following inequalities hold :*

$$\|u\|_\varepsilon \le \|u\|_1 \le c_1 \|u\|_\varepsilon\,, \qquad u \in H_\varepsilon. \tag{4}$$

Proof. In (4), it is not obvious that the second inequality holds. Suppose the contrary. Then there exist sequences $\{\varepsilon_m : m \in \mathbf{N}\}$, $\{v_m : m \in \mathbf{N}\} \in H_{\varepsilon_m}$, such that $\lim_{m\to 0} \varepsilon_m = 0\,$,

$$\|v_m\|_1 = 1\,, \tag{5}$$

$$\|v_m\|_{\varepsilon_m} = \int_{\Omega_{\varepsilon_m}} |\nabla v_m|^2 \, dx < m^{-1}. \tag{6}$$

Since the sequence $\{v_m\}$ is bounded in $H^1(\Omega_0)$, we may assume without loss of generality, that it is a Cauchy sequence in $L_2(\Omega_0)$. From inequality (6) it follows that the sequence $\{v_m\}$ is a Cauchy sequence also in $H^1(\Omega_0)$:

$$\|v_m - v_n; H^1(\Omega_0)\|^2 \leq \|v_m - v_n; L_2(\Omega_0)\|^2 + m^{-1} + n^{-1}.$$

Hence, $\{v_m\}$ converges in this space to some element $v_0 \in H^1(\Omega_0)$.

By virtue of the Friedrich inequality we have

$$\varepsilon^{-\alpha} \int\limits_{G_{\varepsilon_m}} v_m^2\, dx \leq \varepsilon^{-\alpha+2} \int\limits_{\Omega_{\varepsilon_m}} (\partial_1 v_m)^2\, dx\,, \tag{7}$$

where $\partial_i = \partial/\partial x_i,\ i = 1, 2$. Granting this estimate, we obtain from (5) and (6) that

$$1 = \|v_m\|_1 \longrightarrow \int_{\Omega_0} v_0^2\, dx \quad \text{as} \quad m \to \infty\,; \qquad \int_{\Omega_0} |\nabla v_0|^2\, dx = 0\,.$$

This means that $v_0 = \text{const} = |\Omega_0|^{-1/2}$ in Ω_0 , where $|\Upsilon|$ is the measure of a domain Υ in $\mathbf{R}^2$.

On the one hand, from the trace theorem for functions in Sobolev spaces and the Corollary 1.7 [5], it follows that

$$\int\limits_{Q_{\varepsilon_m}} v_m^2\, dx \longrightarrow h\, |\Omega_0|\, a \quad \text{as} \quad m \to \infty\,,$$

where $Q_\varepsilon = G_\varepsilon \cap \{x_2 = 0\}$. On the other hand, we have

$$\int\limits_{Q_{\varepsilon_m}} v_m^2\, dx \leq \int\limits_{G_{\varepsilon_m}} (\partial_2 v_m)^2\, dx < m^{-1} \to 0 \quad \text{as} \quad m \to \infty\,.$$

The lemma is proved.

Remark 1. It should be noted that here and further all constants $\{c_i\}$ in asymptotic inequalities are independent of the parameter ε.

Definition 1. *A number $\lambda(\varepsilon)$ is called an eigenvalue of problem (1) if there exists a function $u(\varepsilon, \cdot) \in H_\varepsilon \setminus \{0\}$ such that for all functions $v \in H_\varepsilon$ the following integral identity holds:*

$$\langle u(\varepsilon, \cdot), v\rangle_\varepsilon = \lambda(\varepsilon)\, (\rho_1(\varepsilon, \cdot) u(\varepsilon, \cdot), v)_{\Omega_\varepsilon}\,. \tag{8}$$

In this case the function $u(\varepsilon, \cdot)$ is called an eigenfunction that corresponds to the eigenvalue $\lambda(\varepsilon)$.

Define an operator $A_\varepsilon^{(1)} : H_\varepsilon \longmapsto H_\varepsilon$ by

$$\langle A_\varepsilon^{(1)} u, v\rangle_\varepsilon = \int\limits_{\Omega_\varepsilon} \rho_1 uv\, dx\,, \qquad u, v \in H_\varepsilon\,. \tag{9}$$

It is easy to verify that this operator is self-adjoint, positive, compact, and

$$\|A_\varepsilon^{(1)} u_\varepsilon\|_\varepsilon \leq c_1 \|u\|_\varepsilon\,, \qquad u \in H_\varepsilon. \tag{10}$$

Now we can rewrite the integral identity (8) as the spectral problem for the operator $A_\varepsilon^{(1)}$: $A_\varepsilon^{(1)} u(\varepsilon,\cdot) = \lambda^{-1}(\varepsilon)\, u(\varepsilon,\cdot)$.

Thus, the eigenvalues of problem (1) form the sequence (2), with the classical convention of repeated eigenvalues. Let us prove some inequalities for these eigenvalues. Let $\mathcal{L}_0(\widetilde{v}_1,\dots,\widetilde{v}_n)$ be the n-dimensional subspace of H_ε that is spanned on n linearly independent functions

$$\widetilde{v}_k = \begin{cases} v_k^+, & x \in \Omega_0; \\ 0, & x \in G_\varepsilon\,, \end{cases} \qquad k = 1,\dots,n\,, \tag{11}$$

where $v_1^+,\dots,v_n^+$ are orthonormal in $L_2(\Omega_0)$ eigenfunctions of the problem

$$\begin{aligned} -\Delta_x\, v_k^+(x) &= \mu_k\, v_k^+(x), && x \in \Omega_0, \\ \partial_\nu\, v_k^+(x) &= 0, && x \in \partial\Omega_0 \cap \{x : x_2 > 0\}, \\ v_k^+(x) &= 0, && x \in \partial\Omega_0 \cap \{x : x_2 = 0\}. \end{aligned} \tag{12}$$

By virtue of the minimax principle for eigenvalues, we have

$$\lambda_n(\varepsilon) = \min_{E\in\mathbf{E_n}} \max_{0\neq v\in E} \frac{\int\limits_{\Omega_\varepsilon} |\nabla v|^2\,dx}{\int_{\Omega_\varepsilon} \rho_1 v^2\,dx} \le \max_{0\neq v\in\mathcal{L}_0} \frac{\int\limits_{\Omega_0} |\nabla v|^2\,dx}{\int_{\Omega_0} v^2\,dx} = \mu_n\,. \tag{13}$$

Here $\mathbf{E_n}$ is a set of all subspaces of H_ε with the dimension n.

Taking into account conditions (3) and the second inequality (4), we obtain from the integral identity (8) the lower estimates for the eigenvalues

$$\lambda_n(\varepsilon) = \|u_n(\varepsilon,\cdot)\|_\varepsilon^2 \ge c_0 \|u_n(\varepsilon,\cdot)\|_1^2 \ge c_0 \int\limits_{\Omega_\varepsilon} \rho_1(\varepsilon,x) u_n^2(\varepsilon,x)\,dx = c_0 > 0\,, \tag{14}$$

where c_0 depend neither on ε nor on n.

Using inequality (13) and conditions (3), we deduce from (8) the following estimates for the eigenfunctions

$$\int\limits_{\Omega_\varepsilon} |\nabla u_n(\varepsilon,x)|^2\,dx \le c(n)\,. \tag{15}$$

The case $\alpha > 2$. Let us consider the following n-dimensional subspace $\mathcal{L}_\varepsilon(\phi_1,\dots,\phi_n)$ of H_ε that is spanned on the linearly independent functions

$$\phi_k = \begin{cases} 0, & x \in \Omega_0; \\ \sin\dfrac{\pi(2x_1 - \varepsilon(1+2j-h))}{2\varepsilon h} \sin\pi k x_2, & x \in G_\varepsilon^j, \end{cases} \qquad k = 1,\dots,n\,.$$

Then we get

$$\lambda_n(\varepsilon) = \min_{E\in\mathbf{E_n}} \max_{0\neq v\in E} \frac{\int\limits_{\Omega_\varepsilon} |\nabla v|^2\,dx}{\int\limits_{\Omega_\varepsilon} \rho_1 v^2\,dx} \le \max_{0\neq v\in\mathcal{L}_\varepsilon} \frac{\int\limits_{G_\varepsilon} |\nabla v|^2\,dx}{\varepsilon^{-\alpha}\int\limits_{G_\varepsilon} v^2\,dx} \le c(n)\varepsilon^{\alpha-2}\,. \tag{16}$$

Taking into account inequality (16), we make the following change of spectral parameter

$$\lambda(\varepsilon) = \varepsilon^{\alpha-2}\widetilde{\lambda}(\varepsilon) \tag{17}$$

in problem (1). As a result, we have the problem

$$\begin{array}{rcll} -\Delta_x\, u(\varepsilon,x) & = & \widetilde{\lambda}(\varepsilon)\, \rho_2(\varepsilon,x)u(\varepsilon,x), & x\in\Omega_\varepsilon, \\ \partial_\nu\, u(\varepsilon,x) & = & 0, & x\in\partial\Omega_\varepsilon\cap\{x: x_2\geq 0\}, \\ u(\varepsilon,x) & = & 0, & x\in\Gamma_\varepsilon, \end{array} \tag{18}$$

where $\rho_2(\varepsilon,x)=\varepsilon^{\alpha-2}$ if $x\in\Omega_0$, and $\rho_2(\varepsilon,x)=\varepsilon^{-2}$ if $x\in G_\varepsilon$.

By analogy with Lemma 1 we prove the following lemmas.

Lemma 2. *For ε small enough, the norms $\|u\|_2 = (\int\limits_{\Omega_\varepsilon}(|\nabla u|^2+\rho_2 u^2)\,dx)^{1/2}$ and $\|\cdot\|_\varepsilon$ are equivalent.*

Lemma 3. *For ε small enough, the following inequality holds $(u,u)_{\Omega_0}\leq\|u\|_\varepsilon,\quad u\in H_\varepsilon$.*

Changing $\rho_2(\varepsilon,\cdot)$ instead of $\rho_1(\varepsilon,\cdot)$, we can repeat definition 1 for problem (18), define an operator $A_\varepsilon^{(2)}: H_\varepsilon\longmapsto H_\varepsilon$ by formula (9) and obtain for one estimate (10). By repeating the previous argument and using Lemma 2, we deduce the following estimates

$$0<c_0\leq\widetilde{\lambda}_n(\varepsilon)\leq c(n), \qquad \|u_n\|_\varepsilon^2\leq c(n) \tag{19}$$

for eigenvalues $\{0<\widetilde{\lambda}_1(\varepsilon)\leq\ldots\leq\widetilde{\lambda}_n(\varepsilon)\leq\ldots\}$ of problem (18) and corresponding eigenfunctions, but in this case these eigenfunctions are orthonormalized by the following way

$$\varepsilon^{\alpha-2}(u_n,u_m)_{\Omega_0}+\varepsilon^{-2}(u_n,u_m)_{G_\varepsilon}=\delta_{n,m}, \quad \{n,\,m\}\in\mathbf{N}. \tag{20}$$

Remark 2. According to Lemma 3 and estimates (19) for the eigenfunctions, the first term in (20) tends 0 as $\varepsilon\to 0$. Taking into account (7), (15), we can state the same for the second term in (3), if $0\leq\alpha<2$. Thus, there are three qualitatively different cases in the asymptotic behavior of the eigenvalues and the eigenfunctions: $0\leq\alpha<2$, $\alpha=2$, $\alpha>2$. As we see below, in the first case the energy of the free vibrations is concentrated in the junction's body. It should be noted that in the other cases the energy is concentrated both in the junction's body and in the fine rods.

3. Junction-layer problems. Let us introduce the „rapid" coordinates $\eta=\varepsilon^{-1}x$ in problem (1). Passing to $\varepsilon=0$, we see that the plane cylinder G_ε^0 is transformed into the semi-infinite strip $\Pi^-=I_h\times(-\infty,0]$, where $I_h=\Big(\dfrac{1-h}{2},\dfrac{1+h}{2}\Big)$; and the set Ω_0 is transformed into the first octant $\{\eta:\eta_i>0\,,i=1,2\}$. Taking into account the periodicity of the cylinders $\{G_\varepsilon^j: j= =0,\ldots,N-1\}$, we can regard that the union Π of the semi-strips Π^- and $\Pi^+=(0,1)\times(0,+\infty)$ is the base domain in which the junction-layer problems have to be considered. Obviously, solutions of these junction-layer problems must be 1-periodic in η_1, i. e.,

$$\partial_{\eta_1}^k Z(\eta)|_{\eta_1=0}=\partial_{\eta_1}^k Z(\eta)|_{\eta_1=1}\,, \qquad \eta\in\partial\Pi^+,\,\eta_2>0,\ k=0,1. \tag{21}$$

Let us investigate some properties of solutions to the following junction-layer problem

$$\begin{array}{rcll} -\Delta_{\eta\eta}\, Z(\eta) & = & F(\eta)\,, & \eta\in\Pi, \\ Z(\eta) & = & 0\,, & \eta\in\partial\Pi^-\setminus I_h, \\ \partial_{\eta_2}Z(\eta_1,0) & = & 0\,, & (\eta_1,0)\in\partial\Pi^+\setminus I_h, \\ \partial_{\eta_1}^k Z(\eta)|_{\eta_1=0} & = & \partial_{\eta_1}^k Z(\eta)|_{\eta_1=1}\,, & \eta\in\partial\Pi^+,\,\eta_2>0,\ k=0,1. \end{array} \tag{22}$$

At first we study the solvability of this problem. In this connection we use the scheme given in [12]. Let $\widehat{C}_0^\infty(\overline{\Pi})$ be a space of infinitely differentiable functions in $\overline{\Pi}$ that satisfy the periodical condition (21), the Dirichlet condition on $\partial\Pi^- \setminus I_h$, and are finite in η_2, i. e., $\forall v \in \widehat{C}_0^\infty(\overline{\Pi}) \quad \exists R > 0 \quad \forall \eta \in \overline{\Pi} \quad |\eta_2| \geq R \, : \, v(\eta) = 0$. Let $\mathcal{H}$ be the completion of the space $\widehat{C}_0^\infty(\overline{\Pi})$ by norm $\|u\|_{\mathcal{H}} = (\|\nabla_\eta u\|^2_{L_2(\Pi)} + \|\rho_0 u\|^2_{L_2(\Pi)})^{1/2}$, where $\rho_0(\eta_2) = (1+\eta_2)^{-1}$ if $\eta_2 \geq 0$, and $\rho_0(\eta_2) = 1$ if $\eta_2 < 0$.

We will call a function Z a generalized solution of problem (22) if for all functions $v \in \mathcal{H}$ the following integral identity holds

$$\int\limits_{\Pi} \nabla_\eta Z \cdot \nabla_\eta v \, d\eta = \int\limits_{\Pi} F v \, d\eta \,. \tag{23}$$

Lemma 4. *Let $\rho_0^{-1} F \in L_2(\Pi)$. Then there exists a unique solution $Z \in \mathcal{H}$ of problem (22).*

Proof. We rewrite identity (23) in the form

$$\langle Z, v \rangle - \int\limits_{\Pi_{0,2}} Z v \, d\eta = \int\limits_{\Pi} F v \, d\eta \,, \tag{24}$$

where $\Pi_{\alpha,\beta} = \{\eta \in \Pi \, : \alpha < \eta_2 < \beta\}$, and

$$\langle u, v \rangle = \int\limits_{\Pi} \nabla_\eta u \cdot \nabla_\eta v \, d\eta + \int\limits_{\Pi_{0,2}} u v \, d\eta \,. \tag{25}$$

We show that the new scalar product (25) generates an equivalent norm in $\mathcal{H}$. It is obvious that $\langle u, u \rangle \leq c \|u\|^2_{\mathcal{H}}, \quad u \in \mathcal{H}$. The inverse inequality with another constant follows from Friedrich's inequality; from Hardy's inequality

$$\int\limits_0^{+\infty} (1+\eta_2)^{-2} \phi^2(\eta_2) \, d\eta_2 \leq 4 \int\limits_0^{+\infty} |\partial_{\eta_2} \phi|^2 \, d\eta_2 \quad \forall \phi \in C^1([0, +\infty)), \quad \phi(0) = 0;$$

and the following inequality

$$\int\limits_{\Pi} \rho_0^2(\eta_2) u^2(\eta) \, d\eta \leq \int\limits_{\Pi^-} u^2 \, d\eta + \int\limits_{\Pi_{0,2}} \rho_0^2 \, u^2 \, d\eta + \int\limits_{\Pi} \rho_0^2((1-\chi(\eta_2))u)^2 \, d\eta \leq$$

$$\leq c_1 \left(\int\limits_{\Pi^-} (\partial_{\eta_1} u)^2 \, d\eta + \int\limits_{\Pi_{0,2}} \rho_0^2 u^2 \, d\eta + \int\limits_{\Pi^+} (\partial_{\eta_2} u)^2 \, d\eta \right) \leq c_2 \langle u, u \rangle. \tag{26}$$

Here $\chi \in C^\infty(\mathbf{R}), \; 0 \leq \chi \leq 1$, and

$$\chi(\eta_2) = \begin{cases} 1, & |\eta_2| \leq 1; \\ 0, & |\eta_2| \geq 2 \,. \end{cases} \tag{27}$$

Due to the conditions of Lemma 4 and to inequality (26), the right-hand side of identity (23) defines a linear continuous functional in $\mathcal{H}$. As the embedding $\mathcal{H} \subset L_2(\Pi_{0,2})$ is compact, there exists a self-adjoint positive compact operator $\mathcal{A} : \mathcal{H} \mapsto \mathcal{H}$ such that

$$\langle \mathcal{A}u, v \rangle = \int\limits_{\Pi_{0,2}} uv\, d\eta\,, \quad \{u, v\} \in \mathcal{H}\ .$$

Thus, we can rewrite identity (24) as the operator equation $Z - \mathcal{A}Z = f$, and apply to it the Fredholm's theorems. It is obvious that every solution of the homogeneous problem (22) in the space $\mathcal{H}$ is trivial. Therefore, the lemma is proved.

Remark 3. Let $\exp(\delta_0|\eta_2|)F \in L_2(\Pi)\,, \delta_0 > 0$. Taking into account the properties of solutions to elliptic problems in semi-cylinders, it is easily seen that the solution Z to problem (22) has the following asymptotics

$$Z(\eta) = \begin{cases} C + O(\exp(-\delta_1\eta_2)), & \eta_2 \to +\infty; \\ \qquad O(\exp(\delta_1\eta_2)), & \eta_2 \to -\infty\,, \end{cases} \tag{28}$$

where δ_1 is some positive number.

Remark 4. If the function F from Lemma 4 is even or odd in η_1 with respect to $1/2$, then the solution Z has the same symmetry. In fact, let for example F be even in η_1 with respect to $1/2$, i. e., $F(\eta_1, \eta_2) = F(1 - \eta_1, \eta_2)$. Then, due to the symmetry of the domain Π and with the substitution $\eta_1 = 1 - \eta_1'$ in problem (22), we obtain that the difference $Z(\eta_1, \eta_2) - Z(1 - \eta_1, \eta_2)$ is a solution of the homogeneous problem (22). By virtue of the uniqueness of such solution in the space $\mathcal{H}$, it follows that this difference vanishes.

Corollary 1. *The homogeneous problem (22) has a solution $\Xi_0 \notin \mathcal{H}$ with the asymptotics*

$$\Xi_0(\eta) = \begin{cases} C_0 + \eta_2 + O(\exp(-\delta_2\eta_2)), & \eta_2 \to +\infty, \\ \qquad O(\exp(\delta_2\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{29}$$

and this solution is even in η_1 with respect to $1/2$.

Proof. The solution Ξ_0 is sought in the form of a sum

$$\Xi_0(\eta) = \chi_+(\eta_2)\eta_2 + Z_0(\eta),$$

where $Z_0 \in \mathcal{H}$, and Z_0 is the solution to the problem (22) with the right-hand sides $F(\eta) = 2\chi_+'(\eta_2) + \chi_+''(\eta_2)\eta_2 =: F_0(\eta_2)$; χ_+ is a smooth cut-off function that equals 1 if $\eta_2 \geq 2$, and vanishes if $\eta_2 \leq 1$.

By virtue of Remarks 3 and 4, this solution Z_0 is even in η_1 with respect to $1/2$, and has the asymptotics

$$Z_0(\eta) = \begin{cases} C_0 + O(\exp(-\delta_2\eta_2)), & \eta_2 \to +\infty; \\ \qquad O(\exp(\delta_2\eta_2)), & \eta_2 \to -\infty\,. \end{cases} \tag{30}$$

In order to find the constant C_0 in (29) and (30), it is necessary to substitute the function Ξ_0 and Z_0 into Green's formula in $\Pi_{-R,R}$, and to pass to the limit as $R \to \infty$. As a result, we obtain

$$C_0 = \int\limits_{\Pi} \Xi_0(\eta)F_0(\eta_2)\, d\eta\ .$$

Remark 5. The solutions Z_0 and Ξ_0 have singularities in the points $\eta_1 = (1 \pm h)/2$ (see [15]). Nevertheless, taking into account the order of these singularities, we can apply to Z_0 and Ξ_0 Green's formula.

Remark 6. By analogy, we can show that the constant C in (28) equals

$$C = \int_{\Pi} \Xi_0(\eta) F(\eta)\, d\eta.$$

4. Asymptotic estimates in the case $0 \le \alpha < 2$. ***Asymptotic approximations.*** Let μ_n and v_n^+ be an eigenvalue and eigenfunction of problem (12). Define the function $\widetilde{v}_n$ by formula (11), and construct the approximation

$$U_n(\varepsilon, x) = \widetilde{v}_n(x) + \varepsilon \chi_0(x_2) \partial_2 v_n^+(x_1, 0) \widetilde{\Xi}_0(x/\varepsilon), \qquad x \in \Omega_\varepsilon, \tag{31}$$

where $\chi_0(x_2) = \chi(2x_2/r_0)$, $r_0 = \min(\gamma_0, 1)$, the function χ is defined by (27);

$$\widetilde{\Xi}_0(\eta) = \begin{cases} \Xi_0(\eta) - \eta_2, & \eta \in \Pi^+; \\ \Xi_0(\eta), & \eta \in \Pi^-, \end{cases}$$

and Ξ_0 is the solution to the homogeneous problem (22) with asymptotic (29).

It is easily seen that $U_n(\varepsilon, \cdot) \in H_\varepsilon$, and due to characteristics of Ξ_0, the function $U_n(\varepsilon, \cdot)$ satisfy the boundary conditions of problem (1). Substituting $\{U_n(\varepsilon, \cdot), \mu_n\}$ into problem (1) in place of $\{u(\varepsilon, \cdot), \lambda(\varepsilon)\}$, we find that for any $\psi \in H_\varepsilon$

$$\int_{\Omega_\varepsilon} (\nabla U_n \cdot \nabla \psi - \mu_n \rho_1 U_n \psi)\, dx = \Phi_\varepsilon(\psi), \tag{32}$$

where

$$\Phi_\varepsilon(\psi) = -\varepsilon \mu_n \int_{\Omega_\varepsilon} \rho_1(\varepsilon, x) \chi_0(x_2) \partial_2 v_n^+(x_1, 0) \widetilde{\Xi}_0(x/\varepsilon) \psi(x)\, dx +$$

$$+ \varepsilon \int_{\Omega_\varepsilon} \chi_0'(x_2) \partial_2 v_n^+(x_1, 0) \left(\widetilde{\Xi}_0(x/\varepsilon) \partial_2 \psi(x) - \varepsilon^{-1} \partial_{\eta_2}(\widetilde{\Xi}_0)(x/\varepsilon) \psi(x) \right) dx +$$

$$+ \varepsilon \int_{\Omega_\varepsilon} \chi_0(x_2) \partial_{12}^2 v_n^+(x_1, 0) \left(\widetilde{\Xi}_0(x/\varepsilon) \partial_1 \psi(x) - \varepsilon^{-1} \partial_{\eta_1}(\widetilde{\Xi}_0)(x/\varepsilon) \psi(x) \right) dx. \tag{33}$$

In order to estimate the terms in (33), we use the following lemma.

Lemma 5. *Assume that Z is a function, 1-periodic in η_1, belonging to the space $L_2(\Pi)$ and exponentially decreasing at infinity, i.e., there exist positive constants C_0, R_0, β_0 such that $|Z(\eta)| \le C_0 \exp(-\beta_0 |\eta_2|)$ if $|\eta_2| \ge R_0$. Then for any $\delta > 0$ there exist positive constants C_1, ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the following inequality is valid:*

$$\left| \int_{\Omega_\varepsilon} Z(x/\varepsilon) \psi(x)\, dx \right| \le C_1 \varepsilon^{1-\delta} \|\psi\|_\varepsilon, \qquad \psi \in H_\varepsilon.$$

Proof. Set $B_{\varepsilon,\delta} = \Omega_\varepsilon \cap (0,a) \times (-\varepsilon^{1-2\delta}, \varepsilon^{1-2\delta}), \quad \delta > 0$. Then

$$\Big|\int\limits_{\Omega_\varepsilon} Z(x/\varepsilon)\psi(x)\,dx\Big| \le \Big|\int\limits_{B_{\varepsilon,\delta}} Z(x/\varepsilon)\psi(x)\,dx\Big| + \Big|\int\limits_{\Omega_\varepsilon\setminus B_{\varepsilon,\delta}} Z(x/\varepsilon)\psi(x)\,dx\Big|.$$

The properties of the function Z lead us to the conclusion that the second summand in this inequality decreases exponentially as $\varepsilon \to 0$. With the help of Lemma 1.5 [5], we estimate the first summand:

$$\Big|\int\limits_{B_{\varepsilon,\delta}} Z(x/\varepsilon)\psi(x)\,dx\Big| \le \Big(\int\limits_{B_{\varepsilon,\delta}} Z^2(x/\varepsilon)\,dx\Big)^{1/2} \|\psi\|_{L_2(B_{\varepsilon,\delta})} \le c\varepsilon^{1-\delta}\|Z\|_{L_2(\Pi)}\|\psi\|_\varepsilon.$$

The lemma is proved.

Using (29) and Lemma 5, we deduce that

$$|\Phi_\varepsilon(\psi)| \le c(\delta)\,\varepsilon^{1-\delta}\|\psi\|_\varepsilon \quad \text{for any} \quad \delta > 0\,. \tag{34}$$

Remark 7. The constant $c(\delta)$ in inequality (34) depends on the quantities $\max_{x_1\in[0,a]} |\partial_{i2}^{i+1} v_n^+(x_1,0)|, \ i = 0,1$. Applying the even extension, with respect to the line $x_1 = 0$ and $x_1 = a$, to problem (12), we establish that the function v_n^+ and its derivatives have no singularities at the points $(0,0)$ and $(0,a)$. Then, by virtue of classical results on the smoothness of solutions to boundary value problems, the quantities mentioned above are bounded.

Thus, the right-hand side of integral equality (32) is a linear bounded functional on the space H_ε, and its norm is bounded by $c(\delta)\,\varepsilon^{1-\delta}, \ \delta > 0$. On the basis of the definition of the operator $A_\varepsilon^{(1)}$ (see (9)) and the Riesz theorem, we get from (32) the inequality

$$\|U_n(\varepsilon,\cdot) - \mu_n A_\varepsilon^{(1)} U_n(\varepsilon,\cdot)\|_\varepsilon \le c(\delta)\,\varepsilon^{1-\delta}\,, \quad \delta > 0\,, \tag{35}$$

which, by virtue of the first part of Lemma 12 [16], partially justifies the constructed asymptotics for the solutions of problem (1) :

$$\min_{k\in\mathbf{N}} |\mu_n^{-1} - \lambda_k^{-1}(\varepsilon)| \le \|U_n\|_\varepsilon^{-1} \|A_\varepsilon^{(1)} U_n - \mu_n^{-1} U_n\|_\varepsilon = O(\varepsilon^{1-\delta})\,. \tag{36}$$

Convergence theorem and asymptotic estimates. To prove the convergence theorem, first we observe that there exists an extension operator

$$\mathbf{P}_\varepsilon : H_\varepsilon \mapsto H^1(\Omega,\Gamma_{-1}) \ \text{ such that } \int\limits_\Omega |\nabla \mathbf{P}_\varepsilon u|^2\,dx \le c\|u\|_\varepsilon, \quad u \in H_\varepsilon. \tag{37}$$

Here Ω is the interior of the union $\overline{\Omega}_0 \cup \overline{D}$; $D = (0,a)\times(-1,0)$; $\Gamma_{-1} = \{x : 0 < x_1 < a, x_2 = -1\}$; and functions that belong to the subspace $H^1(\Omega,\Gamma_{-1})$ of $H^1(\Omega)$ vanish on Γ_{-1}.

We construct this operator in the following way. At first a function $u \in H_\varepsilon$ is prolonged by zero on the set $\Omega_\varepsilon \cup D_\varepsilon$, where $D_\beta = [0,a]\times[-1,-\beta]$. Further extension of u to Ω is performed similarly as for perforated domains [5].

Theorem 1. *Let* $\{\lambda_n(\varepsilon) : n \in \mathbf{N}\}$ *and* $\{0 < \mu_1 < \mu_2 \le \ldots \le \mu_n \ldots\}$ *be the ordered sequences sequences of eigenvalues of problems (1) and (12) respectively; let* $\{u_n(\varepsilon,\cdot) : n \in \mathbf{N}\}$ *be the corresponding sequence of eigenfunctions satisfying condition (3). Then for any* $n \in \mathbf{N}$

$$\lambda_n(\varepsilon) \to \mu_n \quad \textit{as} \quad \varepsilon \to 0\,;$$

there is a subsequence of $\{\varepsilon\}$ *(still denoted by* $\{\varepsilon\}$*) such that*

$$\mathbf{P}_\varepsilon u_n(\varepsilon,\cdot) \to \widetilde{\varphi}_n \quad \textit{weakly in} \quad H^1(\Omega,\Gamma_{-1}) \quad \textit{as} \quad \varepsilon \to 0\,,$$

where $\widetilde{\varphi}_n(x) = \varphi_n^+(x)$ *if* $x \in \Omega_0$, *and* $\widetilde{\varphi}_n(x) = 0$ *if* $x \in D$; $\{\varphi_n^+\}$ *are eigenfunctions of problem (12) that are orthonormal in the space* $L_2(\Omega_0)$.

Proof. Bearing in mind the boundedness of $\lambda_n(\varepsilon)$ in ε for fixed n (see (13), (14)), and the inequalities (15) and (37), with the help of the diagonal process, one can choose a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) such that $\lambda_n(\varepsilon) \to \mu_n^*$, and $\mathbf{P}_\varepsilon u_n(\varepsilon,\cdot) \to \to \widetilde{\varphi}_n$ weakly in $H^1(\Omega,\Gamma_{-1})$ as $\varepsilon \to 0$. From inequality (14), it follows that $0 < \mu_1^* \le \mu_2^* \le \le \ldots \le \mu_n^* \le \ldots$.

According to the Remark 2, we have that

$$\delta_{n,m} = (u_n,u_m)_{\Omega_0} + \varepsilon^{-\alpha}(u_n,u_m)_{G_\varepsilon} \;\to\; (\varphi_n^+,\varphi_m^+)_{\Omega_0} \quad \text{as} \quad \varepsilon \to 0\,,$$

whence $\varphi_n^+ \ne 0$.

Write the integral identity (8) for the eigenfunction $u_n(\varepsilon,\cdot)$ with a test function $v \in H^1(\Omega_0)$ that is equal to 0 on the interval $[0,a]$ and on the set G_ε, and pass to the limit as $\varepsilon \to 0$. We get

$$\int\limits_{\Omega_0} \nabla\varphi_n^+(x)\cdot\nabla v(x)\,dx = \mu_n^* \int_{\Omega_0} \varphi_n^+(x)v(x)\,dx.$$

This means that μ_n^* is an eigenvalue of problem (12), and φ_n^+ is the corresponding eigenfunction.

Now we write (8) with the following test function

$$v(x) = \begin{cases} 0\,, & x \in \Omega_0; \\ u_n(\varepsilon,x)\psi(x)\,, & x \in G_\varepsilon\,, \end{cases}$$

where $\psi \in C^\infty(\overline{D})$; $\psi(x) > 0$ if $x \in [0,a]\times[-1,0)$ and $\psi(x_1,0) = 0$, $x_1 \in [0,a]$. We obtain that

$$\int\limits_{G_\varepsilon} \psi|\nabla u_n|^2\,dx = -\int\limits_{G_\varepsilon} u_n\nabla u_n\cdot\nabla\psi\,dx + \lambda_n(\varepsilon)\varepsilon^{-\alpha}\int\limits_{G_\varepsilon} \psi\; u_n^2\,dx. \tag{38}$$

Fixing some $\beta > 0$ and taking into account (7), (13), (15), we deduce from (38)

$$0 < c_\beta \int\limits_{D_\beta} |\nabla\mathbf{P}_\varepsilon u_n|^2\,dx = c_\beta \int\limits_{G_\varepsilon\cap\{x:\,x_2<-\beta\}} |\nabla u_n|^2\,dx \le C_\beta(\varepsilon + \varepsilon^{2-\alpha})\|u_n\|_\varepsilon^2,$$

if ε is small enough. Since $\mathbf{P}_\varepsilon u_n \to 0$ in $H^1(D_\beta,\Gamma_{-1})$ as $\varepsilon \to 0$ and β is arbitrary positive number, $\widetilde{\varphi}_n = 0$ in D.

In order to complete the proof, it remains to show that

$$\mu_n^* = \mu_n\,, \quad n \in \mathbf{N}\,. \tag{39}$$

Let $\mu_k = \mu_{k+1} = \ldots = \mu_{k+q-1}$ be an eigenvalue of multiplicity q. Let us show that there exist exactly q eigenvalues of problem (1) with regard to multiplicity which tend to μ_k as $\varepsilon \to 0$. This will mean that relations (39) are true.

Assume that there exist r eigenvalues $\{\lambda_{n_i}(\varepsilon) : i = 1, \dots, r\}$ of problem (1) which tend to μ_k and $r > q$. By the preceding arguments, we have for the corresponding eigenfunctions that $u_{n_i}(\varepsilon, \cdot) \to \varphi^+_{n_i}$ weakly in $H^1(\Omega_0)$ as $\varepsilon \to 0$, where $\{\varphi^+_{n_i} : i = 1, \dots, r\}$ are orthonormal in $L_2(\Omega_0)$ eigenfunctions of problem (12). Thus, the eigenvalue μ_k has multiplicity r, but it is a contradiction.

Now, let r be less then q and let v^+_{k+i}, $i = 0, 1, \dots, q-1$ be eigenfunctions of problem (12) that correspond to the eigenvalue μ_k. With the help of these eigenfunctions, we construct the approximations U_{k+i}, $i = 0, 1, \dots, q-1$, by formula (31), and arrive at inequality (35). Applying the second part of Lemma 12 [16] to this inequality, we conclude that there exists a linear combination of the eigenfunctions $u_{n_1}, \dots, u_{n_r}$ of problem (1)

$$R^{(i)}_\varepsilon = \sum_{j=1}^{r} d_{ij}(\varepsilon) u_{n_j}(\varepsilon, \cdot), \quad 0 < c_1 \le \sum_{j=1}^{r} d^2_{ij}(\varepsilon) \le c_2, \quad r < q,$$

such that $\|v^+_{k+i} - R^{(i)}_\varepsilon; L_2(\Omega_0)\| \le c_i \varepsilon^{1-\delta}$, $i = 1, \dots, q$. Passing to the limit in these inequalities over a suitable subsequence of $\{\varepsilon\}$, we get

$$v^+_{k+i}(x) = \sum_{j=1}^{r} d^*_{ij} \varphi^+_{n_j}(x), \quad x \in \Omega, \quad 0 < c_1 \le \sum_{j=1}^{r} (d^*_{ij})^2 \le c_2, \quad i = 0, 1, \dots, q-1.$$

But this contradicts to the linear independence of the functions $v^+_k, \dots, v^+_{k+q-1}$.

Since the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof, we have $\lambda_n(\varepsilon) \to \mu_n$ as $\varepsilon \to 0$. The theorem is proved.

The above theorem allows us to obtain asymptotic estimates for the eigenvalues and eigenfunctions immediately from (36) and Lemma 12 [16].

Theorem 2. *For any $\delta > 0$, $n \in \mathbf{N}$, and ε small enough, we have*

$$|\lambda_n(\varepsilon) - \mu_n| \ \le \ c_1(n, \delta)\, \varepsilon^{1-\delta}.$$

Theorem 3. *Assume that $\mu_n = \mu_{n+1} = \ldots = \mu_{n+q}$ is an eigenvalue of problem (12) with multiplicity q, and that $v^+_n, \dots, v^+_{n+q-1}$ are the corresponding eigenfunctions. Then there exist constants ε_0, $c_2(n, \delta)$, and $\{d_{ik}\}$ such that for $\varepsilon \in (0, \varepsilon_0)$ the following inequalities hold :*

$$\Big\| U_{n+i}(\varepsilon, \cdot) - \sum_{k=0}^{q-1} d_{ik}\, u_{n+k}(\varepsilon, \cdot) \Big\|_\varepsilon \ \le \ c_2(n, \delta)\, \varepsilon^{1-\delta}, \qquad i = 0, 1, \dots, q-1,$$

where $\{U_{n+i} : i = 0, 1, \dots, q-1\}$ are defined by (31).

It follows from Theorem 3, Lemma 1, and Lemma 5 the following corollary.

Corollary 2. *Let μ_n be a simple eigenvalue of problem (12). Then*

$$\Big\| \|U_n\|^{-1}_\varepsilon U_n(\varepsilon, \cdot) - \lambda_n^{-1/2}(\varepsilon)\, u_n(\varepsilon, \cdot) \Big\|_{H^1(\Omega_0)} \ \le \ c_2(n, \delta)\, \varepsilon^{1-\delta},$$

$$\int_{G_\varepsilon} |\nabla u_n(\varepsilon, \cdot)|^2\, dx \le c_3(n, \delta)\, \varepsilon^{1-\delta}, \quad \delta > 0.$$

Conclusion. It follows from above obtained results that the energy of the free vibrations is concentrated in the junction's body, and there exists no reduction of the frequencies in the case $0 \le \alpha < 2$. Similar situations were observed in [1 – 10], when the density on small sets is not so big.

In the other case we observe the big reduction. If $\alpha = 2$, then the eigenvalues $\{\lambda_n(\varepsilon)\}$ tend to $\pi^2 h^{-2}$ (h is the width of the strip Π^-) as $\varepsilon \to 0$, and their „splitting" occurs only in the second term of the asymptotics, i.e., $\lambda_n(\varepsilon) = \pi^2 h^{-2} + \varepsilon^2 \tau_n + O(\varepsilon^3)$, where τ_n is an eigenvalue of some operator-function. If $\alpha > 2$, then all eigenvalues $\{\lambda_n(\varepsilon)\}$ tend to zero, and have the asymptotics $\lambda_n(\varepsilon) = \varepsilon^{\alpha-2}\pi^2 h^{-2} + \varepsilon^\alpha \beta_n + O(\varepsilon^{\alpha+1})$.

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