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DIRICHLET PROBLEM WITH Φ-LAPLACIAN AND MIXED SINGULARITIES*
ЗАДАЧА ДІРІХЛЕ З Φ-ЛАПЛАСІАНОМ ТА МІШАНИМИ ОСОБЛИВОСТЯМИ

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We assume that $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}$ are closed intervals containing 0, ϕ is an increasing odd homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$ and $T \in (0, \infty)$. We will study the singular Dirichlet problem of the form

$$(\phi(u'))' + f(t, u, u') = 0\,, \quad u(0) = u(T) = 0,$$

and we will prove the existence of its smooth solution satisfying

$$u(t) \in \mathcal{A}_1\,, \quad u'(t) \in \mathcal{A}_2 \quad for \quad t \in [0, T].$$

Here f satisfies the Carathéodory conditions on the set $(0, T) \times \mathcal{D}$ and can have time singularities at $t = 0, t = T$ and space singularities at $x = 0, y = 0$.

Для замкнених інтервалів $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}$, які містять 0, та зростаючого непарного гомеоморфізму ϕ, який задовольняє умови $\phi(\mathbb{R}) = \mathbb{R}$ і $T \in (0, \infty)$, вивчено сингулярну задачу Діріхле вигляду

$$(\phi(u'))' + f(t, u, u') = 0\,, \quad u(0) = u(T) = 0,$$

і доведено існування гладкого розв'язку, що задовольняє умови

$$u(t) \in \mathcal{A}_1\,, \quad u'(t) \in \mathcal{A}_2 \quad для \quad t \in [0, T].$$

Тут f задовольняє умови Каратеодорі на множині $(0, T) \times \mathcal{D}$ і може мати особливості в $t = 0, t = T$ та просторові особливості в $x = 0, y = 0$.

1. Introduction. Let $T \in (0, \infty)$ and $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}$ be closed intervals containing 0. Assume that ϕ is an increasing odd homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$. We will study the singular Dirichlet problem of the form

$$\big(\phi(u')\big)' + f(t, u, u') = 0, \quad u(0) = u(T) = 0, \tag{1.1}$$

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and prove the existence of a solution of problem (1.1) satisfying

$$u(t) \in \mathcal{A}_1, \quad u'(t) \in \mathcal{A}_2 \quad \text{for} \quad t \in [0,T]\,.$$

Denote $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_i = \mathcal{A}_i \setminus \{0\}$, $i = 1, 2$.

We assume that

$$\begin{gathered} f \text{ satisfies the Carathéodory conditions on the set } (0,T) \times \mathcal{D} \\ \text{and that } f \text{ can have time singularities at } t = 0,\ t = T \\ \text{and space singularities at } x = 0,\ y = 0. \end{gathered} \tag{1.2}$$

Definition 1.1. *A function f has a time singularity at $t = 0$ $(t = T)$ if there exists $(x,y) \in \mathcal{D}$ such that*

$$\int_0^{\varepsilon} |f(t,x,y)|dt = \infty \quad \left(\int_{T-\varepsilon}^{T} |f(t,x,y)|dt = \infty \right)$$

for any sufficiently small $\varepsilon > 0$.

Definition 1.2. *A function f has a space singularity at $x = 0$ $(y = 0)$ if there exists a set $J \subset [0,T]$ with a positive Lebesgue measure such that the condition*

$$\limsup_{x \to 0} |f(t,x,y)| = \infty \quad \left(\limsup_{y \to 0} |f(t,x,y)| = \infty \right)$$

holds for a.e. $t \in J$ and some $y \in \mathcal{D}_2$ $(x \in \mathcal{D}_1)$.

Notation.

Let $[a,b] \subset \mathbb{R}$. $J \subset \mathbb{R}$, $\mathcal{M} \subset \mathbb{R}^2$.

We let $\operatorname{meas} \mathcal{A}$ denote the Lebegue measure of $\mathcal{A} \subset \mathbb{R}$;

$C[a,b]$ the Banach space of functions continuous on $[a,b]$ with the norm $||x||_C = \max\{|x(t)|; t \in [a,b]\}$;

$C^1[a,b]$ the Banach space of functions having continuous first derivatives on $[a,b]$ with the norm $||x||_{C^1} = ||x||_C + ||x'||_C$;

$AC[a,b]$ the set of absolutely continuous functions on $[a,b]$;

$AC^1[a,b]$ the set of functions having absolutely continuous derivatives on $[a,b]$;

$AC_{\text{loc}}(J)$ the set of functions $x \in AC[c,d]$ for each $[c,d] \subset J$;

$L[a,b]$ the Banach space of functions Lebesgue integrable on $[a,b]$ with the norm

$$||x||_L = \int_a^b |x(t)|dt;$$

Car$([a,b] \times \mathcal{M})$ the set of functions $f : [a,b] \times \mathcal{M} \to \mathbb{R}$ satisfying the Carathéodory conditions on $[a,b] \times \mathcal{M}$, i.e., $f(\cdot,x,y) : [a,b] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathcal{M}$, $f(t,\cdot,\cdot) : \mathcal{M} \to \mathbb{R}$ is continuous for a.e., $t \in [a,b]$, and for each compact set $\mathcal{K} \subset \mathcal{M}$ there is a function $m_{\mathcal{K}} \in L[a,b]$ such that

$$|f(t,x,y)| \le m_{\mathcal{K}}(t) \text{ for a.e. } t \in [a,b] \text{ and all } (x,y) \in \mathcal{K};$$

Car $((a,b)\times\mathcal{M})$ the set of functions $f\in$ Car $([c,d]\times\mathcal{M})$ for each $[c,d]\subset(a,b)$.

Definition 1.3. *A function $u:[0,T]\to\mathbb{R}$ with $\phi(u')\in AC[0,T]$ is a solution of problem (1.1) if u satisfies $(\phi(u'(t)))'+f(t,u(t),u'(t))=0$ for a.e. $t\in[0,T]$ and fulfils the boundary conditions $u(0)=u(T)=0$.*

In some works dealing with singular problems (see e.g. [1] or [2]) a little different definition of a solution is used. In particular, $\phi(u')$ need not belong to $AC[0,T]$. To avoid the misunderstanding we call such functions w-solutions and define them as follows.

Definition 1.4. *A function $u\in C[0,T]$ is a w-solution of problem (1.1) if there exists a finite number of points $t_\nu\in[0,T]$, $\nu=1,\ldots,r$, such that if we denote $J=[0,T]\setminus\{t_\nu\}_{\nu=1}^{r}$, then $\phi(u')\in AC_{\mathrm{loc}}(J)$, u satisfies $(\phi(u'(t)))'+f(t,u(t),u'(t))=0$ for a.e. $t\in[0,T]$ and fulfils the boundary conditions $u(0)=u(T)=0$.*

In this paper we generalize the existence principle of [3] which was proved for problem (1.1) where $\phi(y)\equiv y$. Here we work with a general ϕ including the case $\phi(y)=|y|^{p-2}y$ for $p>1$. Combining this existence principle (Theorem 3.1) with the lower and upper functions method we prove a new existence result (Theorem 4.1) for problem (1.1). Theorem 4.1 extends earlier results by Agarwall, Lü and O'Regan [4], Jiang [5], Staněk [6] and Wang, Gao [7].

2. Regular Dirichlet problem. Singular problems are usually studied by means of approximate regular problems. Therefore we recall here some results for the auxiliary regular problem

$$\big(\phi(u')\big)'+g(t,u,u')=0,\quad u(0)=u(T)=0,\tag{2.1}$$

where $g\in$ Car $\big([0,T]\times\mathbb{R}^2\big)$.

The first one is the Fredholm type existence theorem well known for problem (2.1) with $\phi(y)\equiv y$, see. e.g. [3]. For readers' convenience we will prove it here for problem (2.1) with a general ϕ.

Theorem 2.1 (Fredholm type existence theorem). *Assume that there is a function $h\in L[0,T]$ such that*

$$|g(t,x,y)|\le h(t)\quad\text{for a.e.}\quad t\in[0,T]\quad\text{and all}\quad x,y\in\mathbb{R}\,.\tag{2.2}$$

Then problem (2.1) *has a solution.*

Proof. *Step 1. Solution of an auxiliary problem.* Consider the auxiliary problem

$$\big(\phi(u')\big)'=b(t),\quad u(0)=u(T)=0,\tag{2.3}$$

where $b\in L[0,T]$. Then u is a solution of problem (2.3) if and only if $u\in C^1[0,T]$ satisfies the equalities

$$u(t)=\int_0^t\phi^{-1}\left(\phi(u'(0))+\int_0^s b(\tau)\,d\tau\right)ds$$

and

$$\int_0^T\phi^{-1}\left(\phi(u'(0))+\int_0^s b(\tau)d\tau\right)ds=0.$$

We can check this by a direct computation.

Step 2. Definition of functional γ. For each $\ell \in C[0,T]$ define

$$\psi_\ell : \mathbb{R} \to \mathbb{R}, \quad \psi_\ell(x) = \int_0^T \phi^{-1}(x + \ell(s))ds.$$

Due to the assumption that ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$, the function ψ_ℓ is continuous, increasing, and $\psi_\ell(\mathbb{R}) = \mathbb{R}$. Thus the equation $\psi_\ell(x) = 0$ has exactly one root $x = \gamma(\ell) \in \mathbb{R}$. Therefore, we can define the functional

$$\gamma : C[0,T] \to \mathbb{R}, \quad \psi_\ell(\gamma(\ell)) = 0.$$

Step 3. Functional γ *maps bounded sets to bounded sets.* Assume that $\mathcal{B} \subset C[0,T]$ and $c \in (0,\infty)$ and such that $||\ell||_C \le c$ for each $\ell \in \mathcal{B}$. Further assume that there exists a sequence $\{\ell_n\} \subset \mathcal{B}$ such that

$$\lim_{n\to\infty} \gamma(\ell_n) = \infty \quad \text{or} \quad \lim_{n\to\infty} \gamma(\ell_n) = -\infty\,.$$

Let the former possibility occur. Then

$$0 = \lim_{n\to\infty} \psi_{\ell_n}(\gamma(\ell_n)) \ge \lim_{n\to\infty} T\phi^{-1}(\gamma(\ell_n) - c) = \infty,$$

a contradiction. The latter possibility can be argued similarly. Thus $\gamma(\mathcal{B})$ is bounded.

Step 4. Functional γ *is continuous.* Consider a sequence $\{\ell_n\} \subset C[0,T]$ and assume that $\lim_{n\to\infty} \ell_n = \ell_0$ in $C[0,T]$.

By Step 3, the sequence $\{\gamma(\ell_n)\} \subset \mathbb{R}$ is bounded and hence we can choose a subsequence such that $\lim_{n\to\infty} \gamma(\ell_{k_n}) = x_0 \in \mathbb{R}$. We get

$$0 = \psi_{\ell_{k_n}}(\gamma(\ell_{k_n})) = \int_0^T \phi^{-1}(\gamma(\ell_{k_n}) + \ell_{k_n}(t))\,dt,$$

which, for $n \to \infty$, yields

$$0 = \int_0^T \phi^{-1}(x_0 + \ell_0(t))\,dt.$$

Thus, with respect to Step 2, we have $x_0 = \gamma(\ell_0)$. It follows that any convergent subsequence of $\{\gamma(\ell_n)\}$ has the same limit $\gamma(\ell_0)$. Since $\{\gamma(\ell_n)\}$ is bounded, we get $\gamma(\ell_0) = \lim_{n\to\infty} \gamma(\ell_n)$.

Step 5. Definition of operator $\mathcal{F}$. Define operators $\mathcal{N} : C^1[0,T] \to C[0,T]$ and $\mathcal{F} : C^1[0,T] \to C^1[0,T]$ by

$$(\mathcal{N}(u))(t) = -\int_0^t g(s, u(s), u'(s))ds$$

and

$$(\mathcal{F}(u))(t) = \int_0^t \phi^{-1}(\gamma(\mathcal{N}(u)) + (\mathcal{N}(u))(s))ds.$$

Step 1 and Step 2 yield that u is a solution of problem (2.1) if and only if $u \in C^1[0,T]$ satisfies

$$u(t) = \int\limits_0^t \phi^{-1}(\phi(u'(0)) + (\mathcal{N}(u))(s))ds, \quad \phi(u'(0)) = \gamma(\mathcal{N}(u)).$$

Therefore the operator equation $u = \mathcal{F}(u)$ is equivalent to problem (2.1). Thus it suffices to prove that the operator $\mathcal{F}$ has a fixed point.

Step 6. Fixed point of operator $\mathcal{F}$. Since the operators γ and $\mathcal{N}$ are continuous, it follows that $\mathcal{F}$ is continuous. Choose an arbitrary sequence $\{u_n\} \subset C^1[0,T]$ and denote $v_n = \mathcal{F}(u_n)$ for $n \in \mathbb{N}$. Then

$$v_n'(t) = \phi^{-1}\left(\gamma\left(\mathcal{N}\left(u_n\right)\right) + \left(\mathcal{N}\left(u_n\right)\right)(t)\right), \quad t \in [0,T], \quad n \in \mathbb{N}.$$

By condition (2.2), there is a $c_1 \in (0,\infty)$ such that $||\mathcal{N}(u_n)||_C \le c_1$. This implies that the sequences $\{v_n\}$ and $\{v_n'\}$ are bounded on $[0,T]$. Consequently the sequence $\{v_n\}$ is equicontinuous on $[0,T]$. Further, for $t_1, t_2 \in [0,T]$,

$$\left|\phi\left(v_n'(t_1)\right) - \phi\left(v_n'(t_2)\right)\right| = \left|\left(\mathcal{N}(u_n)\right)(t_1) - \left(\mathcal{N}(u_n)\right)(t_2)\right| \le \left|\int\limits_{t_1}^{t_2} h(s)ds\right|.$$

Thus the sequence $\{\phi(v_n')\}$ is bounded and equicontinuous on $[0,T]$. Making use of the Arzelà — Ascoli theorem we can find subsequences $\{v_{k_n}\}$ and $\{\phi(v_{k_n}')\}$ uniformly convergent on $[0,T]$. Then $\{v_{k_n}'\}$ is also uniformly convergent on $[0,T]$ and so, $\{v_{k_n}\}$ is convergent in $C^1[0,T]$. We have proved that the operator $\mathcal{F}$ is compact on $C^1[0,T]$. By the Schauder fixed theorem, $\mathcal{F}$ has a fixed point, which is a solution of problem (2.1).

The theorem is proved.

In the investigation of the regular problem (2.1), the lower and upper functions method is a profitable instrument, see. e.g. De Coster, Habets [8], Kiguradze, Shekhter [2] or Vasiljev, Klokov [9]. Note that in some works lower and upper functions are called lower and upper solutions.

Definition 2.1. *A function $\sigma \in C[0,T]$ is called an upper function of problem (2.1) if there exists a finite set $\Sigma \subset (0,T)$ such that*

$$\phi(\sigma') \in AC_{\text{loc}}([0,T] \setminus \Sigma), \quad \sigma'(\tau+) := \lim_{t \to \tau+} \sigma'(t) \in \mathbb{R},$$

$$\sigma'(\tau-) := \lim_{t \to \tau-} \sigma'(t) \in \mathbb{R} \textit{ for each } \tau \in \Sigma,$$

$$\begin{cases} (\phi(\sigma'(t)))' + g(t,\sigma(t),\sigma'(t)) \le 0 \textit{ for a.e. } t \in [0,T], \\ \sigma(0) \ge 0, \quad \sigma(T) \ge 0, \quad \sigma'(\tau-) > \sigma'(\tau+) \textit{ for each } \tau \in \Sigma. \end{cases} \tag{2.4}$$

If the inequalities in (2.4) are reversed, then σ is called a lower function of problem (2.1).

The second auxiliary result is contained in the following theorem.

Theorem 2.2 (Lower and upper functions method). *Let σ_1 and σ_2 be a lower function and an upper function of problem (2.1) and let $\sigma_1(t) \le \sigma_2(t)$ for $t \in [0,T]$. Assume that there exists a function $h \in L[0,T]$ such that*

$$|g(t,x,y)| \le h(t) \text{ for a.e. } t \in [0,T] \text{ and all } x \in [\sigma_1(t), \sigma_2(t)]\,,\ y \in \mathbb{R}.$$

Then problem (2.1) has a solution u such that

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \text{ for } t \in [0,T]. \tag{2.5}$$

Proof. *Step 1. Construction of an auxiliary problem.* For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$, $\epsilon \in [0,1]$, define

$$\tilde g(t,x,y) = \begin{cases} g(t,\sigma_1(t),y) + \omega_1\left(t, \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\right) + \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1} & \text{if} \quad x < \sigma_1(t), \\ g(t,x,y) & \text{if} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ g(t,\sigma_2(t),y) - \omega_2\left(t, \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\right) - \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1} & \text{if} \quad x > \sigma_2(t), \end{cases}$$

where

$$\omega_i(t,\epsilon) = \sup\left\{|g(t,\sigma_i(t),\sigma_i'(t)) - g(t,\sigma_i(t),y)| : \ |y-\sigma_i'(t)| < \epsilon\right\}, \quad i = 1,2.$$

We see that $\omega_i \in \operatorname{Car}([0,T]\times[0,1])$ is nonnegative, nondecreasing in its second variable, and $\omega_i(0,t) = 0$ for a.e. $t \in [0,T]$, $i = 1,2$. Further, $\tilde g \in \operatorname{Car}([0,T]\times\mathbb{R}^2)$ and there exists $\tilde h \in L[0,T]$ such that

$$|\tilde g(t,x,y)| \le \tilde h(t) \text{ for a.e. } t \in [0,T] \text{ and all } x,y \in \mathbb{R}.$$

Thus, by Theorem 2.1, the problem

$$\big(\phi(u')\big)' + \tilde g(t,u,u') = 0, \quad u(0) = u(T) = 0,$$

has a solution u.

Step 2. Solution u of the auxiliary problem lies between σ_1 and σ_2. We will prove that estimate (2.5) holds. Denote $v(t) = u(t) - \sigma_2(t)$ for $t \in [0,T]$ and assume, on the contrary, that $\max\{v(t) : t \in [0,T]\} = v(t_0) > 0$. Since $u(0) = u(T) = 0$ and $\sigma_2(0) \ge 0$, $\sigma_2(T) \ge 0$, we have $t_0 \in (0,T)$. Moreover, Definition 2.1 implies that $t_0 \notin \Sigma$, because $v'(\tau-) < v'(\tau+)$ for $\tau \in \Sigma$. So, we have $t_0 \in (0,T)\setminus\Sigma$ and $v'(t_0) = 0$. This guarantees the existence of $t_1 \in (t_0,T)$ such that

$$v(t) > 0 \text{ and } |v'(t)| < \frac{v(t)}{v(t)+1} < 1$$

for $t \in [t_0,t_1]$ and $[t_0,t_1]\cap\Sigma = \varnothing$. Then

$$\begin{aligned} &(\phi(u'(t)))' - (\phi(\sigma_2'(t)))' = -\tilde g(t,u(t),u'(t)) - (\phi(\sigma_2'(t)))' = \\ &\quad = -g(t,\sigma_2(t),u'(t)) + \omega_2\left(t, \frac{v(t)}{v(t)+1}\right) + \frac{v(t)}{v(t)+1} - (\phi(\sigma_2'(t)))' > \\ &\quad > -g(t,\sigma_2(t),u'(t)) + \omega_2\left(t,|v'(t)|\right) - (\phi(\sigma_2'(t)))' \ge \\ &\quad \ge -g(t,\sigma_2(t),u'(t)) + g(t,\sigma_2(t),u'(t)) - g(t,\sigma_2(t),\sigma_2'(t)) - (\phi(\sigma_2'(t)))' \ge 0 \end{aligned}$$

for a.e. $t \in [t_0, t_1]$. Hence,

$$0 < \int_{t_0}^{t} (\phi(u'(s)))' - (\phi(\sigma_2'(s)))' ds = \phi(u'(t)) - \phi(\sigma_2'(t)), \quad t \in [t_0, t_1].$$

Therefore $v' = u' - \sigma_2' > 0$ on $(t_0, t_1]$, which contradicts the assumption that v has its maximum value at t_0. The inequality $\sigma_1 \le u(t)$ can be proved similarly. Thus u fulfils estimate (2.5) and so, u is a solution of problem (2.1).

The theorem is proved.

3. Existence principle for singular Dirichlet problem. We will use the following approach in the investigation of singular problem (1.1):

we approximate problem (1.1) by a sequence of solvable regular problems;

we find a sequence $\{u_n\}$ of approximate solutions;

we investigate convergence of a suitable subsequence $\{u_{k_n}\}$.

The type of this convergence determines the properties of its limit u and, among others, determines whether u is a w-solution or a solution of the original singular problem (1.1).

There are many ways of constructing an approximate sequence of regular problems. The main properties of such a sequence are given in the next theorem.

We will consider the sequence of regular problems

$$(\phi(u'))' + f_n(t, u, u') = 0, \quad u(0) = u(T) = 0, \tag{3.1}$$

where $f_n \in \operatorname{Car}([0, T] \times \mathbb{R}^2)$, $n \in \mathbb{N}$.

Theorem 3.1 (Existence principle for singular problem). *Let (1.2) hold. Let $\varepsilon_n > 0$, $\eta_n > 0$ for $n \in \mathbb{N}$ and let $\lim_{n\to\infty} \varepsilon_n = 0$, $\lim_{n\to\infty} \eta_n = 0$. Assume that*

$$\begin{gathered} f_n(t, x, y) = f(t, x, y) \text{ for a.e. } t \in \left[\frac{1}{n}, T - \frac{1}{n}\right], \text{ for each } n > \frac{2}{T} \\ \text{and for each } (x, y) \in \mathcal{A}_1 \times \mathcal{A}_2, \quad |x| \ge \varepsilon_n, \quad |y| \ge \eta_n, \end{gathered} \tag{3.2}$$

$$\begin{gathered} \text{there exists a bounded set } \Omega \subset C^1[a, b] \text{ such that} \\ \text{for each } n \ge \frac{2}{T} \text{ the regular problem (3.1) has a solution} \\ u_n \in \Omega \text{ and } (u_n(t), u_n'(t)) \in \mathcal{A}_1 \times \mathcal{A}_2 \text{ for } t \in [0, T]. \end{gathered} \tag{3.3}$$

Then there exist $u \in C[0, T]$ and a subsequence $\{u_k\} \subset \{u_n\}$ such that

$$\lim_{k\to\infty} u_k(t) = u(t) \text{ uniformly on } [0, T]. \tag{3.4}$$

Further assume that there is a finite set $S = \{s_1, \ldots, s_\nu\} \subset (0, T)$ such that

$$\text{on each interval } [a, b] \subset (0, T) \setminus S \text{ the sequence } \{\phi(u_n')\} \text{ is equicontinuous.} \tag{3.5}$$

Then $u \in C^1((0, T) \setminus S)$ and

$$\lim_{k\to\infty} u_k'(t) = u'(t) \text{ locally uniformly on } (0, T) \setminus S. \tag{3.6}$$

Assume, in addition, that the set S has the form

$$S = \{s \in (0,T): \ u(s) = 0 \text{ or } u'(s) = 0 \text{ or } u'(s) \text{ does not exist}\}. \tag{3.7}$$

Then $\phi(u') \in AC_{\mathrm{loc}}((0,T)\setminus S)$ and u is a w-solution of problem (1.1).

Denote $s_0 = 0$ and $s_{\nu+1} = T$. If there exist $\eta \in \left(0, \dfrac{T}{2}\right)$, $\lambda_0, \mu_0, \lambda_1, \mu_1, \ldots, \lambda_{\nu+1}, \mu_{\nu+1} \in \in \{-1,1\}$, $k_0 \in \mathbb{N}$ and $\psi \in L[0,T]$ such that

$$\begin{gathered} \lambda_i f_k\left(t, u_k(t), u_k'(t)\right) \operatorname{sign} u_k'(t) \geq \psi(t) \quad \text{for a.e. } t \in (s_i - \eta, s_i) \cap (0,T), \\ \mu_i f_k(t, u_k(t), u_k'(t)) \operatorname{sign} u_k'(t) \geq \psi(t) \quad \text{for a.e. } t \in (s_i, s_i + \eta) \cap (0,T), \\ \text{for all } \ i \in \{0, \ldots \nu+1\}, \ \ k \in \mathbb{N}, \ \ k \geq k_0, \end{gathered} \tag{3.8}$$

then $\phi(u') \in AC[0,T]$ and u is a solution of problem (1.1). *Moreover,* $(u(t), u'(t)) \in \mathcal{A}_1 \times \mathcal{A}_2$ *for* $t \in [0,T]$.

Proof. By (3.3) there exist $r > 0$ and a sequence $\{u_n\}$ of solutions of (3.1) such that

$$||u_n||_{C^1} \leq r \text{ for each } n \in \mathbb{N}, \ \ n > \frac{2}{T}. \tag{3.9}$$

Therefore the sequence $\{u_n\}$ is bounded in $C[0,T]$ and equicontinuous on $[0,T]$. By Arzelà – Ascoli theorem we can choose a subsequence $\{u_\ell\}$ such that

$$\lim_{\ell \to \infty} ||u_\ell - u||_C = 0, \ \ u \in C[0,T]. \tag{3.10}$$

Now assume also (3.5) and choose an interval $[a,b] \subset (0,T) \setminus S$ arbitrarily. Then $\{\phi(u_\ell')\}$ is equicontinuous on $[a,b]$. By (3.9) the sequence $\{u_\ell'\}$ is bounded in $C[a,b]$. Since ϕ is homeomorphism, the sequence $\{\phi(u_\ell')\}$ is bounded in $C[a,b]$ too. Arzelà – Ascoli theorem implies that we can choose a subsequence $\{\phi(u_k)\} \subset \{\phi(u_\ell)\}$ such that

$$\lim_{k \to \infty} \phi(u_k'(t)) = \phi(u'(t)) \text{ uniformly on } [a,b]$$

and consequently we get

$$\lim_{k \to \infty} u_k'(t) = u'(t) \text{ uniformly on } [a,b].$$

By virtue of (3.10), the sequence $\{u_k\}$ satisfies (3.4). Using the diagonalization method we can choose such $\{u_k\}$ that (3.6) holds, as well. Therefore $u \in C^1((0,T)\setminus S)$.

By (3.4), u satisfies $u(0) = u(T) = 0$.

Let (3.7) be true. Define sets

$$V_1 = \{t \in (0,T): \ f(t,\cdot,\cdot) \ : \ \ \mathcal{D} \to \mathbb{R} \text{ is not continuous}\},$$

$$V_2 = \{t \in (0,T): \ \text{the equality in (3.2) is not satisfied}\},$$

and let

$$U = (0,T) \setminus (S \cup V_1 \cup V_2).$$

We see that

$$\text{meas}(S \cup V_1 \cup V_2) = 0. \tag{3.11}$$

Choose an arbitrary $t \in U$. Then there exists $k_t \in \mathbb{N}$, such that for each $k \in \mathbb{N}$, $k \geq k_t$, we have

$$t \in \left[\frac{1}{k}, T - \frac{1}{k}\right], \ \ |u_k(t)| > \varepsilon_k, \ \ |u_k'(t)| > \eta_k,$$

and

$$f_k(t, u_k(t), u_k'(t)) = f(t, u_k(t), u_k'(t)).$$

Since t is an arbitrary element of U, by (3.4), (3.6) and (3.11) we get

$$\lim_{k\to\infty} f_k(t, u_k(t), u_k'(t)) = f(t, u(t), u'(t)) \text{ a.e. on } [0, T]. \tag{3.12}$$

Now choose an arbitrary interval $[a, b] \subset (0, T) \setminus S$ and integrate the equality

$$-(\phi(u'(t)))' = f_k(t, u(t), u'(t)) \text{ for a.e. } t \in [0, T]. \tag{3.13}$$

We get

$$-\phi(u_k'(t)) + \phi(u_k'(a)) = \int_a^t f_k(s, u_k(s), u_k'(s))ds \text{ for each } t \in [a, b]. \tag{3.14}$$

Moreover there exists $k^* \in \mathbb{N}$, $\varepsilon^* > 0$, $\eta^* > 0$, such that for each $k \in \mathbb{N}$, $k \geq k^*$,

$$|f_k(t, u_k(t), u_k'(t))| \leq m(t) \text{ for a.e. } t \in [a, b],$$

where

$$m(t) = \sup\{|f(t, x, y)| : \varepsilon^* \leq |x| \leq r, \ \ \eta^* \leq |y| \leq r\} \in L[a, b].$$

Since $m \in L[a, b]$ we can apply the Lebesgue convergence theorem on $[a, b]$ and get $f(\cdot, u(\cdot), u'(\cdot)) \in L[a, b]$. Moreover,

$$\lim_{k\to\infty} \int_a^b f_k(s, u_k(s), u_k'(s))ds = \int_a^b f(s, u(s), u'(s))ds,$$

which, by (3.14), yields

$$-\phi(u'(t)) + \phi(u'(a)) = \int_a^t f(s, u(s), u'(s))ds \text{ for each } t \in [a, b]. \tag{3.15}$$

Since $[a, b]$ is an arbitrary interval in $(0, T) \setminus S$, we get that $\phi(u') \in AC_{\text{loc}}((0, T) \setminus S)$ and u is a w-solution of (1.1).

Now assume also that there exist $\eta \in \left(0, \frac{T}{2}\right)$, $\lambda_0, \mu_0, \lambda_1, \mu_1, \ldots, \lambda_{\nu+1}, \mu_{\nu+1} \in \{-1, 1\}$, $k_0 \in \mathbb{N}$ and $\psi \in L[0, T]$ such that (3.8) holds. Since u is a w-solution of (1.1), it remains to prove that $\phi(u') \in AC[0, T]$.

Choose $i \in \{0, \ldots, \nu+1\}$ and denote $(c_i, d_i) = (s_i - \eta, s_i) \cap (0, T)$. For $k \in \mathbb{N}$ and for a.e. $t \in (c_i, d_i) \setminus S$ we denote

$$h_k(t) = \lambda_i f_k(t, u_k(t), u_k'(t)) \operatorname{sign} u_k'(t) + |\psi(t)|,$$

$$h(t) = \lambda_i f(t, u(t), u'(t)) \operatorname{sign} u'(t) + |\psi(t)|.$$

Due to (3.7) we have $u'(t) \neq 0$. Further, $h_k \in L[c_i, d_i]$ and according to (3.6) and (3.12) we have

$$\lim_{k \to \infty} h_k(t) = h(t) \text{ for a.e. } t \in [c_i, d_i].$$

If we multiply (3.13) by $\operatorname{sign} u_k'(t)$ and then integrate over $[c_i, d_i]$ we get, for $k \geq k_0$,

$$\left| \int_{c_i}^{d_i} f_k(s, u_k(s), u_k'(s)) \operatorname{sign} u_k'(s) ds \right| \leq \phi(|u_k'(d_i)|) + \phi(|u_k'(c_i)|).$$

By (3.9) we get that the sequence $\{\phi(u_k')\}$ is bounded. By (3.8)

$$\int_{c_i}^{d_i} |h_k(s)| ds = \int_{c_i}^{d_i} h_k(s) ds \leq \left| \int_{c_i}^{d_i} f_k(s, u_k(s), u_k'(s)) \operatorname{sign} u_k'(s) ds \right| +$$

$$+ \int_{c_i}^{d_i} |\psi(s)| ds \leq \phi(|u_k'(d_i)|) + \phi(|u_k'(c_i)|) + \int_{c_i}^{d_i} |\psi(s)| ds \leq c.$$

Fatou lemma implies that $h \in L[c_i, d_i]$ and $f(\cdot, u(\cdot), u'(\cdot)) \in L[c_i, d_i]$.

If $(c_i, d_i) = (s_i, s_i + \eta) \cap (0, T)$ we argue similarly.

Since $f(\cdot, u(\cdot), u'(\cdot)) \in L[a, b]$ for each $[a, b] \subset (0, T) \setminus S$, we get $f(\cdot, u(\cdot), u'(\cdot)) \in L[0, T]$ and the equality in (3.15) is fulfilled for each $t \in [0, T]$ and $\phi(u') \in AC[0, T]$. We have proved that u is a solution of (1.1).

By (3.3) and (3.4) we have $(u(t), u'(t)) \in \mathcal{A}_1 \times \mathcal{A}_2$ for $t \in [0, T]$.

4. Application of existence principle. Existence principle in Theorem 3.1 is applicable to singular problems where their nonlinearity $f(t, x, y)$ can have singularities in all its variables t, x, y. If f has no singularity at $y = 0$, then we can put $\eta_k = 0$ for $k \in \mathbb{N}$ in Theorem 3.1. Moreover, due to the proof of Theorem 3.1, the set S in (3.7) consists only of the zeros of u. This will be accounted for in the next theorem where we will assume that

$$\begin{gathered} f \in \operatorname{Car}((0, T) \times \mathcal{D}) \text{ can change its sign, } \mathcal{D} = (0, \infty) \times \mathbb{R}, \\ f \text{ has mixed singularities at } t = 0,\ t = T,\ x = 0. \end{gathered} \tag{4.1}$$

Theorem 4.1. *Let (4.1) hold. Let σ_1 and σ_2 be a lower function and an upper function of problem (1.1) and let*

$$0 < \sigma_1(t) \leq \sigma_2(t) \textit{ for } t \in (0, T).$$

Assume that there exist $a_1, a_2 \in [0,T]$, $a_1 < a_2$, $b \in (0,\infty)$, *a nonnegative function* $h \in L[0,T]$, *and a function* $\omega \in C[0,\infty)$ *fulfilling*

$$\int_0^\infty \frac{ds}{\omega(s)} = \infty, \quad \omega(s) \geq b \text{ for } s \in [0,\infty) \tag{4.2}$$

and

$$f(t,x,y)\operatorname{sign} y \leq \omega(|\phi(y)|)(h(t)+|y|)$$

for a.e. $t \in [0,a_2]$ *and all* $x \in [\sigma_1(t),\sigma_2(t)]$, $y \in \mathbb{R}$,

$$f(t,x,y)\operatorname{sign} y \geq -\omega(|\phi(y)|)(h(t)+|y|)$$

for a.e. $t \in [a_1,T]$ *and all* $x \in [\sigma_1(t),\sigma_2(t)]$, $y \in \mathbb{R}$.

Then problem (1.1) has a solution satisfying estimate (2.5).

Remark 4.1. Lower and upper functions of problem (1.1) are understood in the sense of Definition 2.1.

The proof of Theorem 4.1 is based on Theorem 3.1 where the existence of a bounded set $\Omega \subset C^1[0,T]$ is needed. Therefore we first prove an apriori estimate.

Lemma 4.1. *Let* $a_1, a_2 \in [0,T]$, $a_1 < a_2$, $r_0, \kappa \in (0,\infty)$. *Further, let* $h_0 \in L[0,T]$ *be nonnegative and let* ω *be positive and fulfil the condition*

$$\int_0^\infty \frac{ds}{\omega(s)} = \infty. \tag{4.3}$$

Then there exists $r > 0$ *such that for each function* u *satisfying*

$$\phi(u') \in AC[0,T], \quad ||u||_C \leq r_0,$$

$$(\phi(u'(t)))' \operatorname{sign} u'(t) \geq -\kappa\omega(|\phi(u'(t))|)(h_0(t)+|u'(t)|) \text{ for a.e. } t \in [0,a_2], \tag{4.4}$$

$$(\phi(u'(t)))' \operatorname{sign} u'(t) \leq \kappa\omega(|\phi(u'(t))|)(h_0(t)+|u'(t)|) \text{ for a.e. } t \in [a_1,T],$$

the estimate $||u'||_C \leq r$ *is valid.*

Proof. Choose an arbitrary u satisfying condition (4.4). By the Mean Value Theorem we can find $\xi \in (a_1,a_2)$ such that

$$|u'(\xi)| \leq \frac{2r_0}{a_2 - a_1} =: c_0.$$

Put $v(t) = \phi(u'(t))$ for $t \in [0,T]$. Then $|v(\xi)| \leq \phi(c_0)$ and $\operatorname{sign} u'(t) = \operatorname{sign} v(t)$ for $t \in [0,T]$. Condition (4.3) implies that there exists $\rho \in (\phi(c_0),\infty)$ such that

$$\int_{\phi(c_0)}^{\rho} \frac{ds}{\omega(s)} > \kappa(||h_0||_L + 2r_0). \tag{4.5}$$

Assume that $\max\{|v(t)| : t \in [0,\xi]\} = |v(\alpha)| > \rho$. Then $\alpha < \xi$ and there exists $\beta \in (\alpha,\xi]$ such that $|v(\beta)| = \phi(c_0)$, $|v(t)| \geq \phi(c_0)$ for $t \in [\alpha,\beta]$. By the inequality in (4.4), which holds on $[0,a_2]$, we get

$$-\frac{v'(t)\operatorname{sign} v(t)}{\omega(|v(t)|)} \leq \kappa(h_0 + |u'(t)|) \text{ for a.e. } t \in [\alpha,\beta].$$

Integrating this inequality over $[\alpha,\beta]$ and using the substitution $s = |v'(t)|$, we have

$$\int\limits_{\phi(c_0)}^{|v(\alpha)|} \frac{ds}{\omega(s)} \leq \kappa\left(\int\limits_{\alpha}^{\beta} h_0(t)dt + \int\limits_{\alpha}^{\beta} |u'(t)|dt\right). \tag{4.6}$$

Since $|v(t)| = |\phi(u'(t))| \geq \phi(c_0)$ for $t \in [\alpha,\beta]$, we see that u' does not change its sign on $[\alpha,\beta]$ and hence

$$\int\limits_{\alpha}^{\beta} |u'(t)|dt = \left|\int\limits_{\alpha}^{\beta} u'(t)dt\right| \leq 2r_0.$$

So, inequality (4.6) leads to

$$\int\limits_{\phi(c_0)}^{\rho} \frac{ds}{\omega(s)} < \int\limits_{\phi(c_0)}^{|v_(\alpha)|} \frac{ds}{\omega(s)} \leq \kappa(||h_0||_L + 2r_0),$$

which contradicts inequality (4.5). Therefore $|v(\alpha)| \leq \rho$ and we have proved

$$|\phi(u'(t))| \leq \rho \text{ for } t \in [0,\xi].$$

The estimate

$$|\phi(u'(t))| \leq \rho \text{ for } t \in [\xi,T]$$

can be proved similarly using the inequality in (4.4) which holds on $[a_1,T]$.

Hence, we get $||u'||_C \leq r$, if we put $r = \phi^{-1}(\rho)$.

The lemma is proved.

Proof Theorem 4.1. Choose an arbitrary $n \in \mathbb{N}$, $n > \dfrac{2}{T}$ and denote

$$\Delta_n = \left[0, \frac{1}{n}\right) \cup \left(T - \frac{1}{n}, T\right],$$

$$\Delta_{n_1} = \{t \in \Delta_n : \ \sigma_1(t) = \sigma_2(t)\},$$

$$\Delta_{n_2} = \{t \in \Delta_n : \ \sigma_1(t) < \sigma_2(t)\}.$$

Define

$$\alpha(t,x) = \begin{cases} \sigma_1(t) & \text{if} \quad x < \sigma_1(t), \\ x & \text{if} \quad \sigma_1(t) \leq x \leq r_0, \\ r_0 & \text{if} \quad x > r_0, \end{cases}$$

for all $t \in [0,T]$, $x \in \mathbb{R}$ and $r_0 = \max\{||\sigma_1||_C, ||\sigma_2||_C\}$,

$$\beta(y) = \begin{cases} y & \text{if} \quad |y| \le r, \\ r \operatorname{sign} y & \text{if} \quad |y| > r, \end{cases}$$

where $r > \max\{||\sigma_1'||_C, ||\sigma_2'||_C\}$ is a constant by Lemma 4.1 for $\kappa = 1 + \frac{1}{b}$,

$$g_n(t,x) = \begin{cases} (\phi(\sigma_2'(t)))' & \text{if} \quad x > \sigma_2(t), \\ \dfrac{(x-\sigma_1(t))(\phi(\sigma_2'(t)))' + (\sigma_2(t)-x)(\phi(\sigma_1'(t)))'}{\sigma_2(t)-\sigma_1(t)} & \text{if} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ (\phi(\sigma_1'(t))' & \text{if} \quad x < \sigma_1 \end{cases}$$

for a.e. $t \in \Delta_{n_2}$ and all $x \in \mathbb{R}$, and

$$f_n(t,x,y) = \begin{cases} (f(t,\alpha(t,x),\beta(y)) & \text{if} \quad t \in [0,T] \setminus \Delta_n, \\ -(\phi(\sigma_1'(t)))' & \text{if} \quad t \in \Delta_{n_1}, \\ -g_n(t,x) & \text{if} \quad t \in \Delta_{n_2} \end{cases}$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$.

Then $f_n \in \operatorname{Car}([0,T] \times \mathbb{R}^2)$, and f_n satisfies the inequalities

$$\begin{gathered} f_n(t,x,y) \operatorname{sign} y \le \left(1 + \frac{1}{b}\right) \omega(|\phi(y)|)(h_0(t) + |y|) \\ \text{for a.e. } t \in [0,a_2] \text{ and all } x \in [\sigma_1(t), \sigma_2(t)],\ y \in \mathbb{R}, \end{gathered} \tag{4.7}$$

$$\begin{gathered} f_n(t,x,y) \operatorname{sign} y \ge -\left(1 + \frac{1}{b}\right) \omega(|\phi(y)|)(h_0(t) + |y|) \\ \text{for a.e. } t \in [a_1,T] \text{ and all } x \in [\sigma_1(t), \sigma_2(t)],\ y \in \mathbb{R}, \end{gathered} \tag{4.8}$$

where $h_0(t) = h(t) + |(\phi(\sigma_1'(t)))'| + |(\phi(\sigma_2'(t)))'|$. Consider the problem

$$(\phi(u'))' + f_n(t,u,u') = 0, \ \ u(0) = u(T) = 0. \tag{4.9}$$

We see that σ_1 and σ_2 are also lower and upper functions to problem (4.9). Moreover there exists $h_n \in L[0,T]$ such that

$$|f_n(t,x,y)| \le h_n(t) \text{ for a.e. } t \in [0,T].$$

Hence, for each $n \in \mathbb{N}$, $n > \frac{2}{T}$, Theorem 2.2 gives a solution u_n of problem (4.9) satisfying (2.5). Moreover u_n fulfils conditions (4.4) with $\kappa = 1 + \frac{1}{b}$. Therefore, by Lemma 4.1, $||u_n'||_C \le$ $\le r$.

Define

$$\Omega = \left\{x \in C^1[0,T] : \ \sigma_1 \le x \le \sigma_2 \text{ on } [0,T],\ ||x'||_C \le r\right\}.$$

Put $\mathcal{A}_1 = [0, r_0]$, $\mathcal{A}_2 = [-r, r]$, $\varepsilon_n = \max\left\{\sigma_1\left(\frac{1}{n}\right), \sigma_1\left(T - \frac{1}{n}\right)\right\}$, $\eta_n = 0$ for $n \in \mathbb{N}$. Then conditions (3.2) and (3.3) are fulfilled and, by Theorem 3.1, we can find a subsequence $\{u_k\} \subset \subset \{u_n\}$ uniformly converging on $[0,T]$ to a function $u \in C[0,T]$.

Choose $[a,b] \subset (0,T)$. Then there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ we have $[a,b] \subset \subset \left[\frac{1}{k}, T - \frac{1}{k}\right]$ and

$$|f_k(t, u_k(t), u_k'(t))| \leq h(t) \text{ for a.e. } t \in [a,b],$$

where

$$h(t) = \sup\{|f(t,x,y)| : \ r_* \leq x \leq \sigma_2(t), \ |y| \leq r\},$$

and $r_* = \min\{\sigma_1(t) : \ t \in [a,b]\} > 0$. Since $h \in L[a,b]$, we see that the sequence $\{\phi(u_k')\}$ is equicontinuous on $[a,b]$. Since f has not singularities at y, the set $S \subset (0,T)$ consists only of the zeros of u. Since u is positive on $(0,T)$, S is empty and we see that conditions (3.5) and (3.7) hold. Hence, by Theorem 3.1, $\phi(u') \in AC_{\mathrm{loc}}((0,T))$ and u is a w-solution of problem (1.1).

Denote $\omega_0 = \max\{\omega(s) : \ s \in [0, \phi(r)]\}$ and $\psi(t) = -\left(1 + \frac{1}{b}\right)\omega_0(h_0(t) + r)$.

Inequality (4.7) implies that

$$-f_k(t, u_k(t), u_k'(t)) \operatorname{sign} u_k'(t) \geq \psi(t)$$

for a.e. $t \in [0, a_2]$ and all $k \geq k_0$, and similarly inequality (4.8) gives

$$f_k(t, u_k(t), u_k'(t)) \operatorname{sign} u_k'(t) \geq \psi(t)$$

for a.e. $t \in [a_1, T]$ and all $k \geq k_0$.

So, if we put $\nu = 0$, $\mu_0 = -1$, $s_0 = 0$, $s_1 = T$, $\lambda_1 = 1$, $\eta = \min\{a_2, T - a_1\}$, we get inequalities (3.8). Therefore, by Theorem 3.1, $\phi(u') \in AC[0,T]$ and u is a solution of problem (1.1).

The theorem is proved.

Example 4.1. Let $\alpha, \beta \in [1, \infty)$, $a \in \mathbb{R}$, $b \in \left(0, \frac{1}{\sqrt{2}}\right)$, $c \in (0, \infty)$, $d \in \left(\frac{1}{b} - 2b\right)$. Consider problem (1.1) where $\phi(y) \equiv y$ and

$$f(t,x,y) = \left((T-t)^{-\beta} - t^{-\alpha} + a\right)(x - bt(T-t))y + cy^2 - d + \frac{t(T-t)}{x}$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. The first term of f has time singularities at $t = 0$, $t = T$ and the last term of f has a space singularity at $x = 0$.

Let us put

$$\sigma_1(t) = bt(T-t), \ \ \sigma_2(t) \equiv r_2 \geq \frac{T^2}{4}\left(\frac{1}{d} + b\right),$$

$$\omega(s) = (s+1)(c+1), \ \ a_1 = \frac{T}{3}, \ \ a_2 = \frac{T}{2}.$$

If we choose a sufficiently large positive constant K and put $h(t) \equiv K$, we can check that all conditions of Theorem 4.1 are fulfilled. Therefore our problem has a solution u satisfying estimate (2.5).

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