UDC 517 . 9

# OSCILLATION OF ALL SOLUTIONS OF ITERATIVE EQUATIONS
# КОЛИВАННЯ ВСІХ РОЗВ'ЯЗКІВ ІТЕРАЦІЙНИХ РІВНЯНЬ

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*The paper contains sufficient conditions for the oscillation of all solutions of linear functional iterative equations.*

*Наведено достатні умови коливання всіх розв'язків лінійних функціональних ітераційних рівнянь.*

**1. Introduction.** Many authors investigate oscillatory properties of solutions of difference equations (see [1] and the references cited therein) with "advanced" arguments,

$$\Delta y(n) = \sum_{i=0}^{m} p_i(n) y(n+i+1), \tag{1}$$

or with "delayed" arguments,

$$\Delta y(n) = \sum_{j=1}^{l} q_j(n) y(n-j), \tag{2}$$

where $l, m, n \in \aleph = \{1, 2, \ldots\}$, $p_i, q_j : \aleph \to \Re$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, l$, are given functions and the difference operator $\Delta y$ is defined by

$$\Delta y(n) = y(n+1) - y(n).$$

If we "join" equations (1) and (2) we obtain a difference equation with "advanced" and "delayed" arguments,

$$\Delta y(n) = \sum_{i=0}^{m} p_i(n) y(n+i+1) + \sum_{j=1}^{l} q_j(n) y(n-j), \tag{3}$$

where $m, n$ and $p_i, q_j$ are as above.

Some kind of generalization of difference and recurrence equations are iterative functional equations. In this paper we consider iterative functional equation of the form

$$\Delta_g x(t) = \sum_{i=0}^{m} a_i(t) x(g^{i+1}(t)) + \sum_{j=1}^{l} b_j(t) x(g^{-j}(t)), \tag{4}$$

where $t \in \Im$, $\Im$ is an unbounded subset of $\Re_+ = [0, \infty)$, $m \geq 0$, $l \geq 1$. The difference operator $\Delta_g$ is defined by $\Delta_g x(t) = x(g(t)) - x(t)$. The functions $a_i, b_j : \Im \to \Re_+$, $i = 0, 1, \ldots, m$; $j = 1, 2, \ldots, l$, and $g : \Im \to \Im$ are given and $x$ is an unknown real-valued function. By $g^m$ we mean the $m$-th iterate of the function $g$, i.e.,

$$g^0(t) = t, \qquad g^{m+1}(t) = g(g^m(t)), \qquad t \in \Im, \qquad m = 0, 1, \ldots .$$

By $g^{-1}$ we mean the inverse function to $g$ and $g^{-m-1}(t) = g^{-1}(g^{-m}(t))$. In the whole paper the upper indices at the sign of a function will denote iterations. In each instance we have the relation $g^1(t) = g(t)$. We also assume that

$$g(t) \neq t \qquad \text{and} \qquad \lim_{t \to \infty} g(t) = \infty, \qquad t \in \Im. \tag{5}$$

Moreover we assume that $g$ has an inverse function.

By a solution of equation (4) we mean a function $x : \Im \to \Re$ such that $\sup\{|x(s)| : s \in \in \Im_{t_0} = [t_0, \infty) \cap \Im\} > 0$ for any $t_0 \in \Re_+$ and $x$ satisfies (4) on $\Im$.

A solution $x$ of equation (4) is called oscillatory if there exists a sequence of points $\{t_n\}_{n=1}^{\infty}$, $t_n \in \Im$, such that $\lim_{n \to \infty} t_n = \infty$ and $x(t_n)x(t_{n+1}) \leq 0$ for $n = 1, 2, \ldots$. Otherwise it is called nonoscillatory.

As usual we take $\sum\limits_{j=k}^{k-1} a_j = 0$ and $\prod\limits_{j=k}^{k-1} a_j = 1$.

In this paper we investigate oscillatory properties of solutions of equation (4). The same problem for functional equations has been considered in [2–7] and for equations (1) and (2) for example in [1, 8–13]. The aim of this paper is to present new oscillation criteria for equation (4).

Let us observe that in the particular case, i.e., $\Im = \aleph$ and $g(t) = t + 1$ from equation (4) we get equation (3). In the end of this paper we give an application of the obtained results to recurrence equations.

In our considerations the following lemma will be useful.

**Lemma 1.** *Consider the functional inequality*

$$x(g(t)) \geq p(t)x(t) + q(t)x(g^{k+1}(t)) \tag{6}$$

*where $k \geq 1$, $p$, $q : \Im \to \Re_+$, and $g$ satisfies condition (5). If*

$$\liminf_{\Im \ni t \to \infty} \sum_{i=0}^{k-1} q(g^i(t)) \prod_{j=1}^{k} p(g^{i+j}(t)) > \left(\frac{k}{k+1}\right)^{k+1} \tag{7}$$

*or*

$$\limsup_{\Im \ni t \to \infty} \sum_{i=0}^{k} q(g^i(t)) \prod_{j=1}^{k} p(g^{i+j}(t)) \times$$

$$\times \left\{1 + \sum_{l=1}^{i} q(g^{k+l}(t)) \prod_{m=1}^{k} p(g^{k+l+m}(t))\right\} > 1, \tag{8}$$

*then the functional inequality (6) has not positive solutions for large $t \in \Im$.*

Condition (7) comes from [3] and (8) follows from Theorem 2 of [4].
Similarly we have the following lemma (see [5]).

**Lemma 2.** *Consider the functional inequality*

$$x(g^k(t)) \geq p(t)x(g^{k+1}(t)) + q(t)x(t) \tag{9}$$

*where* $k \geq 1$, *and* $p, q$ *and* $g$ *are as previously. If*

$$\liminf_{\Im\ni t\to\infty} \sum_{i=0}^{k-1} q(g^{-i}(t)) \prod_{j=1}^{k} p(g^{-i-j}(t)) > \left(\frac{k}{k+1}\right)^{k+1} \tag{10}$$

*or*

$$\limsup_{\Im\ni t\to\infty} \sum_{i=0}^{k} q(g^{-i}(t)) \prod_{j=1}^{k} p(g^{-i-j}(t)) \times$$

$$\times \left\{1 + \sum_{l=1}^{i} q(g^{-k-l}(t)) \prod_{m=1}^{k} p(g^{-k-l-m}(t))\right\} > 1, \tag{11}$$

*then the functional inequality (9) has not positive solutions for large* $t \in \Im$.
To prove our main results we will also need the following.

**Lemma 3.** *Let, for sufficiently large* $t \in \Im_{t_1}$,

$$\sum_{i=0}^{k-1} q(g^{i}(t)) \prod_{j=1}^{k} p(g^{i+j}(t)) \geq \delta > 0, \qquad \delta < \left(\frac{k}{k+1}\right)^{k+1}. \tag{12}$$

*Then every nonoscillatory solution* $x(t) > 0$, $t \in \Im_{t_2}$, $t_2 \geq t_1$ *of inequality (6) satisfies the following inequality:*

$$p(t)x(t) \geq \delta x(g(t)) \qquad \text{for} \qquad t \in \Im_{t_3}, \qquad t_3 \geq t_2.$$

***Proof.*** Suppose that $x(t) > 0$, $t \in \Im_{t_2}$, is a nonoscillatory solution of inequality (6). Then also in view of assumption (5) imposed on the function $g$ there exists a point $t_3 \geq t_2$ such that $x(g^i(t)) > 0$, $i \in \{1, 2, \ldots, k+1\}$, and $t \in \Im_{t_3}$. Thus from inequality (6) we get

$$x(g(t)) \geq p(t)x(t)$$

which gives, for $i \in \{1, 2, \ldots, k+1\}$,

$$x(g^i(t)) \geq x(t) \prod_{j=0}^{i-1} p(g^j(t))$$

and

$$x(g^{i+k+1}(t)) \geq x(g^{k+1}(t)) \prod_{j=k+1}^{k+i} p(g^j(t)), \qquad i = 1, 2, \ldots, k+1. \tag{13}$$

From (6) we obtain, for $i = 1, 2, \ldots, k+1$ and $t \in \Im_{t_4}$, $t_4 \geq t_3$,

$$x(g^{i+1}(t)) \geq p(g^i(t))x(g^i(t)) + q(g^i(t))x(g^{k+i+1}(t)). \tag{14}$$

Multiplying both sides of this inequality by $\prod_{j=i+1}^{k-1} p(g^j(t))$ and summing up from $i = 0$ to $k-1$ we obtain

$$x(g^k(t)) \geq x(t) \prod_{j=0}^{k-1} p(g^j(t)) + \sum_{i=0}^{k-1} q(g^i(t)) \prod_{j=i+1}^{k-1} p(g^j(t)) x(g^{k+i+1}(t)).$$

Multiplying both sides of above inequality by $p(g^k(t))$ we get

$$p(g^k(t))x(g^k(t)) \geq \sum_{i=0}^{k-1} q(g^i(t)) \prod_{j=i+1}^{k} p(g^j(t)) x(g^{k+i+1}(t))$$

and from (13)

$$p(g^k(t))x(g^k(t)) \geq \sum_{i=0}^{k-1} q(g^i(t)) \prod_{j=i+1}^{k+i} p(g^j(t)) x(g^{k+1}(t)).$$

In view of assumption (12) we have

$$p(g^k(t))x(g^k(t)) \geq \delta x(g^{k+1}(t)).$$

Hence

$$p(t)x(t) \geq \delta x(g(t)).$$

Above inequality concludes the proof.

Similarly we can prove the following lemma.

**Lemma 4.** *Suppose that for sufficiently large $t \in \Im_{t_1}$ inequality*

$$\sum_{i=0}^{k-1} q(g^{-i}(t)) \prod_{j=1}^{k} p(g^{-i-j}(t)) \geq \delta > 0, \qquad \delta < \left(\frac{k}{k+1}\right)^{k+1},$$

*is true. Then every nonoscillatory solution $x(t) > 0$, $t \in \Im_{t_2}$, of inequality (9) satisfies for sufficiently large $t \in \Im_{t_3}$, $t_3 \geq t_2$, the following inequality:*

$$p(t)x(g^{k+1}(t)) \geq \delta x(g^k(t)).$$

**2. Main results.** Notice that if in equation (4) one of the coefficients satisfies $a_k(t) > 1$, $k = 0, 1, \ldots, m$, then equation (4) has only oscillatory solutions.

So, further we will consider equation (4) with the assumption $a_k(t) < 1$, $k = 0, 1, \ldots, m$.

We may observe that equation (4) has the form

$$[1 - a_0(t)]x(g(t)) = a_1(t)x(g^2(t)) + a_2(t)x(g^3(t)) + \ldots + a_m(t)x(g^{m+1}(t)) +$$

$$+ x(t) + b_1(t)x(g^{-1}(t)) + b_2(t)x(g^{-2}(t)) + \ldots + b_l(t)x(g^{-l}(t)),$$

where the coefficients $a_k(t)$, $b_j(t)$ and $l, m$ are as before. Thus,

$$x(g(t)) = \sum_{i=1}^{m} A_i(t)x(g^{i+1}(t)) + \sum_{j=0}^{l} B_j(t)x(g^{-j}(t)), \qquad l \geq 0, \quad m \geq 1, \tag{15}$$

where

$$A_i(t) = \frac{a_i(t)}{1 - a_0(t)} \geq 0, \qquad i = 1, 2, \ldots, m,$$

and

$$B_j(t) = \frac{b_j(t)}{1 - a_0(t)} \geq 0, \quad j = 1, 2, \ldots, l, \quad B_0(t) = \frac{1}{1 - a_0(t)} > 0.$$

Further we will assume that inequalities are satisfied for sufficiently large $t \in \Im$.

Now we present sufficient conditions for all solutions of equation (15) to by oscillatory. Let us start with the following.

**Theorem 1.** *If* $l \leq m$ *and*

$$\liminf_{\Im \ni t \to \infty} \sum_{i=0}^{m-1} Q(g^i(t)) \prod_{j=1}^{m} P(g^{i+j}(t)) > \left(\frac{m}{m+1}\right)^{m+1} \tag{16}$$

*or*

$$\limsup_{\Im \ni t \to \infty} \sum_{i=0}^{m} Q(g^i(t)) \prod_{j=1}^{m} P(g^{i+j}(t)) \times$$

$$\times \left\{ 1 + \sum_{k=1}^{i} Q(g^{k+m}(t)) \prod_{s=1}^{m} P(g^{m+k+s}(t)) \right\} > 1, \tag{17}$$

*where*

$$P(t) = B_0(t) + \sum_{k=1}^{l} B_k(t) A_k(g^{-k-1}(t)) \tag{18}$$

*and*

$$Q(t) = \sum_{k=1}^{m-1} A_k(t)A_{m-k}(g^k(t)) + A_m(t), \tag{19}$$

*then equation (15) possesses only oscillatory solutions.*

***Proof.*** Suppose that $x$ is a nonoscillatory solution of (15) and let $x(t) > 0$. Then, in view of assumption (5) about the function $g$ and positivity of the functions $A_k(t)$ and $B_k(t)$, from equation (15) we have

$$x(g(t)) \geq A_i(t)x(g^{i+1}(t)), \qquad i = 1, 2, \ldots, m. \tag{20}$$

Hence,

$$x(g^{k+1}(t)) \geq A_i(g^k(t))x(g^{k+1+i}(t)) \qquad \text{for} \quad 1 \leq k \leq m.$$

Thus

$$x(g^{k+1}(t)) \geq A_{m-k}(g^k(t))x(g^{m+1}(t)) \qquad \text{for} \quad 0 \leq k \leq m-1. \tag{21}$$

Similarly from inequality (20) we have for $l \leq m$ that

$$x(g^{-k}(t)) \geq A_k(g^{-k-1}(t))x(t) \qquad \text{for} \quad 1 \leq k \leq m. \tag{22}$$

Using now inequalities (21) and (22) in (15) we obtain

$$x(g(t)) \geq \left\{ \sum_{k=1}^{m-1} A_k(t)A_{m-k}(g^k(t)) + A_m(t) \right\} x(g^{m+1}(t)) +$$

$$+ \left\{ B_0(t) + \sum_{k=1}^{l} B_k(t)A_k(g^{-k-1}(t)) \right\} x(t)$$

and

$$x(g(t)) \geq P(t)x(t) + Q(t)x(g^{m+1}(t)). \tag{23}$$

Applying now Lemma 1 to the above inequality, in view of assumptions (16) and (17) we obtain a contradiction to the fact that $x(t)$ is a positive solution of equation (15). Thus the theorem is proved.

***Remark 1.*** From conditions (16) and (17) of Theorem 1 it follows that the coefficients $a_i$, $i = 0, 1, \ldots, m$, make an essential influence on oscillation of solutions of equation (15) the coefficients $b_j$, $j = 1, 2, \ldots, l$. Let us observe that if in equation (4) all coefficients $b_j = 0$ for $j = 1, 2, \ldots, l$ (then in equation (15) $B_j(t) \equiv 0$ for $k = 1, 2, \ldots, l$ and $B_0 \not\equiv 0$ ) conditions (16) and (17) take the following respective forms:

$$\liminf_{\Im \ni t \to \infty} \sum_{i=0}^{m-1} Q(g^i(t)) \prod_{j=1}^{m} B_0(g^{i+j}(t)) > \left( \frac{m}{m+1} \right)^{m+1}$$

or

$$\limsup_{\Im\ni t\to\infty}\sum_{i=0}^{m}Q(g^{i}(t))\prod_{j=1}^{m}B_0(g^{i+j}(t))\times$$

$$\times\left\{1+\sum_{k=1}^{i}Q(g^{k+m}(t))\prod_{s=1}^{m}B_0(g^{m+k+s}(t))\right\}>1,$$

where $Q(t)$ is as before. The above conditions could be satisfied and depend only on the coefficients $a_i$. On the other hand, in case where in equation (4) all the coefficients $a_i \equiv 0$ for $i = 0, 1, \ldots, m$, the left-hand sides of conditions (16) and (17) equal to zero independly of the coefficients $b_j$.

Now we give sufficient conditions for all solutions of equation (15) to be oscillatory which can be applied when inequality $l \le m$ is not satisfied.

**Theorem 2.** *Suppose that* $l \ge m-2$, $m \ge 3$, *and*

$$\liminf_{\Im\ni t\to\infty}\sum_{i=0}^{l}S(g^{-i}(t))\prod_{j=1}^{l+1}R(g^{-i-j}(t)) > \left(\frac{l+1}{l+2}\right)^{l+2} \tag{24}$$

*or*

$$\limsup_{\Im\ni t\to\infty}\sum_{i=0}^{l+1}S(g^{-i}(t))\prod_{j=1}^{l+1}R(g^{-i-j}(t))\times$$

$$\times\left\{1+\sum_{k=1}^{i}S(g^{-k-l-1}(t))\prod_{s=1}^{l+1}R(g^{-k-l-s-1}(t))\right\}>1, \tag{25}$$

*where*

$$R(t) = A_1(g^l(t)) + \sum_{k=2}^{m}A_k(g^l(t))B_{k-2}(g^{l+k}(t)) \tag{26}$$

*and*

$$S(t) = \sum_{k=0}^{l-1}B_k(g^l(t))B_{l-k}(g^{l-k-1}(t)) + B_l(g^l(t)). \tag{27}$$

*Then every solution of equation (15) oscillates.*

***Proof.*** Suppose that $x(t) > 0$ is a nonoscillatory solution of equation (15). Then, similarly as in the proof of Theorem 1, in view of assumption (5) on the function $g$ and positivity of the functions $A_k(t)$ and $B_k(t)$ from equation (15) we obtain

$$x(g(t)) \ge B_i(t)x(g^{-i}(t)), \qquad i = 0, 1, \ldots, l.$$

Hence we get

$$x(g^{-k}(t)) \geq B_{l-k-1}(g^{-k-1}(t))x(g^{-l}(t)), \qquad 0 \leq k \leq l-1, \tag{28}$$

and, for $m \leq l+2$,

$$x(g^{k+1}(t)) \geq B_{k-2}(g^{k}(t))x(g^{2}(t)), \qquad 2 \leq k \leq m. \tag{29}$$

Applying now inequalities (28) and (29) in (15) we have

$$x(g(t)) \geq \left\{ A_1(t) + \sum_{k=2}^{m} A_k(t)B_{k-2}(g^k(t)) \right\} x(g^2(t)) +$$

$$+ \left\{ \sum_{k=0}^{l-1} B_k(t)B_{l-k-1}(g^{-k-1}(t)) + B_l(t) \right\} x(g^{-l}(t))$$

and

$$x(g^{l+1}(t)) \geq R(t)x(g^{l+2}(t)) + S(t)x(t). \tag{30}$$

Thus, in view of (24), (25) and Lemma 2, the above inequality cannot possess a positive solution. We get a contradiction which completes the proof.

***Remark 2.*** Let us observe that Theorems 1 and 2 have "common area", i.e., both could be applied for $l = m-2, \quad l = m-1 \quad \text{and} \quad l = m$. But Theorems 1 and 2 are independent. To prove this, we consider a functional equation of the form

$$x(t+1) = \frac{1}{10}x(t+2) + \frac{1}{t}x(t+3) + [t]^2 x(t+4) + t x(t) + \frac{1}{2[t]^2}x(t-1), \qquad t \geq 2.$$

The above equation has only oscillatory solutions because condition (16) of Theorem 1 is satisfied. However assumption (24) of Theorem 2 is not satisfied. On the other hand, for the functional equation

$$x(t+1) = \frac{1}{[t]^2}x(t+2) + \frac{t}{2}x(t+3) + \frac{1}{3}x(t+4) + \frac{1}{5}x(t) + \frac{t}{2}x(t-1), \qquad t \geq 2,$$

condition (16) of Theorem 1 is not fulfilled but the above equation has only oscillatory solutions because condition (24) of Theorem 2 is true.

We present now conditions for all solutions of equation (15) to be oscillatory. These conditions can be applied in the case where the assumptions of Theorems 1 and 2, respectively, are not satisfied.

**Theorem 3.** *Let $l \leq m$ and*

$$\sum_{i=0}^{m-1} Q(g^i(t)) \prod_{j=1}^{m} P(g^{i+j}(t)) \geq \delta > 0, \qquad \delta < \left(\frac{m}{m+1}\right)^{m+1}, \tag{31}$$

*where $P(t)$ and $Q(t)$ are given by (18) and (19). If*

$$\limsup_{\Im\ni t\to\infty}\sum_{i=0}^{m} Q(g^i(t))\prod_{j=1}^{m} P(g^{i+j}(t))\times$$

$$\times\left\{1+\sum_{k=1}^{i} Q(g^{k+m}(t))\prod_{s=1}^{m} P(g^{m+k+s}(t))\right\} > 1-\delta^{m+1}, \qquad (32)$$

*then every solution of equation (15) is oscillatory.*

***Proof.*** Suppose that $x$ is a nonoscillatory solution of (15) and let $x(t) > 0$. Then, similarly as in the proof of Theorem 1, inequality (23) is true. Hence we have

$$x(g(t)) \geq P(t)x(t)$$

and similarly as in the proof of Lemma 3,

$$x(g^{i+m+1}(t)) \geq x(g^{m+1}(t))\prod_{j=m+1}^{m+i} P(g^j(t)), \qquad i = 1,2,\ldots,m+1. \qquad (33)$$

From (23) for $i \in \{1,2,\ldots,m\}$ we get

$$x(g^{i+1}(t)) \geq P(g^i(t))x(g^i(t)) + Q(g^i(t))x(g^{m+i+1}(t)). \qquad (34)$$

Multiplying both sides of the above inequality by $\prod_{j=i+1}^{m} P(g^j(t))$ and a subsequent summation from $i = 1$ to $m$ we obtain

$$x(g^{m+1}(t)) \geq x(g(t))\prod_{j=1}^{m} P(g^j(t)) + \sum_{i=1}^{m} Q(g^i(t))\prod_{j=i+1}^{m} P(g^j(t))x(g^{m+i+1}(t)).$$

Applying now inequality (23) we get

$$x(g^{m+1}(t)) \geq x(t)\prod_{j=0}^{m} P(g^j(t)) + Q(t)\prod_{j=1}^{m} P(g^j(t))x(g^{m+1}(t))+$$

$$+\sum_{i=1}^{m} Q(g^i(t))\prod_{j=i+1}^{m} P(g^j(t))x(g^{m+i+1}(t)). \qquad (35)$$

From (34) we obtain for $i \in \{1,2,\ldots,m\}$ and $j \in \{0,1,\ldots,m\}$ that

$$x(g^{m+i+1-j}(t)) \geq P(g^{m+i-j}(t))x(g^{m+i-j}(t)) + Q(g^{m+i-j}(t))x(g^{2m+i+1-j}(t))$$

and, in view of (33),

$$x(g^{m+i+1-j}(t)) \geq$$

$$\geq P(g^{m+i-j}(t))x(g^{m+i-j}(t)) + Q(g^{m+i-j}(t)) \prod_{l=m+1}^{2m+i-j} P(g^l(t))x(g^{m+1}(t)). \tag{36}$$

Applying now (36) for $j = 0$ in (35) we have

$$x(g^{m+1}(t)) \geq x(t) \prod_{j=0}^{m} P(g^j(t)) + Q(t) \prod_{j=1}^{m} P(g^j(t))x(g^{m+1}(t)) +$$

$$+ \sum_{i=1}^{m} Q(g^i(t)) \prod_{j=i+1}^{m} P(g^j(t)) \times$$

$$\times \left\{ P(g^{m+i}(t))x(g^{m+i}(t)) + Q(g^{m+i}(t)) \prod_{l=m+1}^{2m+i} P(g^l(t))x(g^{m+1}(t)) \right\}.$$

Hence,

$$x(g^{m+1}(t)) \geq x(t) \prod_{j=0}^{m} P(g^j(t)) +$$

$$+ x(g^{m+1}(t)) \left\{ \sum_{i=0}^{1} Q(g^i(t)) \prod_{j=1}^{m} P(g^{i+j}(t)) + \right.$$

$$\left. + \sum_{i=1}^{m} Q(g^i(t)) Q(g^{m+i}(t)) \prod_{j=1}^{2m} P(g^{i+j}(t)) \right\} +$$

$$+ \sum_{i=2}^{m} Q(g^i(t)) \prod_{j=i+1}^{m} P(g^j(t)) P(g^{m+i}(t)) x(g^{m+i}(t)).$$

Using now (36) for $j = 1$ in the above inequality we obtain

$$x(g^{m+1}(t)) \geq x(t)\prod_{j=0}^{m} P(g^j(t)) + x(g^{m+1}(t))\times$$

$$\times\left\{\sum_{i=0}^{2} Q(g^i(t))\prod_{j=1}^{m} P(g^{i+j}(t)) + \sum_{i=1}^{m} Q(g^i(t))Q(g^{m+i}(t))\prod_{j=1}^{2m} P(g^{i+j}(t)) + \right.$$

$$\left. + \sum_{i=2}^{m} Q(g^i(t))Q(g^{m+i-1}(t))P(g^{m+i}(t))\prod_{j=1}^{2m-1} P(g^{i+j}(t))\right\} +$$

$$+ \sum_{i=3}^{m} Q(g^i(t))\prod_{l=0}^{1} P(g^{m+i-l}(t))\prod_{j=i+1}^{m} P(g^j(t))x(g^{m+i-1}(t)).$$

Finally we have

$$x(g^{m+1}(t)) \geq x(t)\prod_{j=0}^{m} P(g^j(t)) + x(g^{m+1}(t))\times$$

$$\times\left\{\sum_{i=0}^{m} Q(g^i(t))\prod_{j=1}^{m} P(g^{i+j}(t)) + \sum_{i=1}^{m} Q(g^i(t))Q(g^{m+i}(t))\prod_{j=1}^{2m} P(g^{i+j}(t)) + \ldots \right.$$

$$\ldots + \sum_{i=m-1}^{m} Q(g^i(t))Q(g^{i+2}(t))\prod_{j=1}^{m+2} P(g^{i+j}(t))\prod_{s=3}^{m} P(g^{i+s}(t)) +$$

$$\left. + Q(g^m(t))Q(g^{m+1}(t))\prod_{j=1}^{m+1} P(g^{m+j}(t))\prod_{s=2}^{m} P(g^{m+s}(t))\right\}. \tag{37}$$

From assumption (31), in view of Lemma 3, we have that a nonoscillatory solution of (23) satisfies the following inequality:

$$P(t)x(t) \geq \delta x(g(t)).$$

Hence

$$\prod_{j=0}^{m} P(g^j(t))x(t) \geq \delta^{m+1} x(g^{m+1}(t)).$$

Using the above inequality in (37) we obtain

$$x(g^{m+1}(t)) \geq \delta^{m+1} x(g^{m+1}(t)) + x(g^{m+1}(t))\times$$

$$\times \left\{ \sum_{i=0}^{m} Q(g^i(t)) \prod_{j=1}^{m} P(g^{i+j}(t)) + \sum_{i=1}^{m} Q(g^i(t)) Q(g^{m+i}(t)) \prod_{j=1}^{2m} P(g^{i+j}(t)) + \ldots \right.$$

$$\ldots + \sum_{i=m-1}^{m} Q(g^i(t)) Q(g^{i+2}(t)) \prod_{j=1}^{m+2} P(g^{i+j}(t)) \prod_{s=3}^{m} P(g^{i+s}(t)) +$$

$$\left. + \, Q(g^m(t)) Q(g^{m+1}(t)) \prod_{j=1}^{m+1} P(g^{m+j}(t)) \prod_{s=2}^{m} P(g^{m+s}(t)) \right\}.$$

Dividing now the above inequality by $x(g^{m+1}(t))$ we obtain

$$\sum_{i=0}^{m} Q(g^i(t)) \prod_{j=1}^{m} P(g^{i+j}(t))\times$$

$$\times \left\{ 1 + \sum_{k=1}^{i} Q(g^{k+m}(t)) \prod_{s=1}^{m} P(g^{m+k+s}(t)) \right\} \leq 1 - \delta^{m+1}.$$

The last inequality contradicts assumption (32). Thus the proof is complete.

**Theorem 4.** *Let* $l \geq m-2$, $m \geq 3$, *and*

$$\sum_{i=0}^{l} S(g^{-i}(t)) \prod_{j=1}^{l+1} R(g^{-i-j}(t)) \geq \delta > 0, \qquad \delta < \left( \frac{l+1}{l+2} \right)^{l+2}, \tag{38}$$

*where* $R(t)$ *and* $S(t)$ *are given by (26) and (27). If*

$$\limsup_{\Im \ni t \to \infty} \sum_{i=0}^{l+1} S(g^{-i}(t)) \prod_{j=1}^{l+1} R(g^{-i-j}(t))\times$$

$$\times \left\{ 1 + \sum_{k=1}^{i} S(g^{-k-l-1}(t)) \prod_{s=1}^{l+1} R(g^{-k-l-s-1}(t)) \right\} > 1 - \delta^{l+2}, \tag{39}$$

*then equation (15) has only oscillatory solutions.*

***Proof.*** Let $x(t) > 0$ be a nonoscillatory solution of equation (15). Then, as in the proof of Theorem 2, inequality (30) is satisfied. Thus for $i \in \{1, 2, \ldots, l+1\}$ we get

$$x(g^{l+1-i}(t)) \geq R(g^{-i}(t)) x(g^{l+2-i}(t)) + S(g^{-i}(t)) x(g^{-i}(t)).$$

Multiplying both sides of the above inequality by $\prod\limits_{j=i+1}^{l+1} R(g^{-j}(t))$ and then summing from $i = 1$ to $l + 1$ we obtain

$$x(t) \geq x(g^{l+1}(t)) \prod_{j=1}^{l+1} R(g^{-j}(t)) + \sum_{i=1}^{l+1} S(g^{-i}(t)) \prod_{j=i+1}^{l+1} R(g^{-j}(t)) x(g^{-i}(t)).$$

Applying now inequality (30) we get

$$x(t) \geq x(g^{l+2}(t)) \prod_{j=0}^{l+1} R(g^{-j}(t)) + S(t) \prod_{j=1}^{l+1} R(g^{-j}(t)) x(t) +$$

$$+ \sum_{i=1}^{l+1} S(g^{-i}(t)) \prod_{j=i+1}^{l+1} R(g^{-j}(t)) x(g^{-i}(t)).$$

Further in the same way as in the proof of Theorem 3 from the above inequality we have

$$x(t) \geq x(g^{l+2}(t)) \prod_{j=0}^{l+1} R(g^{-j}(t)) +$$

$$+ x(t) \left\{ \sum_{i=0}^{l+1} S(g^{-i}(t)) \prod_{j=1}^{l+1} R(g^{-i-j}(t)) + \right.$$

$$+ \sum_{i=1}^{l+1} S(g^{-i}(t)) S(g^{-l-1-i}(t)) \prod_{j=1}^{2l+2} R(g^{-i-j}(t)) + \ldots$$

$$\ldots + \sum_{i=l}^{l+1} S(g^{-i}(t)) S(g^{-i-2}(t)) \prod_{j=1}^{l+3} R(g^{-i-j}(t)) \prod_{s=0}^{l-2} R(g^{s-l-1-i}(t)) +$$

$$\left. + S(g^{-l-1}(t)) S(g^{-l-2}(t)) \prod_{j=1}^{l+2} R(g^{-l-1-j}(t)) \prod_{s=0}^{l-1} R(g^{s-2l-2}(t)) \right\}. \qquad (40)$$

From assumption (38), in view of Lemma 4, we get

$$R(t) x(g^{l+2}(t)) \geq \delta x(g^{l+1}(t)).$$

Thus

$$\prod_{j=0}^{l+1} R(g^{-j}(t)) x(g^{l+2}(t)) \geq \delta^{l+2} x(t).$$

Applying now the above inequality in (40) and dividing both sides of the obtained inequality by $x(t)$ we obtain

$$S(g^{-i}(t))\prod_{j=1}^{l+1}R(g^{-i-j}(t))\times$$

$$\times\left\{1+\sum_{k=1}^{i}S(g^{-k-l-1}(t))\prod_{s=1}^{l+1}R(g^{-k-l-s-1}(t))\right\}\leq 1-\delta^{l+2}.$$

This contradicts assumption (39). Thus the proof is complete.

**3. Final remarks.** As it was mentioned, functional equations are a generalization of recurrence equations. So, from oscillation criteria given for the functional equations we also obtain sufficient conditions for oscillations of solutions of the recurrence equations. Consider a recurrence equation of the form

$$x(n-1) = \sum_{i=1}^{m}A_i(n)x(n-i-1)+\sum_{j=0}^{l}B_j(n)x(n+j), \qquad l\geq 0, \quad m\geq 1. \tag{41}$$

Applying now the results obtained, for example, in Theorem 3 we obtain the following condition for equation (41). Let $l\leq m$ and

$$\sum_{i=0}^{m-1}Q(n-i)\prod_{j=1}^{m}P(n-i-j)\geq\delta>0, \qquad \delta<\left(\frac{m}{m+1}\right)^{m+1},$$

where

$$P(n) = B_0(n)+\sum_{k=1}^{l}B_k(n)A_k(n+k+1) \tag{42}$$

and

$$Q(n) = \sum_{k=1}^{m-1}A_k(n)A_{m-k}(n-k)+A_m(n). \tag{43}$$

If for $l\leq m$

$$\limsup_{n\to\infty}\sum_{i=0}^{m}Q(n-i)\prod_{j=1}^{m}P(n-i-j)\times$$

$$\times\left\{1+\sum_{k=1}^{i}Q(n-k-m)\prod_{s=1}^{m}P(n-m-k-s)\right\}>1-\delta^{m+1}, \tag{44}$$

then equation (41) possesses only oscillatory solutions.

Conditions similar to the above were presented by Chatzarakis and Stavroulakis [8] and Stavroulakis [13] for a difference equation of the form

$$x(n+1) - x(n) + p(n)x(n-m) = 0, \qquad m > 0, \qquad n = 0, 1, 2, \ldots, \tag{45}$$

where $p : \aleph \to \Re_+ \setminus \{0\}$. From [8] and [13] it follows that if

$$\liminf_{n\to\infty} \sum_{i=1}^{m} p(n-i) = \alpha \le \left(\frac{m}{m+1}\right)^{m+1}$$

and one of the conditions

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p(n-i) > 1 - \frac{\alpha^2}{4}, \tag{46}$$

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p(n-i) > 1 - \alpha^m, \tag{47}$$

or

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p(n-i) > 1 - \frac{\alpha^2}{2(2-\alpha)}, \tag{48}$$

hold, then all solutions of equation (45) oscillate. It was shown in [8] that for any $m$ condition (48) is better than (46) and for $m = 1, 2$ condition (48) implies (47), for $m \ge 4$ condition (47) implies (48) but for $m = 3$ conditions (47) and (48) are independent. Now we show that our condition (44), in many cases, is better than conditions (47) and (48). For $m = 1$ condition (47) is better than (48), so it suffices to prove that condition (44) is better than (47). Let us consider an equation of the form

$$x(n+1) - x(n) + p(n)x(n-1) = 0, \qquad n = 0, 1, 2, \ldots,$$

where

$$p(n) = \frac{4(2 + (-1)^n)}{19} + \frac{1}{n}.$$

For this equation condition (44) is fulfilled but condition (47) is not satisfied because $\alpha = \dfrac{4}{19}$ and

$$\limsup_{n\to\infty} p(n-1) = \frac{12}{19} < 1 - \alpha.$$

Now let $m = 3$. In [8] it was shown that conditions (47) and (48) are independent because, for the difference equation

$$x(n+1) - x(n) + p(n)x(n-3) = 0, \qquad n = 0, 1, 2, \ldots, \tag{49}$$

where

$$p(2n) = \frac{1}{10} \qquad \text{and} \qquad p(2n+1) = \frac{1}{10} + \frac{6731}{10000} \sin^2 \frac{n\pi}{2},$$

condition (47) is satisfied and (48) is not but if, in equation (49),

$$p(2n) = \frac{8}{100} \qquad \text{and} \qquad p(2n+1) = \frac{8}{100} + \frac{744}{1000} \sin^2 \frac{n\pi}{2},$$

then condition (48) is fulfilled and (47) is not. Observe that our condition (44) for equation (49) of the form

$$\sum_{i=0}^{2} p(n-i) \geq \delta, \qquad \delta < \left(\frac{3}{4}\right)^4,$$

and

$$\limsup_{n\to\infty} \sum_{i=0}^{3} p(n-i) \left\{ 1 + \sum_{k=1}^{i} p(n-k-3) \right\} > 1 - \delta^4 \tag{50}$$

is satisfied for both sequences $p(n)$ defined above. But if we take

$$p(2n) = \frac{9}{100} \qquad \text{and} \qquad p(2n+1) = \frac{9}{100} + \frac{632}{1000} \sin^2 \frac{n\pi}{2},$$

then conditions (47) and (48) are not satisfied but condition (50) is true. On the other hand, for $m \geq 4$ it suffices to show that condition (44) is better than (48). For example, the difference equation

$$x(n+1) - x(n) + p(n)x(n-4) = 0, \qquad n = 0, 1, 2, \ldots,$$

where

$$p(5n) = p(5n+1) = p(5n+2) = p(5n+3) = \frac{3}{40} \qquad \text{and} \qquad p(5n+4) = \frac{27}{40},$$

has only oscillatory solutions since condition (44) is satisfied. However condition (48) is not fulfilled.

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*Received 05.01.2007*