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IMPLICIT DIFFERENCE METHODS FOR PARABOLIC FUNCTIONAL DIFFERENTIAL PROBLEMS OF THE NEUMANN TYPE

НЕЯВНІ РІЗНИЦЕВІ МЕТОДИ ДЛЯ ПАРАБОЛІЧНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ ЗАДАЧ НЕЙМАНІВСЬКОГО ТИПУ

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Nonlinear parabolic functional differential equations with initial boundary conditions of the Neumann type are considered. A general class of difference methods for the problem is constructed. Theorems on the convergence of difference schemes and error estimates of approximate solutions are presented. The proof of the stability of the difference functional problem is based on a comparison technique. Nonlinear estimates of the Perron type with respect to the functional variable for given functions are used. Numerical examples are given.

Розглянуто нелінійні параболічні функціонально-диференціальні рівняння з початковими граничними умовами нейманівського типу. Побудовано загальний клас різницевих методів для розв'язку задачі. Доведено теореми про збіжність різницевих схем та встановлено оцінки похибок наближених розв'язків. Доведення стійкості різницевої функціональної задачі базується на техніці порівняння. Використано нелінійні оцінки перронівського типу відносно функціональної змінної для фіксованої функції. Наведено числові приклади.

1. Introduction. For any two metric spaces X and Y we denote by $\mathbf{C}(X,Y)$ the class of all continuous functions defined on X and taking values in Y. Let $M[n]$ denote the set of all $n \times n$ real matrices. We will use vector inequalities, understanding that the same inequalities hold between their corresponding components. Let $E = [0,a] \times [-b,b]$, where $a > 0$, $b = (b_1, \ldots, b_n)$, $b_i > 0$ for $1 \le i \le n$, and

$$\partial_0 E = [0,a] \times ([-b,b] \setminus (-b,b)).$$

Write $\Sigma = E \times \mathbf{C}(E,\mathbf{R}) \times \mathbf{R}^n \times M[n]$ and

$$\partial_0 E_j = \{(t,x) \in \partial_0 E : \ x_j = b_j\} \cup \{(t,x) \in \partial_0 E : \ x_j = -b_j\}, \quad 1 \le j \le n,$$

and suppose that

$$f : \Sigma \to \mathbf{R}, \quad \varphi : [-b,b] \to \mathbf{R}, \quad \varphi_j : \partial_0 E_j \to \mathbf{R}, \quad 1 \le j \le n,$$

are given functions. We consider the problem consisting of the functional differential equation

$$\partial_t z(t,x) = f(t,x,z,\partial_x z(t,x), \partial_{xx} z(t,x)) \tag{1}$$

with the initial boundary condition of Neumann type,

$$z(0,x) = \varphi(x) \quad \text{for} \quad x \in [-b,b], \tag{2}$$

$$\partial_{x_j} z(t,x) = \varphi_j(t,x) \quad \text{for} \quad (t,x) \in \partial_0 E_j, \quad 1 \le j \le n, \tag{3}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, and $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\ldots,n}$.

For $t \in [0,a]$ we write $E_t = [0,t] \times [-b,b]$. The function f is said to satisfy the Volterra condition if for each $(t,x,q,s) \in E \times \mathbf{R}^n \times M[n]$ and $z, \bar{z} \in C(E,\mathbf{R})$ such that $z(\tau,y) = \bar{z}(\tau,y)$ for $(\tau,y) \in E_t$ we have $f(t,x,z,q,s) = f(t,x,\bar{z},q,s)$. Note that the Volterra condition means that the value of f at the point (t,x,z,q,s) of the space Σ depends on (t,x,q,s) and on the restriction of z to the set E_t.

Our purpose is to investigate a numerical method for the approximation of classical solutions to problem (1) – (3) assuming that f satisfies the Volterra condition. We wish to approximate these classical solutions with solutions of associated implicit difference functional equations and to estimate the difference between these solutions.

In recent years a number of papers concerned with numerical methods for parabolic differential or functional differential equations were published.

Difference methods for nonlinear parabolic problems have the following property. It is easy to construct an explicit Euler's type difference scheme which satisfies consistency conditions on all classical solutions of the original problem. The main task in these considerations is to find a finite difference scheme which is stable. The method of difference inequalities or simple theorems on recurrent inequalities are used in the investigations of the stability. The convergence results were also based on a general theorem on the error estimate of numerical solutions for functional difference equations of the Volterra type with unknown functions of several variables.

Finite difference approximations of the initial boundary-value problems for parabolic differential or functional equations were considered by many authors under various assumptions. Difference methods for nonlinear parabolic differential equations with initial boundary conditions of the Dirichlet type were considered in [1 – 3]. Numerical treatment of the Cauchy problem can be found in [4 – 7].

The paper [8] is concerned with initial boundary-value problems of the Neumann type.

Difference methods for nonlinear parabolic equations with nonlinear boundary condition are investigated in [9 – 12].

The papers [13 – 16] initiated the theory of implicit difference methods for nonlinear parabolic differential equations. Classical solutions of initial boundary-value problems of the Dirichlet type for nonlinear equations without mixed derivatives are approximated in [14, 15] by solutions of difference schemes which are implicit with respect to time variable. The paper [16] deals with initial boundary-value problems of the Neumann type for nonlinear equations with mixed derivatives. The proofs of the convergence of implicit difference schemes are based on the method of difference inequalities. It is assumed that given functions have partial derivatives with respect to all variables except for (t,x). Our assumptions are more general. In the paper we introduce nonlinear estimates of the Perron type with respect to the functional variable. Note that our theorems are new also in the case of parabolic equations without a functional variable.

The paper is organized as follows. In Section 2 we not only set up the notation and terminology, but we construct a class of difference schemes for (1) – (3) as well. The existence and uniqueness of implicit difference schemes, which are not obvious in contrary to the explicit schemes, are proved in Section 3. The third section is also devoted to the study of error estimates for approximate solutions of implicit difference functional problems. The main part of the paper, Section 4, deals with the convergence of a difference method for (1) – (3). Finally the numerical examples are presented in the last part of the paper.

Natural specification of given operator allows to apply the results of this paper to differential equations with deviated variables and integral-differential problems.

2. Discretization of mixed problems. We will denote by $\mathbf{F}(X, Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Let $\mathbf{N}$ and $\mathbf{Z}$ denote the set of natural numbers and the set of integers, respectively. For $x, y \in \mathbf{R}^n$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, we write $\|x\| = |x_1| + \ldots + |x_n|$ and $x * y = (x_1 y_1, \ldots, x_n y_n)$. We formulate now a difference problem corresponding to (1) – (3). We define a mesh on E in the following way. Let (h_0, h') where $h' = (h_1, \ldots, h_n)$ stand for steps of the mesh. For $h = (h_0, h')$ and $(r, m) \in \mathbf{Z}^{1+n}$ where $m = (m_1, \ldots, m_n)$ we define nodal points as follows

$$t^{(r)} = r h_0, \quad x^{(m)} = m * h', \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}).$$

Let us denote by H the set of all $h = (h_0, h')$ such that there exist $(N_1, \ldots, N_n) = N \in \mathbf{N}^n$ satisfying the condition $N * h' = b$. We write $\|h\| = h_0 + h_1 + \ldots + h_n$. Let $N_0 \in \mathbf{N}$ be defined by the relation $N_0 h_0 \le a < (N_0 + 1) h_0$. For $h \in H$ we put

$$\mathbf{R}_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbf{Z}^{1+n}\}$$

and

$$E_h = E \cap \mathbf{R}_h^{1+n}, \quad \partial_0 E_h = \partial_0 E \cap \mathbf{R}_h^{1+n},$$

$$\partial_0 E_{h.j} = \partial_0 E_j \cap \mathbf{R}_h^{1+n}, \quad j = 1, \ldots, n,$$

$$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : 0 \le r \le N_0 - 1\},$$

$$\Sigma_h = E'_h \times F(E_h, \mathbf{R}) \times \mathbf{R}^n \times M[n].$$

Put $E_{h.r} = E \cap \left([0, t^{(r)}] \times \mathbf{R}^n\right)$, where $0 \le r \le N_0$, and

$$\|z\|_{h.r} = \max\{|z^{(\tilde{r}, m)}| : (t^{(\tilde{r})}, x^{(m)}) \in E_{h.r}\}, \quad 0 \le r \le N_0.$$

Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{R}^n$ be the vector with 1 in the i-th position. Write

$$J = \{(i, j) : 1 \le i, j \le n, \ i \ne j\}$$

and suppose that we have defined the sets $J_+, J_- \in J$ such that $J_+ \cup J_- = J$, $J_+ \cap J_- = \varnothing$ (in particular, it may happen that $J_+ = \varnothing$ or $J_- = \varnothing$). We assume that $(i, j) \in J_+$ when $(j, i) \in J_+$.

For each $m \in \mathbf{Z}^n$ such that $x^{(m)} \in [-b, b] \backslash (-b, b)$ we consider the class of $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n$ satisfying the conditions:

i) $\|\alpha\| = 1$ or $\|\alpha\| = 2$,

ii) if $m = (m_1, \ldots, m_n)$ and there is j, $1 \le j \le n$, such that $m_j = N_j$ then $\alpha_j \in \{0, 1\}$,

iii) if $m = (m_1, \ldots, m_n)$ and there is j, $1 \le j \le n$, such that $m_j = -N_j$ then $\alpha_j \in \{-1, 0\}$.

The set of all $\alpha \in \mathbf{Z}^n$ satisfying the above conditions will be denoted by $A^{(m)}$. Let us define the following sets:

$$\partial E_h^+ = \{(t^{(r)}, x^{(m+\alpha)}) : (t^{(r)}, x^{(m)}) \in \partial_0 E_h \quad \text{and} \quad \alpha \in A^{(m)}\},$$

$$E_h^+ = \partial E_h^+ \cup E_h.$$

Let $z : E_h^+ \to \mathbf{R}$ and $-N \le m \le N$. We define

$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i}\left(z^{(r,m+e_i)} - z^{(r,m)}\right), \quad \delta_i^- z^{(r,m)} = \frac{1}{h_i}\left(z^{(r,m)} - z^{(r,m-e_i)}\right),$$

where $1 \le i \le n$. We apply the difference operators δ_0, and the operators

$$\delta = (\delta_1, \ldots, \delta_n), \quad \delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$$

given by

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\left(z^{(r+1,m)} - z^{(r,m)}\right), \tag{4}$$

$$\delta_i z^{(r,m)} = \frac{1}{2}\left(\delta_i^+ z^{(r,m)} + \delta_i^- z^{(r,m)}\right), \quad 1 \le i \le n. \tag{5}$$

The difference operators of the second order δ_{ij}, $i, j = 1, \ldots, n$, are defined in the following way:

$$\delta_{ii} z^{(r,m)} = \delta_i^+ \delta_i^- z^{(r,m)}, \quad 1 \le i \le n, \tag{6}$$

and

$$\delta_{ij} z^{(r,m)} = \frac{1}{2}\left(\delta_i^+ \delta_j^- z^{(r,m)} + \delta_i^- \delta_j^+ z^{(r,m)}\right), \quad (i,j) \in J_-, \tag{7}$$

$$\delta_{ij} z^{(r,m)} = \frac{1}{2}\left(\delta_i^+ \delta_j^+ z^{(r,m)} + \delta_i^- \delta_j^- z^{(r,m)}\right), \quad (i,j) \in J_+. \tag{8}$$

Suppose that the functions

$$f_h : \Sigma_h \to \mathbf{R}, \quad \varphi_h : [-b, b] \to \mathbf{R}, \quad \varphi_{h.j} : \partial_0 E_{h.j} \to \mathbf{R}, \quad 1 \le j \le n,$$

are given. We consider the difference equations

$$\delta_0 z^{(r,m)} = f_h(t^{(r)}, x^{(m)}, z, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}), \quad -N \le m \le N, \tag{9}$$

$$z(t^{(r)}, x^{(m+\alpha)}) = z(t^{(r)}, x^{(m-\alpha)}) + 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{h.j}(t^{(r)}, x^{(m)}) \quad \text{on} \quad \partial_0 E_h, \quad \alpha \in A^{(m)}, \quad (10)$$

with the initial condition

$$z^{(0,m)} = \varphi_h^{(m)} \quad \text{for} \quad x^{(m)} \in [-b, b]. \quad (11)$$

The function f_h is said to satisfy the Volterra condition if for each $(t^{(r)}, x^{(m)}, q, s) \in \Sigma'_h \times \times \mathbf{R}^n \times M[n]$ and $z, \bar{z} \in F(E_h, \mathbf{R})$ such that $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_{h.r}$ we have

$$f_h\left(t^{(r)}, x^{(m)}, z, q, s\right) = f_h\left(t^{(r)}, x^{(m)}, \bar{z}, q, s\right).$$

The difference functional problem (9) – (11) with δ_0, δ, $\delta^{(2)}$ defined by (4) – (8) is considered as an implicit difference method for (1) – (3). It is important in our considerations that the difference expressions δz and $\delta^{(2)} z$ appear in (9) at the point $(t^{(r+1)}, x^{(m)})$. The corresponding explicit difference scheme consist of (10), (11) and the equation

$$\delta_0 z^{(r,m)} = f_h(t^{(r)}, x^{(m)}, z, \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)}), \quad -N \le m \le N. \quad (12)$$

We assume that f_h satisfies the Volterra condition. It is clear that there exists exactly one solution of problem (10) – (12). We prove that under natural assumptions on given functions there exists exactly one solution $u_h : E_h^+ \to \mathbf{R}$ of the implicit difference problem (9) – (11).

3. Approximate solutions of difference functional problems. We will denote by F_h the Niemycki operator corresponding to (9), i.e.,

$$F_h[z]^{(r,m)} = f_h\left(t^{(r)}, x^{(m)}, z, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}\right), \quad (t^{(r)}, x^{(m)}) \in E'_h.$$

Assumption $H[f_h]$. The function $f_h : \Sigma_h \to \mathbf{R}$ of variables (t, x, w, q, s), where

$$q = (q_1, \ldots, q_n), \quad s = [s_{ij}]_{i,j=1,\ldots,n},$$

satisfies the conditions

1) $f_h(t, x, z, \cdot) \in \mathbf{C}(\mathbf{R}^n \times M[n], \mathbf{R})$ and the derivatives

$$\partial_q f_h = (\partial_{q_1} f_h, \ldots, \partial_{q_n} f_h), \quad \partial_s f_h = [\partial_{s_{ij}} f_h]_{i,j=1,\ldots,n},$$

exist on Σ_h and

$$\partial_q f_h(t, x, z, \cdot) \in C(\mathbf{R}^n \times M[n], \mathbf{R}^n), \quad \partial_s f_h(t, x, z, \cdot) \in C(\mathbf{R}^n \times M[n], M[n])$$

for each $(t, x, z) \in E'_h \times \mathbf{F}(E_h, \mathbf{R})$;

2) the functions $\partial_q f_h : \Sigma_h \to \mathbf{R}^n$, $\partial_s f_h : \Sigma_h \to M[n]$ are bounded;

3) the matrix $\partial_s f_h$ is symmetric and

$$\partial_{s_{ij}} f_h(P) \ge 0 \quad \text{for} \quad (i,j) \in J_+, \quad \partial_{s_{ij}} f_h(P) \le 0 \quad \text{for} \quad (i,j) \in J_-, \quad (13)$$

$$-\frac{1}{2}|\partial_{q_i} f_h(P)| + \frac{1}{h_i}\partial_{s_{ii}} f_h(P) - \sum_{j=1, j\neq i}^{n} \frac{1}{h_j}|\partial_{s_{ij}} f_h(P)| \geq 0, \quad 1 \leq i \leq n, \tag{14}$$

where $P = (x, y, z, q, s) \in \Sigma$.

Remark 1. It is assumed in condition $H[f_h]$ 3) that the functions

$$g_{h.ij} = \operatorname{sign} \partial_{s_{ij}} f_h, \quad (i,j) \in J,$$

are constant on Σ_h. Relations (13) can be considered as definitions of the sets J_+ and J_-.

Remark 2. Suppose that
(i) conditions 1), 2) of Assumption $H[f_h]$ are satisfied;
(ii) there is $\tilde{p} > 0$ such that

$$\partial_{s_{ii}} f_h(P) - \sum_{\substack{j=1\\ j\neq i}}^{n} \left|\partial_{s_{ij}} f_h(P)\right| \geq \tilde{p}, \quad i = 1, \ldots, n, \tag{15}$$

where $P = (t, x, w, q, s) \in \Sigma$.

Then there is $\tilde{\varepsilon} > 0$ such that for $\|h'\| < \tilde{\varepsilon}$ assumption (14) is satisfied.

It is also worth noting that condition (15) implies that the function f_h is parabolic in the sense of Walter, i.e.,

$$\text{if } \bar{s}, \tilde{s} \in M_{n\times n} \quad \text{and} \quad \bar{s} \leq \tilde{s} \quad \text{then} \quad \sum_{i,j=1}^{n} \partial_{s_{ij}} f_h(P)(\tilde{s}_{ij} - \bar{s}_{ij})\xi_i\xi_j > 0,$$

$$\text{for} \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n, \ \xi \neq 0.$$

We first prove a lemma on existence and uniqueness of a solution for problem (9) – (11). The proof is based on the Banach fixed point theorem.

Lemma 1. *If assumption $H[f_h]$ is satisfied and $\varphi_h : [-b, b] \to \mathbf{R}$, $\varphi_{h.j} : \partial_0 E_{h.j} \to \mathbf{R}$, $j = 1, \ldots, n$, then there is exactly one solution $u_h : E_h^+ \to \mathbf{R}$ of problem (9) – (11).*

Proof. Suppose that $0 \leq r \leq N_0 - 1$ is fixed and that the solution of (9) – (11) is defined on $E_h^+ \cap \left([0, t^{(r)}] \times \mathbf{R}^n\right)$. We prove that the numbers $u_h^{(r+1,m)}$, where $(t^{(r+1)}, x^{(m)}) \in E_h^+$, exist and that they are unique. There is $Q_h > 0$ such that

$$Q_h \geq 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2}\partial_{s_{ii}} f_h(P) - h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j}|\partial_{s_{ij}} f_h(P)|, \quad P \in \Sigma_h. \tag{16}$$

Then equation (9) is equivalent to the system of equations

$$z^{(r+1,m)} = \frac{1}{Q_h + 1}\left[Q_h z^{(r+1,m)} + u_h^{(r,m)} + h_0 f_h(t^{(r)}, x^{(m)}, u_h, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)})\right], \tag{17}$$

where $-N \le m \le N$, and

$$z(t^{(r+1)}, x^{(m+\alpha)}) = z(t^{(r+1)}, x^{(m-\alpha)}) + 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{h.j}(t^{(r+1)}, x^{(m)}), \tag{18}$$

where $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_h$, $\alpha \in A^{(m)}$, with the initial condition

$$z^{(0,m)} = \varphi_h^{(m)} \quad for \quad x^{(m)} \in [-b, b] \tag{19}$$

and $z^{(r+1,m)}$ are unknown. Write

$$S_h = \left\{ x^{(m)} : (t^{(r+1)}, x^{(m)}) \in E_h^+ \right\}.$$

We consider the space $\mathbf{F}(S_h, \mathbf{R})$. Elements of $\mathbf{F}(S_h, \mathbf{R})$ are denoted by $\xi, \bar{\xi}$. For $\xi \in \mathbf{F}(S_h, \mathbf{R})$ we write $\xi^{(m)} = \xi(x^{(m)})$ and

$$\delta\xi^{(m)} = (\delta_1\xi^{(m)}, \dots, \delta_n\xi^{(m)}), \quad \delta^{(2)}\xi^{(m)} = \left[\delta_{ij}\xi^{(m)}\right]_{i,j=1,\dots,n},$$

where δ_i and δ_{ij}, $1 \le i, j \le n$, are defined by (5)–(8). The norm in the space $\mathbf{F}(S_h, \mathbf{R})$ is defined by

$$\|\xi\|_* = \max\{|\xi^{(m)}| : x^{(m)} \in S_h\}.$$

Set

$$X_h = \{\xi \in \mathbf{F}(S_h, \mathbf{R}) : \xi^{(m+\alpha)} = \xi^{(m-\alpha)} + 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{h.j}(t^{(r+1)}, x^{(m)}) \text{ on } \partial_0 E_h, \ \alpha \in A^{(m)}\}.$$

Let $W_{r.h} : X_h \to X_h$ be the operator defined by

$$W_{r.h}[\xi]^{(m)} = \frac{1}{Q_h + 1}\left[Q_h\xi^{(m)} + u_h^{(r,m)} + h_0 f_h(t^{(r)}, x^{(m)}, u_h, \delta\xi^{(m)}, \delta^{(2)}\xi^{(m)})\right],$$

where $-N \le m \le N$ and

$$W_{r.h}[\xi]^{(m+\alpha)} = W_{r.h}[\xi]^{(m-\alpha)} + 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{h.j}(t^{(r+1)}, x^{(m)}) \text{ on } \partial_0 E_h, \ \alpha \in A^{(m)}. \tag{20}$$

We prove that for $\xi, \bar{\xi} \in X_h$ we have

$$\|W_{r.h}[\xi] - W_{r.h}[\bar{\xi}]\|_* \le \frac{Q_h}{1 + Q_h}\|\xi - \bar{\xi}\|_*. \tag{21}$$

Write

$$A_{i.+}(Q) = \frac{h_0}{2h_i}\partial_{q_i} f_h(Q) + \frac{h_0}{h_i^2}\partial_{s_{ii}} f_h(Q) - \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{h_0}{h_i h_j}|\partial_{s_{ij}} f_h(Q)|,$$

$$A_{i.-}(Q) = -\frac{h_0}{2h_i}\partial_{q_i} f_h(Q) + \frac{h_0}{h_i^2}\partial_{s_{ii}} f_h(Q) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_0}{h_i h_j}|\partial_{s_{ij}} f_h(Q)|,$$

$$B(Q) = -2\sum_{i=1}^{n} \frac{h_0}{h_i^2}\partial_{s_{ii}} f_h(Q) + \sum_{\substack{i,j=1\\ j\neq i}}^{n} \frac{h_0}{h_i h_j}|\partial_{s_{ij}} f_h(Q)|,$$

where $Q \in \Sigma_h$ and $1 \leq i \leq n$.

It follows from assumption $H[f_h]$ that for each m, $-N \leq m \leq N$, there is $P^{(r,m)} \in \Sigma_h$ such that

$$|W_{r.h}[\xi]^{(m)} - W_{r.h}[\bar{\xi}]^{(m)}|(Q_h + 1) \leq |(Q_h + B(P^{(r,m)}))(\xi - \bar{\xi})^{(m)}| +$$

$$+ \left|\sum_{i=1}^{n} A_{i.+}(P^{(r,m)})(\xi - \bar{\xi})^{(m+e_i)}\right| + \left|\sum_{i=1}^{n} A_{i.-}(P^{(r,m)})(\xi - \bar{\xi})^{(m-e_i)}\right| +$$

$$+ h_0 \sum_{(i,j)\in J_+}^{n} \frac{1}{2h_i h_j}\partial_{s_{ij}} f_h(Q)\Big[|(\xi - \bar{\xi})^{(m+e_i+e_j)}| + |(\xi - \bar{\xi})^{(m-e_i-e_j)}\Big] -$$

$$- h_0 \sum_{(i,j)\in J_-}^{n} \frac{1}{2h_i h_j}\partial_{s_{ij}} f_h(Q)\Big[|(\xi - \bar{\xi})^{(m+e_i-e_j)}| + |(\xi - \bar{\xi})^{(m-e_i+e_j)}|\Big].$$

It follows from assumption $H[f_h]$ and from (16) that

$$Q_h + B(P^{(r,m)}) \geq 0,\ A_{i.+}(P^{(r,m)}) \geq 0,\ A_{i.-}(P^{(r,m)}) \geq 0,\ 1 \leq i \leq n,$$

and

$$B(P^{(r,m)}) + \sum_{i=1}^{n} A_{i.+}(P^{(r,m)}) + \sum_{i=1}^{n} A_{i.-}(P^{(r,m)}) +$$

$$+ h_0 \sum_{(i,j)\in J_+}^{n} \frac{1}{2h_i h_j}\partial_{s_{ij}} f_h(Q) - h_0 \sum_{(i,j)\in J_-}^{n} \frac{1}{2h_i h_j}\partial_{s_{ij}} f_h(Q) = 0.$$

Thus we get

$$\Big|W_{r.h}[\xi]^{(m)} - W_{r.h}[\xi]^{(m)}\Big| \leq \frac{Q_h}{Q_h + 1}\|(\xi - \bar{\xi})\|_*, \quad -N \leq m \leq N.$$

We conclude from (20) that the above inequality is satisfied for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_h, \alpha \in A^{(m)}$. This completes the proof of (21). The Banach fixed point theorem implies that there exists exactly one solution of (17) – (19). Since u_h is given on the initial set $\{0\} \times [-b, b]$, the proof of the lemma is completed by induction with respect to r, $0 \leq r \leq N_0$.

Let us suppose that $u_h : E_h^+ \to \mathbf{R}$ is a solution of problem (9) – (11) and $v_h : E_h^+ \to \mathbf{R}$ satisfies the following conditions:

$$|\delta_0 v_h^{(r,m)} - F_h[v_h]^{(r,m)}| \leq \gamma(h) \text{ on } E_h', \tag{22}$$

$$|v_h^{(r,m+\alpha)} - v_h^{(r,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j.h}^{(r,m)}| \leq \gamma_1(h)\|h'\|^2 \text{ on } \partial_0 E_h,\ \alpha \in A^{(m)}, \tag{23}$$

$$|(v_h^{(0,m)} - \varphi_h^{(m)})| \leq \gamma_0(h), \quad x^{(m)} \in [-b, b], \tag{24}$$

where $\gamma, \gamma_0, \gamma_1 : H \to \mathbf{R}_+$ and

$$\lim_{h\to 0} \gamma(h) = 0, \quad \lim_{h\to 0} \gamma_0(h) = 0, \quad \lim_{h\to 0} \gamma_1(h) = 0. \tag{25}$$

The function v_h satisfying the above relations is considered as an approximate solution of problem (9) – (11). We prove a theorem on an estimate of the difference between the exact and approximate solutions of (9) – (11). Put

$$I_h = \{t^{(r)} : 0 \leq r \leq N_0)\}, \quad I_h' = I_h \setminus \{t^{(N_0)}\}.$$

For a function $\eta : I_h \to \mathbf{R}$ we write $\eta^{(r)} = \eta(t^{(r)})$.

Assumption $H[f_h, \sigma_h]$. Assumption $H[f_h]$ is satisfied and there is a function $\sigma_h : I_h' \times \mathbf{R}_+ \to \mathbf{R}_+$ is such that

1) σ_h is nondecreasing with respect to the second variable and $\sigma_h(t, 0) = 0$ for $t \in I_h'$;

2) the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \sigma_h(t^{(r)}, \eta^{(r)}), \quad 0 \leq r \leq N_0 - 1, \tag{26}$$

$$\eta^{(0)} = 0, \tag{27}$$

is stable in the following sense: if $\bar{\gamma}, \bar{\gamma}_0 : H \to \mathbf{R}_+$ are functions such that

$$\lim_{h\to 0} \bar{\gamma}(h) = 0, \quad \lim_{h\to 0} \bar{\gamma}_0(h) = 0, \tag{28}$$

and $\eta_h : I_h \to \mathbf{R}_+$ is a solution of the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \sigma_h(t^{(r)}, \eta^{(r)}) + h_0 \bar{\gamma}(h), \quad 0 \leq r \leq N_0 - 1, \tag{29}$$

$$\eta^{(0)} = \bar{\gamma}_0(h), \tag{30}$$

then there is $\tilde{\alpha} : H \to \mathbf{R}_+$ such that $\eta_h^{(r)} \leq \tilde{\alpha}(h)$ for $t^{(r)} \in I_h$ and $\lim_{h\to 0} \tilde{\alpha}(h) = 0$;

3) the estimate

$$\|f_h(t,x,z,q,s) - f_h(t,x,\bar{z},q,s)\| \leq \sigma_h(t, \|z-\bar{z}\|_{h.r})$$

is satisfied on Σ_h.

Theorem 1. *Suppose that assumption $H[f_h, \sigma_h]$ is satisfied and*
1) $u_h : E_h^+ \to \mathbf{R}$ is a solution of (9) – (11) and the function $v_h : E_h^+ \to \mathbf{R}$ satisfies (22) – (24);
2) there is $\tilde{c} \in \mathbf{R}_+$ such that $\|h'\|^2 \leq \tilde{c} h_0$.
Then there is $\alpha : H \to \mathbf{R}_+$ such that

$$|(u_h - v_h)^{(r,m)}| \leq \alpha(h) \quad \text{on} \quad E_h \tag{31}$$

and

$$\lim_{h\to 0} \alpha(h) = 0. \tag{32}$$

Proof. Let $\Gamma_h : E'_h \to \mathbf{R}$, $\Gamma_{0.h} : E_{0.h} \to \mathbf{R}$, $\Gamma_{\partial.h} : \partial_0 E_h \to \mathbf{R}$ be defined by the relations

$$\delta_0 v_h^{(r,m)} = F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)} \quad \text{on} \quad E'_h,$$

$$v_h^{(r,m+\alpha)} - v_h^{(r,m-\alpha)} = 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j.h}^{(r,m)} + \Gamma_{\partial.h}^{(r,m)} \quad \text{on} \quad \partial_0 E_h \quad \text{and} \quad \alpha \in A^{(m)},$$

$$v_h^{(0,m)} = \varphi_h^{(m)} + \Gamma_{0.h}^{(m)}, \quad x^{(m)} \in [-b,b].$$

It follows from (22) – (25) that

$$\left|\Gamma_h^{(r,m)}\right| \leq \gamma(h) \quad \text{on} \quad E'_h, \ \left|\Gamma_{\partial.h}^{(r,m)}\right| \leq \gamma_1(h)\|h'\|^2 \quad \text{on} \quad \partial_0 E_h,$$

$$\left|\Gamma_{0.h}^{(m)}\right| \leq \gamma_0(h) \quad \text{for} \quad x^{(m)} \in [-b,b]$$

and

$$\lim_{h\to 0} \gamma(h) = 0, \quad \lim_{h\to 0} \gamma_0(h) = 0, \quad \lim_{h\to 0} \gamma_1(h) = 0.$$

Write $z_h = u_h - v_h$ and

$$\Xi_h^{(r,m)} = h_0 \left[f_h(t^{(r)}, x^{(m)}, v_h, \delta u_h^{(r+1,m)}, \delta^{(2)} u_h^{(r+1,m)}) - f_h(t^{(r)}, x^{(m)}, v_h, \delta v_h^{(r+1,m)}, \delta^{(2)} v_h^{(r+1,m)})\right],$$

$$\Lambda_h^{(r,m)} = h_0 \left[f_h(t^{(r)}, x^{(m)}, u_h, \delta u_h^{(r+1,m)}, \delta^{(2)} u_h^{(r+1,m)}) - f_h(t^{(r)}, x^{(m)}, v_h, \delta u_h^{(r+1,m)}, \delta^{(2)} u_h^{(r+1,m)})\right].$$

Then we have

$$z_h^{(r+1,m)} = z_h^{(r,m)} + \Xi_h^{(r,m)} + \Lambda_h^{(r,m)} - h_0 \Gamma_h^{(r,m)} \quad \text{on} \quad E'_h \tag{33}$$

and

$$z_h^{(r,m+\alpha)} = z_h^{(r,m-\alpha)} - \Gamma_{\partial.h}^{(r,m)} \quad \text{on} \quad \partial_0 E_h, \quad \alpha \in A^{(m)}.$$

Our first goal is to estimate the function Λ_h. According to condition 3) of assumption $H[f_h]$, we have

$$|\Lambda_h^{(r,m)}| \le h_0 \sigma_h(t^{(r)}, \|z\|_{h.r}) \quad \text{on} \quad E'_h.$$

The task is now to find an estimate for $\Xi_h^{(r,m)}$.

It follows from the definition of difference operators and from condition 1) of assumption $H[f_h]$ that there is $Q \in \Sigma_h$ such that

$$\begin{aligned}
\Xi_h^{(r,m)} &= B(Q) z_h^{(r+1,m)} + \\
&\quad + \sum_{i=1}^{n} A_{i.+}(Q) z_h^{(r+1,m+e_i)} + \sum_{i=1}^{n} A_{i.-}(Q) z_h^{(r+1,m-e_i)} + \\
&\quad + h_0 \sum_{(i,j)\in J_+} \frac{1}{2h_i h_j} |\partial_{s_{ij}} f(Q)| \big[z_h^{(r,m+e_i+e_j)} + z_h^{(r,m-e_i-e_j)} \big] + \\
&\quad + h_0 \sum_{(i,j)\in J_-} \frac{1}{2h_i h_j} |\partial_{s_{ij}} f(Q)| \big[z_h^{(r,m+e_i-e_j)} + z_h^{(r,m-e_i+e_j)} \big],
\end{aligned} \tag{34}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$.

Write

$$\varepsilon_h^{(r)} = \max\left\{ |z_h^{(r,m)}| : \ (t^{(r)}, x^{(m)}) \in E_{h.r} \right\},$$

$$\tilde{\varepsilon}_h^{(r)} = \max\left\{ |z_h^{(r,m)}| : \ (t^{(r)}, x^{(m)}) \in E_h^+ \cap \big([0, t^{(r)}] \times \mathbf{R}^n \big) \right\},$$

where $0 \le r \le N_0$.

It follows from (13), (14), condition 1) of assumption $H[f_h]$ that

$$A_{i.+}(Q) \ge 0, \quad A_{i.-}(Q) \ge 0. \tag{35}$$

Thus we get

$$\begin{aligned}
\varepsilon_h^{(r+1)}[1 - B(Q)] &\le \tilde{\varepsilon}_h^{(r,m)} + \\
&\quad + \varepsilon_h^{(r+1)} \Bigg[\sum_{i=1}^{n} A_{i.+}(Q) + \sum_{i=1}^{n} A_{i.-}(Q) + \sum_{\substack{i,j=1 \\ j \ne i}}^{n} \frac{h_0}{h_i h_j} |\partial_{s_{ij}} f_h(Q)| \Bigg] + \\
&\quad + h_0 \sigma_h(t^{(r)}, \varepsilon_h^{(r)}) + h_0 \gamma(h).
\end{aligned}$$

One can note that

$$\sum_{i=1}^{n} A_{i.+}(Q) + \sum_{i=1}^{n} A_{i.-}(Q) + B(Q) + \sum_{\substack{i,j=1 \\ j \ne i}}^{n} \frac{h_0}{h_i h_j} |\partial_{s_{ij}} f_h(Q)| = 0 \tag{36}$$

and

$$1 - B(Q) > 0.$$

The above estimates and (33) imply

$$\varepsilon_h^{(r+1)} \leq \tilde{\varepsilon}_h^{(r)} + h_0 \sigma_h(t^{(r)}, \varepsilon_h^{(r)}) + h_0 \gamma(h),$$

where $0 \leq r \leq N_0 - 1$. It is easily seen that

$$\tilde{\varepsilon}_h^{(r)} \leq \varepsilon_h^{(r)} + h_0 \gamma_1(h) \tilde{c}, \quad 0 \leq r \leq N_0 - 1.$$

Thus we see that the function ε_h satisfies the recurrence inequality

$$\varepsilon_h^{(r+1)} \leq \varepsilon_h^{(r)} + h_0 \sigma_h(t^{(r)}, \varepsilon_h^{(r)}) + h_0 \left(\gamma(h) + \tilde{c}\gamma_1(h)\right), \quad 0 \leq r \leq N_0 - 1,$$

and $\varepsilon_h^{(0)} \leq \gamma_0(h)$.

Let us denote by $\bar{\eta}_h : I_h \to \mathbf{R}_+$ a solution of the initial problem

$$\eta_h^{(r+1)} = \eta_h^{(r)} + h_0 \sigma_h(t^{(r)}, \eta_h^{(r)}) + h_0 \left(\gamma(h) + \tilde{c}\gamma_1(h)\right), \quad 0 \leq r \leq N_0 - 1,$$

$$\eta_h^{(0)} = \gamma_0(h).$$

It follows easily that $\varepsilon_h^{(r)} \leq \eta_h^{(r)}$ for $0 \leq r \leq N_0$. Then the assertion of the theorem follows from the stability of problem (26), (27).

4. Convergence of implicit difference methods. Now we give an example of the operator f_h associated with (1)–(3), and we prove that the corresponding difference method is convergent. For any $z \in \mathbf{C}(E, \mathbf{R})$ we put

$$\|z\|_t = \max\{|z(\tau, x)| : (\tau, x) \in E_t\}, \quad 0 \leq t \leq a.$$

Equation (1) contains the functional variable z which is an element of the space $C(E, \mathbf{R})$. Then we need an interpolating operator $T_h : \mathbf{F}(E_h, \mathbf{R}) \to \mathbf{C}(E, \mathbf{R})$. We give an example of such an operator as follows. Put

$$\Im = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \in \{0, 1\} \text{ for } 0 \leq i \leq n\}.$$

Let $z \in F(E_h, \mathbf{R})$ and $(t, x) \in E$. There exists $(r, m) \in \mathbf{Z}^{1+n}$ such that $t^{(r)} \leq t \leq t^{(r+1)}$, $x^{(m)} \leq x \leq x^{(m+1)}$ and $(t^{(r)}, x^{(m)})$, $(t^{(r+1)}, x^{(m+1)}) \in E_h$ where $m+1 = (m_1+1, \ldots, m_n+1)$. We define

$$T_h[z](t, x) = \frac{t - t^{(r)}}{h_0} \sum_{\lambda \in \Im} z^{(r+1, m+\lambda)} \left(\frac{x - x^{(m)}}{h}\right)^{\lambda} \left(1 - \frac{x - x^{(m)}}{h}\right)^{1-\lambda} +$$

$$+ \left(1 - \frac{t - t^{(r)}}{h_0}\right) \sum_{\lambda \in \Im} z^{(r, m+\lambda)} \left(\frac{x - x^{(m)}}{h}\right)^{\lambda} \left(1 - \frac{x - x^{(m)}}{h}\right)^{1-\lambda}$$

where

$$\left(\frac{x-x^{(m)}}{h}\right)^{\lambda} = \prod_{i=1}^{n}\left(\frac{x_i-x_i^{(m_i)}}{h_i}\right)^{\lambda_i},$$

$$\left(1-\frac{x-x^{(m)}}{h}\right)^{1-\lambda} = \prod_{i=1}^{n}\left(1-\frac{x_i-x_i^{(m_i)}}{h_i}\right)^{1-\lambda_i},$$

and we take $0^0 = 1$ in the above formulas. Then we have defined $T_h z$ on E. It follows easily that $T_h z \in C(E,\mathbf{R})$, and that $\|T_h[z]\|_{t^{(r)}} = \|z\|_{h.r}$, $0 \le r \le N_0$.

We approximate solutions of (1)–(3) with solutions of the difference equation

$$\delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}) \tag{37}$$

with initial boundary condition (10), (11).

Lemma 2. *Suppose that $z : E \to \mathbf{R}$ and*
1) $z(t,\cdot) : [-b,b] \to \mathbf{R}$ is of class C^2 for $t \in [0,a]$ and $z_h = z|_{E_h}$,
2) $\tilde{d} \in \mathbf{R}_+$ is such a constant that

$$|\partial_{x_j x_k} z(t,x)| \le \tilde{d}, \quad (t,x) \in E, \quad j,k = 1,\ldots,n, \tag{38}$$

3) there is $L \in \mathbf{R}_+$ such that

$$|z(t,x) - z(\bar{t},x)| \le L|t-\bar{t}|. \tag{39}$$

Then

$$\|T_h[z_h] - z\|_E \le L h_0 + \tilde{d}\|h'\|^2.$$

Proof. Let $(t,x) \in E$ and $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$ where $(t^{(r)}, x^{(m)})$, $(t^{(r+1)}, x^{(m+1)}) \in E_h$. Write

$$U(t,x) = \frac{t-t^{(r)}}{h_0}\left\{\sum_{\lambda\in\Im} z^{(r+1,m+\lambda)}\left(\frac{x-x^{(m)}}{h}\right)^{\lambda}\left(1-\frac{x-x^{(m)}}{h}\right)^{1-\lambda} - z(t^{(r+1)},x)\right\},$$

$$V(t,x) = \left(1-\frac{t-t^{(r)}}{h_0}\right)\left\{\sum_{\lambda\in\Im} z^{(r,m+\lambda)}\left(\frac{x-x^{(m)}}{h}\right)^{\lambda}\left(1-\frac{x-x^{(m)}}{h}\right)^{1-\lambda} - z(t^{(r)},x)\right\},$$

$$W(t,x) = \frac{t-t^{(r)}}{h_0}[z(t^{(r+1)},x) - z(t,x)] + \left(1-\frac{t-t^{(r)}}{h_0}\right)[z(t^{(r)},x) - z(t,x)].$$

Then we have

$$T_h[z](t,x) - z(t,x) = U(t,x) + V(t,x) + W(t,x).$$

It follows from Theorem 5.27 ([3], Chapter 5) that

$$|U(t,x)| + |V(t,x)| \leq \tilde{d}\|h'\|^2.$$

According to condition (39) we have $|W(t,x)| \leq Lh_0$. Hence, the proof is completed.

Assumption $H[\sigma]$. Suppose that the function $\sigma : [0,a] \times \mathbf{R}_+ \to \mathbf{R}_+$ is such that
1) σ is nondecreasing with respect to both variables,
2) $\sigma(t,0) = 0$ for $t \in [0,a]$ and the maximal solution of the Cauchy problem

$$\zeta'(t) = \sigma(t, \zeta(t)), \quad \zeta(0) = 0,$$

is $\zeta(t) = 0$ for $t \in [0,a]$.

Assumption $H[f]$. The function $f : \Sigma \to \mathbf{R}$ of variables (t,x,z,q,s) satisfies the conditions:
1) $f(t,x,z,\,\cdot\,) \in \mathbf{C}(\mathbf{R}^n \times M[n], \mathbf{R})$, the derivatives

$$\partial_q f = (\partial_{q_1} f, \ldots, \partial_{q_n} f), \quad \partial_s f = [\partial_{s_{ij}} f]_{i,j=1,\ldots,n},$$

exist on Σ and

$$\partial_q f(t,x,z,\cdot) \in C(\mathbf{R}^n \times M[n], \mathbf{R}^n), \quad \partial_s f(t,x,z,\cdot) \in C(\mathbf{R}^n \times M[n], M[n])$$

for each $(t,x,z) \in E' \times \mathbf{F}(E_h, \mathbf{R})$,
2) the matrix $\partial_s f$ is symmetric and

$$\partial_{s_{ij}} f(P) \geq 0 \quad \text{for} \quad (i,j) \in J_+, \quad \partial_{s_{ij}} f(P) \leq 0 \quad \text{for} \quad (i,j) \in J_-, \tag{40}$$

$$-\frac{1}{2}|\partial_{q_i} f(P)| + \frac{1}{h_i}\partial_{s_{ii}} f(P) - \sum_{j=1, j\neq i} \frac{1}{h_j}|\partial_{s_{ij}} f(P)| \geq 0, \quad 1 \leq i \leq n, \tag{41}$$

where $P = (t,x,z,q,s) \in \Sigma$,
3) there is a function σ satisfying assumption $H[\sigma]$ such that

$$\|f(t,x,z,q,s) - f(t,x,\bar{z},q,s)\| \leq \sigma(t, \|z-\bar{z}\|_t)$$

on Σ_h.

We can now formulate our main results.

Theorem 2. *Suppose that assumption $H[f]$ is satisfied and*
1) the function $v : E \to \mathbf{R}$ is a solution of (1) – (3) and

$$v_h = v|_{E_h}, \quad \varphi_{h.j} = \varphi_j|_{\partial_0 E_h}, \quad 1 \leq j \leq n,$$

2) the function $u_h : E_h \to \mathbf{R}$ is a solution of (10), (11), (37),
3) there exists $c \in \mathbf{R}_+$ such that $h_k \leq ch_j$ for $1 \leq k, j \leq n$,

4) there is $\gamma_0 : H \to \mathbf{R}_+$ *such that*

$$|\varphi_0^{(r,m)} - \varphi_{0.h}^{(r,m)}| \leq \gamma_0(h) \quad on \quad E_{0.h} \quad and \quad \lim_{h\to 0} \gamma_0(h) = 0, \tag{42}$$

5) $v(\cdot, x)$ *is of class* C^1 *and* $v(t, \cdot)$ *is of class* C^2.

Then there exists $\varepsilon_0 > 0$ *and a function* $\alpha : H \to \mathbf{R}_+$ *such that for* $\|h\| < \varepsilon_0$, $h \in H$ *we have*

$$|(u_h - v_h)^{(r,m)}| \leq \alpha(h) \ on \ E'_h \quad and \quad \lim_{h\to 0} \alpha(h) = 0. \tag{43}$$

Proof. We will use Theorem 1 on the error estimation. Write

$$f_h(t, x, z, q, s) = f(t, x, T_h[z], q, s) \quad \text{on} \quad \Sigma_h,$$

and

$$\sigma_h(t, p) = \sigma(t, p) \quad \text{on} \quad I'_h \times R_+.$$

The conditions (22)–(24) are satisfied. Now we prove that problem (26), (27) is stable. Let $\eta_h : I_h \to \mathbf{R}_+$ be a solution of (29), (30) where $\gamma_0, \bar{\gamma} : H \to \mathbf{R}_+$ and $\lim_{h\to 0} \gamma_0(h) = 0$, $\lim_{h\to 0} \bar{\gamma}(h) = 0$. Let $\tilde{\eta}_h : [0, a] \to \mathbf{R}_+$ be the maximal solution of the Cauchy problem

$$\zeta'(t) = \sigma(t, \zeta(t)) + \bar{\gamma}(h), \quad \zeta(0) = \alpha_0(h). \tag{44}$$

Then $\lim_{h\to 0} \tilde{\eta}_h(t) = 0$ uniformly on $[0, a]$. The function $\tilde{\eta}_h$ is convex on $[0, a]$, therefore we have

$$\tilde{\eta}_h^{(r+1)} \geq \tilde{\eta}_h^{(r)} + h_0\sigma(t^{(r)}, \tilde{\eta}_h^{(r)}) + h_0\bar{\gamma}(h), \quad 0 \leq r \leq N_0 - 1.$$

Since η_h satisfies (29), we have $\eta_h^{(r)} \leq \tilde{\eta}_h^{(r)} \leq \tilde{\eta}_h(a)$ for $0 \leq i \leq N_0$, which completes the proof of the stability of problem (26), (27). It follows from assumption $H[f]$ that

$$\begin{aligned} |f_h(t, x, z, q, s) - f_h(t, x, \bar{z}, q, s)| &= \\ &= |f(t, x, T_h[z], q, s) - f(t, x, T_h[\bar{z}], q, s)| \leq \\ &\leq \sigma(t, \|T_h[z] - T_h[\bar{z}]\|_t \leq \sigma(t, \|z - \bar{z}\|_{h.r}) = \sigma_h(t, \|z - \bar{z}\|_{h.r}). \end{aligned}$$

Thus we see that all the assumptions of Theorem 1 are satisfied and the proof of (43) is complete.

Remark 3. Suppose that assumption $H[f]$ is satisfied with

$$\sigma(t, p) = Lp, \ (t, p) \in [0, a] \times \mathbf{R}_+ \quad \text{where} \quad L \in \mathbf{R}_+.$$

Then assuming that f satisfies the Lipschitz condition with respect to the functional variable we obtain the following error estimates:

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \leq \alpha_0(h)e^{La} + \bar{\gamma}(h)\frac{e^{La} - 1}{L} \quad \text{on} \quad E_h \quad \text{if} \quad L > 0$$

and

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \le \alpha_0(h) + a\bar{\gamma}(h) \quad \text{on} \quad E_h \quad \text{if} \quad L = 0.$$

The above inequality follows from (43) with $\alpha(h) = \tilde{\eta}_h(a)$ where $\tilde{\eta}_h : [0,a] \to \mathbf{R}_+$ is a solution of (44).

Remark 4. Let us consider the explicit difference method (10)–(12). Then we need the following assumption on f and on the steps of the mesh [8, 16]:

$$1 - 2h_0 \sum_{j=1}^{n} \frac{1}{h_j^2} \partial_{s_{jj}} f(P) + h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |\partial_{s_{ij}} f(P)| \ge 0, \tag{45}$$

where $P \in \Sigma$. If the partial derivatives $\partial_{s_{ij}} f$, $i, j = 1, \ldots, 1$, are bounded on Σ then inequality (45) states relations between h_0 and $h' = (h_1, \ldots, h_n)$. It is important in our considerations that condition (45) be omitted in the convergence theorem.

5. Numerical examples.

Example 1. Write

$$E = [0, 0.2] \times [-1, 1] \times [-1, 1],$$

$$\partial_0 E = [0, 0.2] \times \big[\big([-1,1] \times [-1,1]\big) \setminus \big((-1,1) \times (-1,1)\big)\big].$$

Consider the differential equation with deviated variables

$$\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) - \frac{1}{2}\partial_{xy} z(t,x,y) +$$

$$+ z\left(t, \frac{x+y}{2}, \frac{x-y}{2}\right) + f(t,x,y) z(t,x,y) + g(t,x,y) \tag{46}$$

and the initial boundary conditions

$$z(0,x,y) = 1 \quad \text{for} \quad (x,y) \in [-1,1] \times [-1,1], \tag{47}$$

$$\partial_x z(t,0,y) = ty, \quad \partial_x z(t,1,y) = tye^{ty} \quad \text{for} \quad t \in [0, 0.2], \quad y \in [-1,1], \tag{48}$$

$$\partial_y z(t,x,0) = tx, \quad \partial_y z(t,x,1) = txe^{tx} \quad \text{for} \quad t \in [0, 0.2], \quad x \in [-1,1], \tag{49}$$

where

$$f(t,x,y) = xy - t^2(x^2+y^2) + \frac{t}{2} + \frac{t^2 xy}{2},$$

$$g(t,x,y) = -e^{\frac{t(x^2-y^2)}{4}}.$$

The solution of (46) – (49) is known, it is

$$v(t,x,y) = e^{txy}.$$

We found the approximate solutions of (46) – (49) using both implicit and explicit numerical method, and taking the following steps of the mesh: $h_0 = 0.0005$, $h_1 = 0.002$, $h_2 = 0.002$.

Note, that the function f and the steps of the mesh do not satisfy condition (45), which is necessary for the explicit method to be convergent. In our numerical example the average errors of the explicit method exceeded 10^{134}, while the average errors ε_h for fixed $t^{(r)}$ of implicit method are given in the following table.

Table of errors (ε_h)
$\boldsymbol{h_0 = 0.0005, \quad h_1 = 0.002, \quad h_2 = 0.002}$

t :	0.525	0.100	0.125	0.150	0.175	0.200
ε_h :	$3 \cdot 10^{-5}$	$4 \cdot 10^{-5}$	$5 \cdot 10^{-5}$	$6 \cdot 10^{-5}$	$6 \cdot 10^{-5}$	$7 \cdot 10^{-5}$

Example 2. Write

$$E = [0, 0.2] \times [0,1] \times [0,1],$$

$$\partial_0 E = [0, 0.2] \times \big[\big([0,1] \times [0,1]\big) \setminus \big((0,1) \times (0,1)\big)\big].$$

Let us consider the integral-differential equation

$$\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) - \frac{1}{\pi^2}\partial_{xy} z(t,x,y) + \\ + \pi^2 \int_0^x \int_0^y z(t,\tau,s)\,ds\,d\tau + \int_0^t z(\tau,x,y)\,d\tau + 2\pi^2 z(t,x,y) + (t+1)\cos\pi x \cos\pi y \tag{50}$$

and the initial boundary conditions

$$z(0,x,y) = 0 \quad \text{for} \quad (x,y) \in [0,1] \times [0,1], \tag{51}$$

$$\partial_x z(t,0,y) = 0, \quad \partial_x z(t,1,y) = 0 \quad \text{for} \quad t \in [0,0.2], \quad y \in [0,1], \tag{52}$$

$$\partial_y z(t,x,0) = 0, \quad \partial_y z(t,x,1) = 0 \quad \text{for} \quad t \in [0,0.2], \quad x \in [0,1]. \tag{53}$$

The solution of (50) – (53) is known, it is

$$v(t,x,y) = (e^t - 1)\cos\pi x \cos\pi y.$$

Likewise in the previous numerical example we chose the steps of the mesh which do not satisfy condition (45). In accordance with our expectations the explicit method is not convergent, and the average errors are bigger than 10^{150}, while the implicit method is convergent and gives the following average errors.

Table of errors (ε_h)

$\boldsymbol{h_0 = 0.0005,\ h_1 = 0.002,\ h_2 = 0.002}$

t :	0.525	0.100	0.125	0.150	0.175	0.200
ε_h :	$3 \cdot 10^{-4}$	$4 \cdot 10^{-4}$	$5 \cdot 10^{-4}$	$6 \cdot 10^{-4}$	$7 \cdot 10^{-4}$	$8 \cdot 10^{-4}$

The above examples show that there are implicit difference schemes which are convergent and the corresponding classical methods are not convergent. This is due to the fact that we need the relation (45) for steps of the mesh in the classical case. We do not need this condition in our implicit method. Implicit difference methods presented in this paper have the potential for applications in the numerical solving of integral-differential equations or equations with deviated variables.

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