Nonlinear Oscillations, Vol. 4, No. 4, 2001

ABOUT ONE CHARACTERISTIC INITIAL VALUE PROBLEM

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The characteristic initial value problem has been studied for the second order nonlinear differential equation, and modifications of the two-sided method of its approximate integration have been constructed.

AMS Subject Classification: 35L15

Let's consider a nonlinear partial differential equation of the hyperbolic type of the form

$$U_{xy}(x,y) = f(x,y,U(x,y),U_x(x,y),U_y(x,y)) \equiv f[U(x,y)], \tag{1}$$

where

$$(x,\ y) \in B,\ B = B_1 \cup B_2 \cup B_3,\ B_1 = \Big\{(x,\ y)\Big| x \in [0,\ x_0),\ y \in (0,\ x]\Big\},$$

$$B_2 = \Big\{(x,\ y)\ \Big| x \in (x_0,\ 1],\ y \in [0,\ x_0)\Big\},$$

$$B_3 = \Big\{(x,\ y)\Big| x \in [x_0,\ 1),\ y \in (x_0,\ x]\Big\},\ f : D \to R,\ D \in R^5.$$

The setting of the problem [1] is as follows: in the functional space $C^2(B) \cap C(\bar{B})$, find a solution of the differential equation (1) that would satisfy the conditions

$$\begin{aligned} &U(x,\ 0) = \psi_1(x),\ x \in [0,\ x_0], \quad U(x_0,\ y) = \varphi_1(y),\ y \in [0,\ x_0], \\ &U(x,\ x_0) = \psi_2(x),\ x \in [x_0,\ 1], \quad U(1,\ y) = \varphi_2(y),\ y \in [x_0,\ 1]. \end{aligned} \tag{2}$$

We assume that $\psi_1(x) \in C^1([0,\ x_0])$, $\varphi_1(y) \in C^1([0,\ x_0])$, $\psi_2(x) \in C^1([x_0,\ 1]), \varphi_2(y) \in C^1([x_0,\ 1])$; moreover they satisfy the consistency conditions

$$\psi_1(x_0) = \varphi_1(0), \quad \varphi_1(x_0) = \psi_2(x_0), \quad \psi_2(1) = \varphi_2(x_0). \tag{3}$$

It is easy to show that the characteristic initial value problem (1) – (3) is equivalent to the integral equation

$$U(x,y) = U_i(x,y), (x,y) \in \bar{B}_i, \tag{4}$$

where $U_i(x,y) = \Phi_i(x,y) + T_i f[U(\xi,\eta)]$, $i = \overline{1,3}$, and

$$\Phi_1(x,y) \equiv \psi_1(x) + \varphi_1(y) - \varphi_1(0), \quad T_1 f[U(\xi,\eta)] \equiv \int_0^y \int_{x_0}^x f[U(\xi,\eta)] d\xi d\eta,$$

$$\Phi_2(x,y) \equiv \psi_2(x) + \varphi_1(y) - \varphi_1(x_0), \quad T_2 f[U(\xi,\eta)] \equiv \int_{x_0}^y \int_{x_0}^x f[U(\xi,\eta)] d\xi d\eta,$$

$$\Phi_3(x,y) \equiv \psi_2(x) + \varphi_2(y) - \psi_2(1), \quad T_3 f[U(\xi,\eta)] \equiv \int_{x_0}^y \int_1^x f[U(\xi,\eta)] d\xi d\eta.$$

It is obvious that $\Phi_i(x,y) \in C^{(1.1)}(\bar{B}_i)$ and since they satisfy the conditions (2), the problem (1) – (3) is reduced by the substitution

$$V_i(x,y) = U_i(x,y) - \Phi_i(x,y), \quad (x,y) \in B_i, \quad i = \overline{1,\, 3},$$

to a problem with homogeneous conditions (2). Hence from now on, without loss of generality, we assume that

$$\varphi_1(y) = \varphi_2(y) = \psi_1(x) = \psi_2(x) = 0.$$

Definition. *Any functions $Z(x,y), V(x,y)$ from the space $C^2(B) \cap C(\bar{B})$ that satisfy the conditions (2) and the inequalities*

$$\begin{gathered} W(x,y) \le 0, (x,y) \in \bar{B}, \;\; W_x(x,y) \ge 0, \;\; W_y(x,y) \le 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ W_x(x,y) \le 0, \;\; W_y(x,y) \ge 0, \quad (x,y) \in \bar{B}_2, \end{gathered} \tag{5}$$

are called comparison functions of the problem (1) – (3).

Let the right-hand side of the equation (1), $f[U(x,y)]$, belong to the space $C_1(\bar{D})$ [2], where $C_1(\bar{D})$ is the space of functions that satisfy the following conditions:

1) $f[U(x,y)] \in C(\bar{D})$;

2) the function $f[U(x,y)]$ can be represented in the form $f[U(x,y)] \equiv f[U^+(x,y); U^-(x,y) \in C(\bar{D}_1), \bar{D}_1 \in R^8$, so that for any functions $Z(x,y), Z^*(x,y), V(x,y), V^*(x,y) \in \bar{D}_1$ from the space $C^2(B) \cap C(\bar{B})$ that satisfy the inequalities

$$Z(x,y) \le Z^*(x,y), \quad V(x,y) \ge V^*(x,y), \quad (x,y) \in \bar{B},$$

$$Z_x(x,y) \ge Z_x^*(x,y), \quad V_x(x,y) \le V_x^*(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$Z_y(x,y) \le Z_y^*(x,y), \quad V_y(x,y) \ge V_y^*(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$Z_x(x,y) \le Z_x^*(x,y), \quad V_x(x,y) \ge V_x^*(x,y), \quad (x,y) \in \bar{B}_2,$$

$$Z_y(x,y) \ge Z_y^*(x,y), \quad V_y(x,y) \le V_y^*(x,y), \quad (x,y) \in \bar{B}_2,$$

the condition

$$f[Z(x,y);V(x,y)] \ge f[Z^*(x,y);V^*(x,y)] \tag{6}$$

is fulfilled;

3) in the set $\bar{D}_1$ the function $f[U^+(x,y);U^-(x,y)]$ satisfies Lipshits' condition with a constant K,

$$\begin{aligned}\Big|f[Z(x,y);V(x,y)] - f[Z^*(x,y);V^*(x,y)]\Big| &\le K(|Z(x,y) - Z^*(x,y)| \\ &+ |V(x,y) - V^*(x,y)| + |Z_x(x,y) - Z_x^*(x,y)| + |V_x(x,y) - V_x^*(x,y)| \\ &+ |Z_y(x,y) - Z_y^*(x,y)| + |V_y(x,y) - V_y^*(x,y)|).\end{aligned} \tag{7}$$

If $f[U(x,y)] \in C(\bar{D})$ and has bounded first order partial derivatives in all its variables starting from the third one, then $f[U(x,y)] \in C_1(\bar{D})$.

Let's denote

$$\begin{gathered} f^p = f[Z_p(x,y);V_p(x,y)], \quad f_p = f[V_p(x,y);Z_p(x,y)], \\ \bar{f}^p = f[\bar{Z}_p(x,y);\bar{V}_p(x,y)], \quad \bar{f}_p = f[\bar{V}_p(x,y);\bar{Z}_p(x,y)], \\ \bar{Z}_p(x,y) = Z_p(x,y) - d_p(x,y)W_p(x,y), \quad \bar{V}_p(x,y) = V_p(x,y) + d_p(x,y)W_p(x,y), \\ \alpha_p(x,y) = Z_{p,xy}(x,y) - f^p, \quad \beta_p(x,y) = V_{p,xy} - f_p, \quad p = 0,\ 1,\ 2,\ \ldots, \end{gathered} \tag{8}$$

where $d_p(x,y)$ are any functions from the space $C^{(1.1)}(\bar{B})$ that satisfy the conditions

$$d_p(x,y) \ge 0, \quad (x,y) \in \bar{B},$$

$$\begin{gathered} d_{p,x}(x,y) \le 0,\ \ d_{p,y}(x,y) \ge 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ d_{p,x}(x,y) \ge 0,\ \ d_{p,y}(x,y) \le 0, \quad (x,y) \in \bar{B}_2, \end{gathered} \tag{9}$$

$$\sup_B d_p(x,y) \le 0,5, \quad \sup_B \Big|d_{p,x}(x,y)\Big| \le 0,5, \quad \sup_B \Big|d_{p,y}(x,y)\Big| \le 0,5, \quad p = 0,\ 1,\ 2,\ \ldots.$$

Let us construct sequences of functions, $\{Z_p(x,y)\}, \{V_p(x,y)\}$, by [4]

$$Z_{i,p+1}(x,y) = T_i\left\{\bar f^{\,p} - c_p(\bar f^{\,p} - \bar f_p)\right\}, \quad (x,y)\in \bar B_i,$$
$$V_{i,p+1}(x,y) = T_i\left\{\bar f_p + c_p(\bar f^{\,p} - \bar f_p)\right\}, \quad (x,y)\in \bar B_i,\ i=\overline{1,3}, \tag{10}$$

where $c_p(x,y)$ are any nonnegative functions from the space $C(\bar B)$ that satisfy the condition

$$\sup_B c_p(x,y) \le 0,5, \quad p = 0,1,2,\ldots. \tag{11}$$

The formulas

$$W_{i,p+1}(x,y) = T_i\left\{(1-2c_p)(\bar f^p - \bar f_p)\right\},\ (x,y)\in \bar B_i, \tag{12}$$

$$Z_{i,p}(x,y) - Z_{i,p+1}(x,y) = T_i\left\{\alpha_p(\xi,\eta) + f^p - \bar f^p + c_p(\bar f^p - \bar f_p)\right\},$$
$$V_{i,p}(x,y) - V_{i,p+1}(x,y) = T_i\left\{\beta_p(\xi,\eta) + f_p - \bar f_p - c_p(\bar f^p - \bar f_p)\right\}, \tag{13}$$

$$(x,y)\in \bar B_i, \quad i=\overline{1,3},$$

$$\alpha_{p+1}(x,y) = \bar f^{\,p} - f^{p+1} - c_p(\bar f^{\,p} - \bar f_p),$$
$$\beta_{p+1}(x,y) = \bar f_p - f_{p+1} + c_p(\bar f^{\,p} - \bar f_p) \tag{14}$$

follow from (8), (10).

As the zero approximation, we choose arbitrary comparison functions $Z_0(x,y)$, $V_0(x,y)$ that satisfy, in the set $\bar B$, the inequalities

$$\alpha_0(x,y) \ge 0,\ \beta_0(x,y) \le 0. \tag{15}$$

Let

$$M = \sup_{D_1} f(x,y,U^+(x,y),U^+_x(x,y),U^+_y(x,y);\ U^-(x,y),U^-_x(x,y),U^-_y(x,y)),$$

$$m = \inf_{D_1} f(x,y,U^-(x,y),U^-_x(x,y),U^-_y(x,y);\ U^+(x,y),U^+_x(x,y),U^+_y(x,y)).$$

Then if the functions

$$Z_{i,0}(x,y) = T_iM = \begin{cases} M(x-x_0)y, & (x,y)\in \bar B_1;\\ M(x-x_0)(y-x_0), & (x,y)\in \bar B_2;\\ M(x-1)(y-x_0), & (x,y)\in \bar B_3,\end{cases}$$

$$V_{i,0}(x,y) = T_i m = \begin{cases} m(x-x_0)y, & (x,y) \in \bar{B}_1; \\ m(x-x_0)(y-x_0), & (x,y) \in \bar{B}_2; \\ m(x-1)(y-x_0), & (x,y) \in \bar{B}_3 \end{cases}$$

belongs to the space $\bar{D}_1$, then they are comparison functions of the problem (1) – (3) that satisfy conditions (15) .

We will assume that the function $d_0(x,y)$ is such that, in the set $\bar{B}$, the inequalities (9) hold and

$$(1-2d_0(x,y))W_{0,x}(x,y) - 2d_{0,x}(x,y)W_0(x,y) \geq 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$(1-2d_0(x,y))W_{0,y}(x,y) - 2d_{0,y}(x,y)W_0(x,y) \leq 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$(1-2d_0(x,y))W_{0,x}(x,y) - 2d_{0,x}(x,y)W_0(x,y) \leq 0, \quad (x,y) \in \bar{B}_2,$$

$$(1-2d_0(x,y))W_{0,y}(x,y) - 2d_{0,y}(x,y)W_0(x,y) \geq 0, \quad (x,y) \in \bar{B}_2.$$

Then we obtain

$$\begin{aligned} &Z_0(x,y) \leq \bar{Z}_0(x,y) \leq \bar{V}_0(x,y) \leq V_0(x,y), \quad (x,y) \in \bar{B}, \\ &Z_{0,x}(x,y) \geq \bar{Z}_{0,x}(x,y) \geq \bar{V}_{0,x}(x,y) \geq V_{0,x}(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ &Z_{0,y}(x,y) \leq \bar{Z}_{0,y}(x,y) \leq \bar{V}_{0,y}(x,y) \leq V_{0,y}(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ &Z_{0,x}(x,y) \leq \bar{Z}_{0,x}(x,y) \leq \bar{V}_{0,x}(x,y) \leq V_{0,x}(x,y), \quad (x,y) \in \bar{B}_2, \\ &Z_{0,y}(x,y) \geq \bar{Z}_{0,y}(x,y) \geq \bar{V}_{0,y}(x,y) \geq V_{0,y}(x,y), \quad (x,y) \in \bar{B}_2. \end{aligned} \tag{16}$$

Taking into account inequalities (6), (11), (16), from (12), for $p = 0$, we have

$$W_{1,xy}(x,y) = (1-2c_0(x,y))(\bar{f}^{\,0} - \bar{f}_0) \geq 0.$$

By integrating the latter inequality with respect to x from x_0 to x and with respect to y from 0 to y in $\bar{B}_1$, with respect to x from x_0 to x and with respect to y from x_0 to y in $\bar{B}_2$, with respect to x from 1 to x and with respect to y from x_0 to y in $\bar{B}_3$ and taking into account conditions (2), (3), we see that the following inequalities hold in the set $\bar{D}_1$:

$$W_1(x,y) \leq 0, \qquad (x,y) \in \bar{B},$$

$$W_{1,x}(x,y) \geq 0, \quad W_{1,y}(x,y) \leq 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$W_{1,x}(x,y) \leq 0, \quad W_{1,y}(x,y) \geq 0, \quad (x,y) \in \bar{B}_2.$$

Let us choose the function $d_0(x,y)$ so that the conditions

$$\bar{Z}_0(x,y) - Z_1(x,y) \le 0, \quad \bar{V}_0(x,y) - V_1(x,y) \ge 0, \quad (x,y) \in \bar{B},$$

$$\bar{Z}_{0,x}(x,y) - Z_{1,x}(x,y) \ge 0, \quad \bar{V}_{0,x}(x,y) - V_{1,x}(x,y) \le 0,$$
$$(x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$\bar{Z}_{0,y}(x,y) - Z_{1,y}(x,y) \le 0, \quad \bar{V}_{0,y}(x,y) - V_{1,y}(x,y) \ge 0,$$
$$(x,y) \in \bar{B}_1 \cup \bar{B}_3, \tag{17}$$

$$\bar{Z}_{0,x}(x,y) - Z_{1,x}(x,y) \le 0, \quad \bar{V}_{0,x}(x,y) - V_{1,x}(x,y) \ge 0, \quad (x,y) \in \bar{B}_2,$$

$$\bar{Z}_{0,y}(x,y) - Z_{1,y}(x,y) \ge 0, \quad \bar{V}_{0,y}(x,y) - V_{1,y}(x,y) \le 0, \quad (x,y) \in \bar{B}_2$$

are fulfilled.

Then, taking into account (13), (11), (15), (16), (17), (6), we obtain

$$\bar{f}^{\,0} - f^1 \ge 0, \quad \bar{f}_0 - f_1 \le 0.$$

By choosing the function $c_0(x,y)$ so that the inequalities

$$\bar{f}^{\,0} - f^1 - c_0(x,y)(\bar{f}^{\,0} - \bar{f}_0) \ge 0, \quad \bar{f}_0 - f_1 + c_0(x,y)(\bar{f}^{\,0} - \bar{f}_0) \le 0,$$

hold in the set $\bar{D}_1$, from (14), for $p = 0$, we obtain $\alpha_1(x,y) \ge 0, \beta_1(x,y) \le 0$.

Starting with the functions $Z_1(x,y), V_1(x,y)$ and repeating previous considerations, by using induction, we see that if the functions $d_p(x,y), c_p(x,y), p = 0, 1, 2, \ldots$, were chosen so that

$$(1 - 2d_p(x,y))W_{p,x}(x,y) - 2d_{p,x}(x,y)W_p(x,y) \ge 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$(1 - 2d_p(x,y))W_{p,y}(x,y) - 2d_{p,y}(x,y)W_p(x,y) \le 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$(1 - 2d_p(x,y))W_{p,x}(x,y) - 2d_{p,x}(x,y)W_p(x,y) \le 0, \quad (x,y) \in \bar{B}_2,$$

$$(1 - 2d_p(x,y))W_{p,y}(x,y) - 2d_{p,y}(x,y)W_p(x,y) \ge 0, \quad (x,y) \in \bar{B}_2,$$

$$\bar{Z}_p(x,y) - Z_{p+1}(x,y) \le 0, \quad \bar{V}_p(x,y) - V_{p+1}(x,y) \ge 0, \quad (x,y) \in \bar{B},$$
$$\bar{Z}_{p,x}(x,y) - Z_{p+1,x}(x,y) \ge 0, \quad \bar{V}_{p,x}(x,y) - V_{p+1,x}(x,y) \le 0, \tag{18}$$
$$(x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$\bar{Z}_{p,y}(x,y) - Z_{p+1,y}(x,y) \le 0, \quad \bar{V}_{p,y}(x,y) - V_{p+1,y}(x,y) \ge 0,$$
$$(x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$\bar{Z}_{p,x}(x,y) - Z_{p+1,x}(x,y) \le 0, \quad \bar{V}_{p,x}(x,y) - V_{p+1,x}(x,y) \ge 0,$$
$$(x,y) \in \bar{B}_2,$$

$$\bar{Z}_{p,y}(x,y) - Z_{p+1,y}(x,y) \ge 0, \quad \bar{V}_{p,y}(x,y) - V_{p+1,y}(x,y) \le 0,$$
$$(x,y) \in \bar{B}_2,$$

$$\bar{f}^{\,p} - f^{p+1} - c_p(x,y)(\bar{f}^{\,p} - \bar{f}_p) \ge 0, \quad \bar{f}_p - f_{p+1} + c_p(x,y)(\bar{f}^{\,p} - \bar{f}_p) \le 0,$$

then the inequalities

$$\begin{gathered} Z_p(x,y) \le Z_{p+1}(x,y) \le V_{p+1}(x,y) \le V_p(x,y), \quad (x,y) \in \bar{B}, \\ Z_{p,x}(x,y) \ge Z_{p+1,x}(x,y) \ge V_{p+1,x}(x,y) \ge V_{p,x}(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ Z_{p,y}(x,y) \le Z_{p+1,y}(x,y) \le V_{p+1,y}(x,y) \le V_{p,y}(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ Z_{p,x}(x,y) \le Z_{p+1,x}(x,y) \le V_{p+1,x}(x,y) \le V_{p,x}(x,y), \quad (x,y) \in \bar{B}_2, \\ Z_{p,y}(x,y) \ge Z_{p+1,y}(x,y) \ge V_{p+1,y}(x,y) \le V_{p,y}(x,y), \quad (x,y) \in \bar{B}_2 \end{gathered} \tag{19}$$

take place in the set $\bar{D}_1$ for any $p = 0, 1, 2, \ldots$.

Theorem 1. *Let there exist comparison functions of the problem (1) – (3), $Z_0(x,y), V_0(x,y)$, that satisfy conditions (15) for $(x,y) \in \bar{B}$ and the right-hand side of the equation (1) $f[U(x,y)] \in C_1(\bar{D})$. Then, if the functions $d_p(x,y), c_p(x,y), p = 0, 1, 2, \ldots$, satisfying conditions (9), (11), are chosen so that inequalities (18) hold in the set $\bar{D}_1$, then the sequences of functions, $\{Z_p(x,y)\}$, $\{V_p(x,y)\}$, constructed according to (10), converge to a unique solution of the problem (1) – (3) in the space $C^2(B) \cap C(\bar{B})$ $U(x,y)$ in the set $\bar{B}$ absolutely and uniformly and*

$$\begin{gathered} Z_p(x,y) \le U(x,y) \le V_p(x,y), \quad (x,y) \in \bar{B}, \\ Z_{p,x}(x,y) \ge U_x(x,y) \ge V_{p,x}(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ Z_{p,y}(x,y) \le U_y(x,y) \le V_{p,y}(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ Z_{p,x}(x,y) \le U_x(x,y) \le V_{p,x}(x,y), \quad (x,y) \in \bar{B}_2, \\ Z_{p,y}(x,y) \ge U_y(x,y) \ge V_{p,y}(x,y), \quad (x,y) \in \bar{B}_2. \end{gathered} \tag{20}$$

Proof. To prove that the respective sequence of functions, $\{Z_p(x,y)\}$, $\{V_p(x,y)\}$, $\{Z_{p,x}(x,y)\}$, $\{V_{p,x}(x,y)\}$, $\{Z_{p,y}(x,y)\}$, $\{V_{p,y}(x,y)\}$, uniformly converges to the same limit, taking into account inequalities (19), it is sufficiently to demonstrate that $W_p(x,y) \underset{p\to\infty}{\overset{(x,y)\in B}{\Longrightarrow}} 0$, $W_{p,x}(x,y) \underset{p\to\infty}{\overset{(x,y)\in B}{\Longrightarrow}} 0$, $W_{p,y}(x,y) \underset{p\to\infty}{\overset{(x,y)\in B}{\Longrightarrow}} 0$.

From (7) we have

$$
\begin{aligned}
\bar{f}^{\,p} - \bar{f}_p &\le 2K\left(|\bar{W}_p(x,y)| + |\bar{W}_{p,x}(x,y)| + |\bar{W}_{p,y}(x,y)|\right) \\
&\le 2K\big((1-2d_p(x,y))\left(|W_p(x,y)| + |W_{p,x}(x,y)| + |W_{p,y}(x,y)|\right) \\
&\quad + 2|W_p(x,y)|(|d_{p,x}(x,y) + d_{p,y}(x,y)|)\big) \\
&\le 2Kl\left(|W_p(x,y)| + |W_{p,x}(x,y)| + |W_{p,y}(x,y)|\right), \qquad (21)\\
l &= \max_p \sup_B \left\{1 - 2d_p(x,y) + 2|d_{p,x}(x,y) + d_{p,y}(x,y)|\right\}.
\end{aligned}
$$

If $p=0$, we have $\bar{f}^{\,0} - \bar{f}_0 \le 2Kl\left(|W_0(x,y)| + |W_{0,x}(x,y)| + |W_{0,y}(x,y)|\right)$.

Let's denote

$$
d = \sup_B \left\{|W_0(x,y)|, |W_{0,x}(x,y)|, |W_{0,y}(x,y)|\right\}, \quad q = \max_p \sup_B (1 - 2c_p(x,y)),
$$

$$
|\Omega_p(x,y)| = \left\{|W_p(x,y)|, |W_{p,x}(x,y)|, |W_{p,y}(x,y)|\right\}.
$$

Then from (12), for $p=0$, it follows that

$$
W_{1,xy}(x,y) = (1 - 2c_0(x,y))(\bar{f}^{\,0} - \bar{f}_0) \le 6Kldq,
$$

$$
|\Omega_1(x,y)| \le \begin{cases} 6Klqd(y + x_0 - x), & (x,y) \in \bar{B}_1; \\ 6Klqd(x-y), & (x,y) \in \bar{B}_2; \\ 6Klqd(1 - x + y - x_0), & (x,y) \in \bar{B}_3. \end{cases}
$$

For $p=1$, from (12) we obtain

$$
\begin{aligned}
W_{2,xy}(x,y) &= (1 - 2c_1(x,y))(\bar{f}^{\,1} - \bar{f}_1) \\
&\le (1 - 2c_1(x,y))2kl\left(|W_1(x,y)| + |W_{1,x}(x,y)| + W_{1,y}(x,y)|\right) \\
&\le \begin{cases} d(6klq)^2(y + x_0 - x), & (x,y) \in \bar{B}_1; \\ d(6klq)^2(x-y), & (x,y) \in \bar{B}_2; \\ d(6klq)^2(1 - x + y - x_0), & (x,y) \in \bar{B}_3, \end{cases}
\end{aligned}
$$

hence,

$$|\Omega_2(x,y)| \le \begin{cases} d(6Klq)^2(y+x_0-x)^2/2!, & (x,y)\in\bar B_1;\\ d(6Klq)^2(x-y)^2/2!, & (x,y)\in\bar B_2;\\ d(6Klq)^2(1-x+y-x_0)^2/2!, & (x,y)\in\bar B_3.\end{cases}$$

Supposo that the recurrence estimates

$$|\Omega_p(x,y)| \le \begin{cases} d(6Klq)^p(y+x_0-x)^p/p!, & (x,y)\in\bar B_1;\\ d(6Klq)^p(x-y)^p/p!, & (x,y)\in\bar B_2;\\ d(6Klq)^p(1-x+y-x_0)^p/p!, & (x,y)\in\bar B_3\end{cases}$$

hold. Then from (12), (21) we have

$$W_{p+1,xy}(x,y) \le \begin{cases} d(6Klq)^{p+1}(y+x_0-x)^p/p!, & (x,y)\in\bar B_1;\\ d(6Klq)^{p+1}(x-y)^p/p!, & (x,y)\in\bar B_2;\\ d(6Klq)^{p+1}(1-x+y-x_0)^p/p!, & (x,y)\in\bar B_3.\end{cases}$$

Integrating the latter inequality with respect to x from x_0 to x and with respect to y from 0 to y in $\bar B_1$, with respect to x from x_0 to x and with respect to y from x_0 to y in $\bar B_2$, with respect to x from 1 to x and with respect to y from x_0 to y in $\bar B_3$ we obtain

$$|\Omega_{p+1}(x,y)| \le \begin{cases} d(6Klq(y+x_0-x))^{p+1}/(p+1)!, & (x,y)\in\bar B_1;\\ d(6Klq(x-y))^{p+1}/(p+1)!, & (x,y)\in\bar B_2;\\ d(6Klq(1-x+y-x_0))^{p+1}/(p+1)!, & (x,y)\in\bar B_3.\end{cases} \tag{22}$$

From estimates (22) it follows that $\lim_{p\to\infty}|\Omega_p(x,y)| = 0$, that is, in the set $\bar B$,

$$\lim_{p\to\infty} Z_p(x,y) = \lim_{p\to\infty} V_p(x,y) = U(x,y),$$

$$\lim_{p\to\infty} Z_{p,x}(x,y) = \lim_{p\to\infty} V_{p,x}(x,y) = U_x(x,y),$$

$$\lim_{p\to\infty} Z_{p,y}(x,y) = \lim_{p\to\infty} V_{p,y}(x,y) = U_y(x,y).$$

Passing to the limit in (10) for $p\to\infty$ and differentiating with respect to x and y, we see that the limit function $U(x,y)$ is solution of the problem (1) – (3).

Let's prove uniqueness of the solution of the problem (1) – (3) in the set $\bar D$. To do this, assume that there exist two solutions, $U(x,y)$ and $Z(x,y)$. We denote $W(x,y) = U(x,y) - Z(x,y)$. Then we have

$$|W_{xy}(x,y)| = 2K(|W(x,y)| + |W_x(x,y)| + |W_y(x,y)|).$$

Denoting $d_1 = \max \sup_B \{|W(x,y)|, |W_x(x,y)|, |W_y(x,y)|\}$, as in the previous case, we see that the following estimate holds,

$$|\Omega(x,y)| \le \begin{cases} (6Kd_1(y + x_0 - x))^p/p!, & (x,y) \in \bar{B}_1; \\ (6Kd_1(x-y))^p/p!, & (x,y) \in \bar{B}_2; \\ (6Kd_1(1 - x + y - x_0))^p/p!, & (x,y) \in \bar{B}_3, \end{cases}$$

where p is any nonnegative number. This is possible only if $W(x,y) \equiv 0$.

It remains to demonstrate that inequalities (20) take place. We will assume that for some number p,

$$Z_p(x,y) > U(x,y), \quad (x,y) \in \bar{B}.$$

Then by (19) we obtain

$$Z_p(x,y) > Z_{p+q}(x,y), \quad (x,y) \in \bar{B}$$

for any $q \in N$, hence, the sequence $\{Z_{p+q}(x,y)\}$ does not converge to a solution of the problem (1)–(3) for $q \to \infty$, which contradicts to what has been proved above. Similarly, another inequalities (20) are proved to hold in the set $\bar{D}$ and the theorem is proved completely.

Theorem 2. *Let the right-hand side of equation (1), $f[U(x,y)] \in C_1(\bar{D})$, and there exist in the space $C^2(B) \cap C(\bar{B})$ a function $Z_0(x,y)(V_0(x,y))$ that satisfies the homogeneous conditions (2) and the inequalities*

$$\begin{gathered} Z_0(x,y) \le 0 \quad (V_0(x,y) \ge 0), \quad (x,y) \in \bar{B}, \\ Z_{0,x}(x,y) \ge 0 \quad (V_{0,x}(x,y) \le 0), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ Z_{0,y}(x,y) \le 0 \quad (V_{0,y}(x,y) \ge 0), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ Z_{0,x}(x,y) \le 0 \quad (V_{0,x}(x,y) \ge 0), \quad (x,y) \in \bar{B}_2, \\ Z_{0,y}(x,y) \ge 0 \quad (V_{0,y}(x,y) \le 0), \quad (x,y) \in \bar{B}_2, \\ Z_{0,xy}(x,y) - f[Z_0(x,y);0] \ge 0, \quad f[0;Z_0(x,y)] \ge 0 \\ \big(V_{0,xy}(x,y) - f[V_0(x,y);0] \le 0, \quad f[0;V_0(x,y)] \le 0\big). \end{gathered} \tag{23}$$

Then a solution of the problem (1), (2) satisfies the inequalities

$$U(x,y) \le 0 \ \ (U(x,y) \ge 0), \quad (x,y) \in \bar{B},$$

$$U_x(x,y) \ge 0, \ \ U_y(x,y) \le 0 \ \ (U_x(x,y) \le 0, U_y(x,y) \ge 0), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \tag{24}$$

$$U_x(x,y) \le 0, \ \ U_y(x,y) \ge 0 \ \ (U_x(x,y) \ge 0, U_y(x,y) \le 0), \quad (x,y) \in \bar{B}_2.$$

Proof. The functions $Z_0(x,y), V_0(x,y) \equiv 0 \ \ (Z_0(x,y) \equiv 0, V_0(x,y))$ are comparison functions of the problem (1) – (3) and, by conditions (23), $\alpha_0(x,y) \ge 0$, $\beta_0(x,y) \le 0$. According to Theorem 1 inequalities (20) take place, hence, for $p = 0$, we obtain (24). Thus, the theorem is proved.

Consider a system of two linear equation of the form

$$Z_{xy}(x,y) = q_1(x,y)Z(x,y) + q_2(x,y)Z_x(x,y) + q_3(x,y)Z_y(x,y) + f_1(x,y), \tag{25}$$

$$V_{xy}(x,y) = p_1(x,y)V(x,y) + p_2(x,y)V_x(x,y) + p_3(x,y)V_y(x,y) + f_2(x,y) \tag{26}$$

with homogeneous conditions (2), where $Z(x,y), V(x,y)$ are the sought functions and $q_j(x,y)$, $p_j(x,y), f_i(x,y), i = 1,2, j = \overline{1,3}$, are known piecewise continuous functions that satisfy the conditions

$$\begin{gathered} f_i(x,y) \ge 0, \quad i = 1,2, \\ q_1(x,y) \le 0, \ \ p_1(x,y) \le 0, \ \ (x,y) \in \bar{B}, \\ q_2(x,y) \ge 0, \ \ p_2(x,y) \ge 0, \ \ (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ q_2(x,y) \le 0, \ \ p_2(x,y) \le 0, \ \ (x,y) \in \bar{B}_2, \\ q_3(x,y) \le 0, \ \ p_3(x,y) \le 0, \ \ (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ q_3(x,y) \ge 0, \ \ p_3(x,y) \ge 0, \ \ (x,y) \in \bar{B}_2. \end{gathered} \tag{27}$$

According to Theorem 2, solutions of the problems (25), (2) and (26), (2) satisfy the inequalities

$$Z(x,y) \le 0, \quad V(x,y) \le 0, \quad (x,y) \in \bar{B},$$

$$Z_x(x,y) \ge 0, \quad V_x(x,y) \ge 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$Z_y(x,y) \le 0, \quad V_y(x,y) \le 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \tag{28}$$

$$Z_x(x,y) \le 0, \quad V_x(x,y) \le 0, \quad (x,y) \in \bar{B}_2,$$

$$Z_y(x,y) \ge 0, \quad V_y(x,y) \ge 0, \quad (x,y) \in \bar{B}_2.$$

Theorem 3. *Let, for piecewise continuous functions* $q_j(x,y), p_j(x,y), f_i(x,y), i = 1,2, j = \overline{1,3}$*, that satisfy conditions (27), the inequalities*

$$\begin{gathered} f_1(x,y) \ge f_2(x,y), \\ q_1(x,y) \le p_1(x,y), \quad (x,y) \in \bar{B}, \\ q_2(x,y) \ge p_2(x,y), \quad q_3(x,y) \le p_3(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3, \\ q_2(x,y) \le p_2(x,y), \quad q_3(x,y) \ge p_3(x,y), \quad (x,y) \in \bar{B}_2 \end{gathered} \tag{29}$$

take place.

Then solutions of the problems (25), (2) and (26), (2) satisfy

$$Z(x,y) \ge V(x,y), \quad (x,y) \in \bar{B},$$

$$Z_x(x,y) \ge V_x(x,y), \quad Z_y(x,y) \le V_y(x,y), \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$Z_x(x,y) \le V_x(x,y), \quad Z_y(x,y) \ge V_y(x,y), \quad (x,y) \in \bar{B}_2.$$

Proof. Denoting $W(x,y) = Z(x,y) - V(x,y)$, from (25), (26) we obtain

$$\begin{aligned} W_{xy}(x,y) = {} & q_1(x,y)W(x,y) + q_2(x,y)W_x(x,y) + q_3(x,y)W_y(x,y) + f(x,y), \\ f(x,y) = {} & \big(q_1(x,y) - p_1(x,y)\big)V(x,y) + \big(q_2(x,y) - p_2(x,y)\big)V_x(x,y) \\ & + \big(q_3(x,y) - p_3(x,y)\big)V_y(x,y) + f_1(x,y) - f_2(x,y). \end{aligned} \tag{30}$$

Taking into account (28), (29), we have $f(x,y) \ge 0$, hence, the solution of the problem (30), (2) satisfies the conditions

$$W(x,y) \le 0, \quad (x,y) \in \bar{B},$$

$$W_x(x,y) \ge 0, \;\; W_y(x,y) \le 0, \quad (x,y) \in \bar{B}_1 \cup \bar{B}_3,$$

$$W_x(x,y) \le 0, \;\; W_y(x,y) \ge 0, \quad (x,y) \in \bar{B}_2,$$

what was to be proved.

Consider an equation of the form

$$U_{xy}(x,y) = f(x,y,U(x,y)) \equiv f\,[U(x,y)]. \tag{31}$$

Lemma. *Let the right-hand side of the equation (31),* $f(x, y, U(x, y)) \in C_1(\bar{D})$*, and the functions* $\psi_i(x), \varphi_i(y), i = 1, 2$*, satisfy the relation*

$$\psi_2(x_0) = \psi_1(x_0) + \int_0^{x_0} f[\varphi_1(\eta)]d\eta,$$

$$\varphi_2(x_0) = \varphi_1(x_0) + \int_{x_0}^{1} f[\psi_2(\xi)]d\xi,$$

and the consistency conditions (3).

Then the solution of the problem (31), (2) is regular in the set $\bar{B}$*.*

REFERENCES

1. *Bitsadze A.V.* Certain Classes Of Partial Differential Equations [in Russian], Nauka, Moscow (1981).
2. *Marynets' V.V. and Troshyna A.V.* "A gGeneralized darboux problem," Naukovyi Visnyk Uzhhorods'koho Universytetu. Ser. Mat., No. 4, 79 – 84 (1999).
3. *Kurpel' N.S. and Shuvar B.A.* Two-Sided Operator Inequalities and Their Applications [in Russian], Naukova Dumka, Kiev (1980).
4. *Marynets' V.V.* "On one mixed problem for deterministic quasilinear partial differential equations with deviating argument," Ukr. Mat. Zh., **46**, No. 11, 1581 – 1585 (1994).

Received 21.08.2001