UDC 539.3

VOLTERRA'S THEORY OF ELASTIC DISLOCATIONS FOR A TRANSVERSALLY ISOTROPIC HOMOGENEOUS HOLLOW CYLINDER

ТЕОРІЯ ВОЛЬТЕРРИ ДЛЯ ПРУЖНИХ ДИСЛОКАЦІЙ ТРАНСВЕРСАЛЬНО ІЗОТРОПНИХ ОДНОРІДНИХ ПОРОЖНИСТИХ ЦИЛІНДРІВ

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This work will consider Volterra's theory of elastic dislocations in the case of a transversally isotropic homogeneous hyperelastic hollow cylinder. We obtain explicit equations for vector field of displacements, for tensor fields of strain and stress, and for forces upon the boundary.

Розглянуто теорію Вольтерри для пружних дислокацій у випадку трансверсально ізотропних однорідних надпружних порожнистих циліндрів. Одержано рівняння для векторного поля переміщень, тензорного поля напружень і сил на межі.

Introduction. In his note on distorsions [1], Volterra studies the equilibrium of multi-connected elastic homogeneous bodies, particularly hollow cylinders, limiting his study only to isotropic bodies (see e.g. [1–3]). Recently G. Caricato proposed an extension of that theory in the case of a transversally isotropic homogeneous hiperelastic hollow cylinder (see e.g. [4, 5]).

In this work we will reconsider and expand the findings [4, 5]; we will obtain explicit formulas for the equilibrium equations, for the boundary conditions, for the vector field of displacements, and for tensor fields of strain and stress [2].

Thus we are presenting the following:

That the hypothesis in [4, 5] of the parallelism of the two vectors $\mathbf{h}$ and $\mathbf{k}$, characteristic of the displacement (3), plays no role in our research.

From the analysis of the equilibrium equations we show that the coefficient l_3, present in the displacement (3) and arbitrarily retained in the notes [4, 5], assumes instead the expression (19), so the displacement (3) depends only on the parameters a_1 and l_4.The strain (35) and the stress (23) are calculated from the following form (21) of the displacement; they only depend on the parameters a_1 and l_4. From examining the boundary conditions we can then deduce that the coefficient l_3 vanishes and, as a consequence, the parameter a_1 assumes the explicit form (32).

[1] Volterra calls the deformations which his theory refers to *„distortions"*. Love prefers to call them *"dislocations"* (see [1, p. 221], art. 156, note).

[2] We have utilized the Computer Algebra System *Mathematica*, which allows not only to verify the calculations rapidly, but also automatically generates the LaTeX sources of formulas.

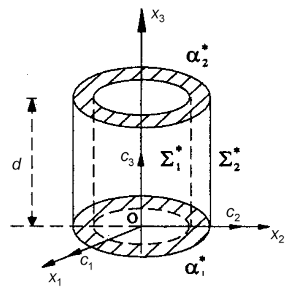

Fig. 1

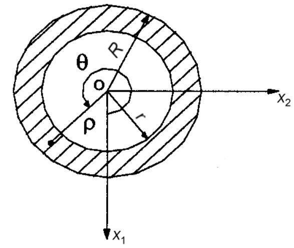

Fig. 2

The final displacement (33), together with the strain (35) and stress (36) tensors, become exclusively dependent on the arbitrary parameter l_4. The only components of the strain and stress tensors depending on the parameter l_4 are ϵ_{13}, ϵ_{23} and σ_{13}, σ_{23}, respectively.

The vector field displacement becomes dependent on the ratio N/A of only two of the five elastic constants, which characterize the transversally isotropic case.

Finally we have found under what conditions Volterra's formulas (2) for the isotropic case can be attained again.

It remains to be calculated, in a following paper, a generic auxiliary displacement $\mathbf{u}'$, to obtain the complete explicit form of Volterra's dislocations in the case under examination (see [5], §2.1, note 5).

1. Volterra's dislocations in the case of an isotropic homogeneous hollow cylinder. Briefly let's refer to Volterra's dislocations theory, limiting it to the case of cylinder $\mathcal{C}$, circular, hollow (therefore doubly connected), homogeneous, hyperelastic and isotropic, which is found, at a certain assigned temperature τ, in a natural state $\mathcal{C}^*_\tau$, and assumed as a reference configuration [3]. So we introduce into an ordinary space a Cartesian rectangular reference $O\,x_1x_2x_3$ of respective versors $\{\mathbf{c_1},\, \mathbf{c_2},\, \mathbf{c_3}\}$. We choose the axis $O\,x_3$ coinciding with the symmetry axis of the cylinder and the coordinate plane $O\,x_1x_2$ placed over the base $\alpha^*{}_1$; $d = (x_3)_{\alpha^*{}_2} > 0$ is the height of the base $\alpha^*{}_2$ (Fig. 1). Finally Σ^* is the lateral surface of $\mathcal{C}^*$, made from the two cylindrical coaxial surfaces $\Sigma^*{}_1$, internal surface of radius r, and $\Sigma^*{}_2$, external surface of radius R (Fig. 2). P^* is the generic point of $\mathcal{C}^*_\tau$, $\theta = \arctan \dfrac{x_2}{x_1}$ is the anomaly of P^* and $\rho = \sqrt{{x_1}^2 + {x_2}^2}$ is the distance of P^* from the axis of the cylinder. Since the cylinder $\mathcal{C}$ is doubly connected, many-valued displacements $\mathbf{u}$ are possible (see e.g. [1, p. 221], art. 156).

Volterra used Weingarten's note [6] as a starting point, where it is shown that an elastic body occupying a dominion, not simply connected, can find itself in a state of tension also in the absence of external forces. Volterra developed a general theory, with some improvements from Cesaro (see e.g. [2] and [1, p. 221], art. 156). Volterra began with the observation that Weingarten's considerations could not be validated in the case of simply-connected bodies in the range of regular deformations. With this in mind he constructed his well-known Volterra's formulas, which obtain the displacements of the points of an elastic body, once assigned the

[3] The theory of dislocations initially has a very general character, but subsequently is substantially focused to obtain explicit results in the study of equilibrium of hollow homogeneous and *isotropic* cylinders.

linearized tensor of deformation. Then he examined the field of displacements whose Cartesian components are [4]

$$u_1 = \frac{1}{2\pi}(\,l + qx_3 - rx_2)\,\theta + (\,ax_1 + bx_2 + cx_3 + e)\,\log\rho^2,$$

$$u_2 = \frac{1}{2\pi}(\,m + rx_1 - px_3)\,\theta + (\,a'x_1 + b'x_2 + c'x_3 + e')\,\log\rho^2, \qquad (1)$$

$$u_3 = \frac{1}{2\pi}(\,n + px_2 - qx_1)\,\theta + (\,a''x_1 + b''x_2 + c''x_3 + e'')\,\log\rho^2,$$

where the two triplets (l, m, n) and (p, q, r) are the respective Cartesian components of the two assigned constant vectors $\mathbf{h} \equiv (l, m, n)$ and $\mathbf{k} \equiv (p, q, r)$. He determined the twelve constants a, b, c, e; a', b', c', e'; a'', b'', c'', e'' so that the three functions (1) would verify the equilibrium equations (see e.g. [2, p. 428]); so he obtained the formulas

$$u_1 = \frac{1}{2\pi}\left\{(l + qx_3 - rx_2)\,\theta + \frac{1}{2}\left(-m + px_3 + \frac{r\mu}{\lambda + 2\mu}x_1\right)\,\log\rho^2\right\},$$

$$u_2 = \frac{1}{2\pi}\left\{(m + rx_1 - px_3)\,\theta + \frac{1}{2}\left(l + qx_3 + \frac{r\mu}{\lambda + 2\mu}x_2\right)\,\log\rho^2\right\}, \qquad (2)$$

$$u_3 = \frac{1}{2\pi}\left\{(n + px_2 - qx_1)\,\theta - \frac{1}{2}(px_1 + qx_2)\,\log\rho^2\right\},$$

where λ and μ are the two Lamé constants (see e.g. [3]).

He observed that the displacement (2) generates a distribution of forces not identically vanishing on the surface of the cylinder. So he calculated a supplementary field of displacements $\mathbf{u}'(P^*)$ single-valued, which would satisfy the indefinite equations of elastic equilibrium in the absence of forces of mass and would generate the same distribution of surface forces on the boundary of the cylinder. The field of displacements,

$$\mathbf{u}''(P^*) = \mathbf{u}(P^*) - \mathbf{u}'(P^*),$$

satisfies the indefinite equations of equilibrium equally, but does not generate any distribution of forces on the boundary of the cylinder and is many-valued like $\mathbf{u}(P^*)$ [5].

The many-valued field of displacements, $\mathbf{u}''(P^*)$, can be physically interpreted in terms of the following operations (see e.g. [1, p. 224]):

1. By making a transversal cut on an axial semi-plane, we make the hollow homogeneous cylinder $\mathcal{C}_\tau$ simply-connected and it assumes a natural state $\mathcal{C}^*_\tau$. We'll denote the two faces of the cut by γ^*_1 and γ^*_2.

[4] Conforming to [3, 5] the p, q, r have opossite signs as compared with Volterra's original work (see e.g. [2, p. 427]).

[5] So we obtain the real Volterra's dislocation, which consists of two parts: the many-valued main displacement $\mathbf{u}(P^*)$ and the single-valued supplementary displacement $-\mathbf{u}'(P^*)$.

2. We'll impose a translatory displacement $\mathbf{h} \equiv (l, m, n)$ and a rotatory displacement $\mathbf{k} \equiv \equiv (p, q, r)$ to one of the two faces, e.g. γ_1^*, with respect to the other. The two characteristic vectors $\mathbf{h}$ and $\mathbf{k}$ together will be parallel to the semi-plane π. In this way making the face γ_1^* penetrate into γ_2^* or distance itself from γ_2^* according to the vector $\mathbf{k}$ will levogyrous or dextrogyrous accordingly.

3. If $\mathbf{k}$ is levogyrous we'll remove a thin slice of matter, a thickness proportional to the distance from the axis of the cylinder. If instead $\mathbf{k}$ is dextrogyrous we'll add a thin slice of matter, the same material as the cylinder, between the two faces of the cut, a thickness still proportional to the distance from the axis of the cylinder. In this way we create a state of deformation in the cylinder and therefore of stress.

4. Finally we'll remake the cylinder doubly connected by soldering the two faces of the cut. In this way the cylinder assumes a helicoidal configuration absent of superficial forces and results in a state of regular internal stress.

Collectively, Volterra called the described operations a *dislocation* whose characteristics are l, m, n, p, q, r (see e.g. [2]).

2. Volterra's dislocations in the case of a transversally isotropic homogeneous hollow cylinder. Now let's consider a *transversally isotropic* [6] hyperelastic homogeneous hollow cylinder and let's suppose it is found in a natural state $\mathcal{C}_\tau^*$[7] at temperature τ.

In analogy to Volterra's procedure, conforming to [5], let's consider a displacement of the following type:

$$\mathbf{u}(P^*) = \frac{1}{2\pi}(\mathbf{h} + \mathbf{k} \wedge OP^*)\,\theta + \\ + \left[(\mathbf{a} \cdot OP^* + a_4)\,\mathbf{c}_1 + (\mathbf{b} \cdot OP^* + b_4)\,\mathbf{c}_2 + (\mathbf{l} \cdot OP^* + l_4)\,\mathbf{c}_3\right]\,\log \rho^2, \tag{3}$$

where we can assign the two vectors $\mathbf{h}$ and $\mathbf{k}$, characteristic of the dislocation, while we have to determine yet the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{l}$ and the constants a_4, b_4, l_4.

If we project (3) onto the axis we obtain

$$\begin{aligned} u_1 &= \frac{1}{2\pi}(h_1 + k_2 x_3 - k_3 x_2)\,\theta + (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4)\,\log \rho^2, \\ u_2 &= \frac{1}{2\pi}(h_2 + k_3 x_1 - k_1 x_3)\,\theta + (b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4)\,\log \rho^2, \\ u_3 &= \frac{1}{2\pi}(h_3 + k_1 x_2 - k_2 x_1)\,\theta + (l_1 x_1 + l_2 x_2 + l_3 x_3 + l_4)\,\log \rho^2. \end{aligned} \tag{4}$$

The displacement gradient $\nabla \mathbf{u} \equiv \left\| \dfrac{\partial u_h}{\partial x_k} \right\|$ relative to the displacement (3), (4) is

[6] Since it conserves its mechanical characteristics along any direction perpendicular to the axis of symmetry.
[7] Therefore absent of external mass and/or superficial forces.

$$\frac{\partial u_1}{\partial x_1} = \frac{1}{2\pi\rho^2}(4\pi a_4 x_1 - h_1 x_2 + 4\pi a_1 x_1^2 + 4\pi a_2 x_1 x_2 + 4\pi a_3 x_1 x_3 + k_3 {x_2}^2 - k_2 x_2 x_3) + + a_1 \log\rho^2,$$

$$\frac{\partial u_1}{\partial x_2} = \frac{1}{2\pi\rho^2}(h_1 x_1 + 4\pi a_4 x_2 - (k_3 - 4\pi a_1) x_1 x_2 + 4\pi a_2 x_2^2 + k_2 x_1 x_3 + 4\pi a_3 x_2 x_3) - - \frac{k_3}{2\pi}\arctan\theta + a_2 \log\rho^2,$$

$$\frac{\partial u_1}{\partial x_3} = \frac{k_2}{2\pi}\arctan\theta + a_3 \log\rho^2,$$

$$\frac{\partial u_2}{\partial x_1} = \frac{1}{2\pi\rho^2}(4\pi b_4 x_1 - h_2 x_2 + 4\pi b_1 x_1^2 - (k_3 - 4\pi b_2) x_1 x_2 + 4\pi b_3 x_1 x_3 + k_1 x_2 x_3) + + \frac{k_3}{2\pi}\arctan\theta + b_1 \log\rho^2,$$

$$\frac{\partial u_2}{\partial x_2} = \frac{1}{2\pi\rho^2}(h_2 x_1 + 4\pi b_4 x_2 + k_3 {x_1}^2 + 4\pi b_1 x_1 x_2 + 4\pi b_2 x_2^2 - k_1 x_1 x_3 + 4\pi b_3 x_2 x_3) + + b_2 \log\rho^2, \tag{5}$$

$$\frac{\partial u_2}{\partial x_3} = -\frac{k_1}{2\pi}\arctan\theta + b_3 \log\rho^2,$$

$$\frac{\partial u_3}{\partial x_1} = \frac{1}{2\pi\rho^2}(4\pi l_4 x_1 - h_3 x_2 + k_2 x_1 x_2 - k_1 x_2^2 + 4\pi l_1 x_1^2 + 4\pi l_2 x_1 x_2 + 4\pi l_3 x_1 x_3) - - \frac{k_2}{2\pi}\arctan\theta + l_1 \log\rho^2,$$

$$\frac{\partial u_3}{\partial x_2} = \frac{1}{2\pi\rho^2}(h_3 x_1 + 4\pi l_4 x_2 - k_2 x_1^2 + (k_1 + 4\pi l_1) x_1 x_2 + 4\pi l_2 x_2^2 + 4\pi l_3 x_3) + + \frac{k_1}{2\pi}\arctan\theta + l_2 \log\rho^2,$$

$$\frac{\partial u_3}{\partial x_3} = l_3 \log\rho^2$$

and the components of the strain tensor

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \mathbf{u}^T + \nabla \mathbf{u}) \quad \Leftrightarrow \quad \epsilon_{hk} = \frac{1}{2}\left(\frac{\partial u^h}{\partial x^k} + \frac{\partial u^k}{\partial x^h}\right)$$

are [8]

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{2\pi\rho^2}(4\pi a_4 x_1 + 4\pi a_1 {x_1}^2 - h_1 x_2 + 4\pi a_2 x_1 x_2 + k_3 {x_2}^2 + 4\pi a_3 x_1 x_3 - k_2 x_2 x_3) + a_1 \log \rho^2, \\
\epsilon_{22} &= \frac{1}{2\pi\rho^2}(h_2 x_1 + k_3 {x_1}^2 + 4\pi b_4 x_2 + 4\pi b_1 x_1 x_2 + 4\pi b_2 {x_2}^2 - k_1 x_1 x_3 + 4\pi b_3 x_2 x_3) + b_2 \log \rho^2, \\
\epsilon_{33} &= l_3 \log \rho^2, \\
\epsilon_{12} &= \frac{1}{4\pi\rho^2}((h_1 + 4\pi b_4)x_1 + 4\pi b_1 {x_1}^2 - (h_2 - 4\pi a_4)x_2 - 2(k_3 - 2\pi a_1 - 2\pi b_2)x_1 x_2 + \\
&\quad + 4\pi a_2 {x_2}^2 + (k_2 + 4\pi b_3)x_1 x_3 + (k_1 + 4\pi a_3)x_2 x_3) + \frac{1}{2}(a_2 + b_1) \log \rho^2, \\
\epsilon_{13} &= \frac{1}{4\pi\rho^2}(4\pi l_4 x_1 + 4\pi l_1 {x_1}^2 - h_3 x_2 + (k_2 + 4\pi l_2)x_1 x_2 - k_1 {x_2}^2 + 4\pi l_3 x_1 x_3) + \\
&\quad + \frac{1}{2}(a_3 + l_1) \log \rho^2, \\
\epsilon_{23} &= \frac{1}{4\pi\rho^2}(h_3 x_1 - k_2 {x_1}^2 + 4\pi l_4 x_2 + (k_1 + 4\pi l_1)x_1 x_2 + 4\pi l_2 {x_2}^2 + 4\pi l_3 x_2 x_3) + \\
&\quad + \frac{1}{2}(b_3 + l_2) \log \rho^2.
\end{aligned} \tag{6}$$

In analogy to Volterra's procedure, to calculate the unknown constants $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$, l_1, l_2, l_3, l_4, we have to impose the verification of the indefinite equations of equilibrium and the boundary conditions on the field of displacements. [9]

2.1. Constitutive equations. If a homogeneous body, linearly elastic and transversally isotropic, experiences an isotermic displacement at an assigned temperature τ, and departs from its natural state $\mathcal{C}_\tau^*$, then its isotermic strain-energy-function W_τ can be written in the form

$$W_\tau(\boldsymbol{\epsilon}) = \frac{1}{2}A(\epsilon_{11}^2 + \epsilon_{22}^2) + \frac{1}{2}C\epsilon_{33}^2 + (A - 2N)\epsilon_{11}\epsilon_{22} + F(\epsilon_{11} + \epsilon_{22})\epsilon_{33} + 2L(\epsilon_{23}^2 + \epsilon_{13}^2) + 2N\epsilon_{12}^2$$

[8] In engineering practice the *characteristics of strain*, $e_{hk} = \epsilon_{hk}$ if $h = k$, $e_{hk} = 2\epsilon_{hk}$ if $h \neq k$, are usually used (see e.g. [1, p. 39], art.10).

[9] It's evident the strain (6) proves to be congruent, in that De Saint-Venant's conditions of congruence are automatically verified, independently of the value of the unknown constants.

(see e.g. [1, p. 160], (16) or [7], Capter V, §2), where the coefficients A, C, F, L, N are the elastic constants[10] of the cylinder $\mathcal{C}_\tau$ and are, by hypotesis, not vanishing and different from each other.

Given the tensor field of stress $\boldsymbol{\sigma} \equiv ||\sigma_{hk}^{(\tau)}||$ at a temperature τ, the constitutive equations take the form[11]

$$\sigma_{hh}^{(\tau)} = -\frac{\partial W_\tau}{\partial \epsilon_{hh}}, \quad h = 1,2,3, \quad \sigma_{hk}^{(\tau)} = -\frac{1}{2}\frac{\partial W_\tau}{\partial(\epsilon_{hk})}, \quad (h,k) = (2,3),(1,3),(1,2) \tag{7}$$

(see e.g. [1, 5]).

Writing (7) in an explicit form, we obtain the stress-strain relations:

$$\begin{aligned} \sigma_{11}^{(\tau)} &= -A\epsilon_{11} - (A-2N)\epsilon_{22} - F\epsilon_{33}, \quad \sigma_{12}^{(\tau)} = -2N\epsilon_{12}, \\ \sigma_{22}^{(\tau)} &= -(A-2N)\epsilon_{11} - A\epsilon_{22} - F\epsilon_{33}, \quad \sigma_{13}^{(\tau)} = -2L\epsilon_{13}, \\ \sigma_{33}^{(\tau)} &= -F(\epsilon_{11}+\epsilon_{22}) - C\epsilon_{33}, \quad \sigma_{23}^{(\tau)} = -2L\epsilon_{23}. \end{aligned} \tag{8}$$

Taking into account the relationship between the displacement gradient and the strain tensor, the preceding relations can also be written:

$$\begin{aligned} \sigma_{11}^{(\tau)} &= -A\frac{\partial u_1}{\partial x_1} - (A-2N)\frac{\partial u_2}{\partial x_2} - F\frac{\partial u_3}{\partial x_3}, & \sigma_{12}^{(\tau)} &= -N\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \\ \sigma_{22}^{(\tau)} &= -A\frac{\partial u_2}{\partial x_2} - (A-2N)\frac{\partial u_1}{\partial x_1} - F\frac{\partial u_3}{\partial x_3}, & \sigma_{13}^{(\tau)} &= -L\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \\ \sigma_{33}^{(\tau)} &= -F\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) - C\frac{\partial u_3}{\partial x_3}, & \sigma_{23}^{(\tau)} &= -L\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \end{aligned} \tag{9}$$

(see e.g. [5]).

Through the stress-strain relations (8) or the preceding equations (10), we obtain the following explicit expression for the components of the stress tensor $\boldsymbol{\sigma}$ (relative to the field of di-

[10] Since the cylinder is homogeneous, the coefficients A, C, F, L, N are constant.

[11] The signs of σ_{hk} are chosen conforming to [4, 7], so a pressure is a positive stress and a tension is a negative stress, as it is usual in theoretical mechanics (see e.g. [8]). Many authors define the stress tensor with the opposite sign from the definition adopted here (see e.g. [1]), as it is almost universal in engineering practice.

splacements (4))

$$\sigma_{11}^{(\tau)} = \frac{1}{2\pi\rho^2}(((A-2\,N)\,h_2+4\,\pi\,A\,a_4)\,x_1+(-A\,h_1+4\,\pi\,b_4\,(A-2\,N))\,x_2+((A-2\,N)k_3+$$

$$+\,4\,\pi\,A\,a_1)x_1{}^2+4\,\pi(A\,a_2+(A\,-2\,N)\,b_1)\,x_1\,x_2+(A\,k_3-\,\pi\,(N-\,A)\,b_2)\,x_2{}^2+$$

$$+\,((2\,N-A)\,k_1+4\,\pi\,A\,a_3)\,x_1\,x_3+(-A\,k_2+4\,\pi(\,A-2\,N\,)\,b_3)\,x_2\,x_3)-$$

$$-\,(Aa_1+(2N-A)b2+Fl_3)\log\rho^2,$$

$$\sigma_{22}^{(\tau)} = -\,\frac{1}{2\pi\rho^2}\,(A\,h_2+4\,a_4\,\,(A-2\,N)\,\,\pi)\,x_1+4a_1\,\,(A-2\,N)\,\pi x_1{}^2+(h_1\,(-A+2\,N)+$$

$$+\,4\,A\,b_4\,\pi)\,x_2+4\,(A\,b_1+a_2\,\,(A-2\,N))\,\pi\,x_1\,x_2-2\,(k_3\,N-2\,A\,b_2\,\pi)\,x_2{}^2+$$

$$+\,(-A\,k_1+4\,a_3\,\,(A-2\,N)\,\,\pi)\,\,x_1\,x_3+(k_2\,\,(-A+2\,N)+4\,A\,b_3\,\pi)\,\,x_2\,x_3)-\frac{k_3}{2\,\pi}\,A-$$

$$-\,(A\,b_2+F\,l_3+a_1\,\,(A-2\,N))\,\,\log\rho^2,$$

$$\sigma_{33}^{(\tau)} = -\,F\frac{k_3}{2\pi}-\frac{F}{2\pi\rho^2}((h_2+4\,a_4\,\pi)\,x_1+4\,a_1\,\pi\,x_1{}^2+(-h_1+4\,b_4\,\pi)\,\,x_2+4\,(a_2+b_1)\,\,\pi\,x_1\,x_2+$$

$$+\,4\,b_2\,\pi\,x_2{}^2\,(-k_1+4\,a_3\,\pi)\,\,x_1\,x_3+(-k_2+4\,b_3\,\pi)\,\,x_2\,x_3)-(F\,(a_1+b_2)+C\,l_3)\,\,\log\rho^2, \tag{10}$$

$$\sigma_{12}^{(\tau)} = -\,\frac{N}{2\pi\rho^2}((h_1+4\,b_4\,\pi)\,x_1+4\,b_1\,\pi\,x_1{}^2+(-h_2+4\,a_4\,\pi)\,x_2+(-2\,k_3+4\,a_1\,\pi+$$

$$+\,4\,b_2\,\pi)\,x_1\,x_2+4\,a_2\,\pi\,x_2{}^2+(k_2+4\,b_3\,\pi)\,x_1\,x_3+(k_1+4\,a_3\,\pi)\,x_2\,x_3)-$$

$$-\,(a_2+b_1)\,N\,\,\log\rho^2,$$

$$\sigma_{13}^{(\tau)} = -\,\frac{L}{2\pi\rho^2}\left(4\,l_4\,\pi\,x_1+4\,l_1\,\pi\,x_1{}^2-h_3\,x_2+(k_2+4\,l_2\,\pi)\,\,x_1\,x_2-k_1\,x_2{}^2+4\,l_3\,\pi\,x_1\,x_3\right)-$$

$$-\,L\,\,(a_3+l_1)\,\,\log\rho^2,$$

$$\sigma_{23}^{(\tau)} = -\,\frac{L}{2\pi\rho^2}\left(h_3\,x_1-k_2\,x_1{}^2+4\,l_4\,\pi\,x_2+(k_1+4\,l_1\,\pi)x_1\,x_2+4\,l_2\,\pi\,x_2{}^2+4\,l_3\,\pi\,x_2\,x_3\right)-$$

$$-\,L\,\,(b_3+l_2)\,\,\log\rho^2.$$

Remark. Since the cylinder is transversally isotropic, the stress-strain relation (8) are invariant with respect to the exchange of the axes x_1 and x_2. We can obtain from (8), (10) the stress-strain relations for a homogeneous and isotropic cylinder by making them invariant with respect to the exchange of any pair of axes or to the directional change of any one of the axes (see e.g. [1, p. 102]). The five elastic constants therefore, must verify the following three conditions of isotropy:

$$C = A, \qquad L = N, \qquad F = A - 2N \tag{11}$$

reduce the independent elastic constants to only A and N.

2.2. Indefinite equation. Because $\mathcal{C}$ is initially found in a natural state, Cauchy's static equations, in absence of the force of mass, must be verified:

$$\operatorname{div} \boldsymbol{\sigma} = 0 \qquad \forall P^* \in \mathcal{C}^*_\tau. \tag{12}$$

If (12) is projected onto the axes, and expressions (9) of the stress-tensor are taken into account, we obtain

$$\begin{gathered}
A\frac{\partial^2 u_1}{(\partial x_1)^2} + N\frac{\partial^2 u_1}{(\partial x_2)^2} + L\frac{\partial^2 u_1}{(\partial x_3)^2} + (A-N)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + (F+L)\frac{\partial^2 u_3}{\partial x_1 \partial x_3} = 0, \\
N\frac{\partial^2 u_2}{(\partial x_1)^2} + A\frac{\partial^2 u_2}{(\partial x_2)^2} + L\frac{\partial^2 u_2}{(\partial x_3)^2} + (A-N)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + (F+L)\frac{\partial^2 u_3}{\partial x_2 \partial x_3} = 0, \\
L\frac{\partial^2 u_3}{(\partial x_1)^2} + L\frac{\partial^2 u_3}{(\partial x_2)^2} + C\frac{\partial^2 u_3}{(\partial x_3)^2} + (F+L)\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + (F+L)\frac{\partial^2 u_2}{\partial x_2 \partial x_3} = 0
\end{gathered} \tag{13}$$

(see e.g. [5]).

Referring to (13) or the expressions of stress (10), the equations of equilibrium (12) can be written in the explicit forms:

$$\begin{aligned}
(\operatorname{div} \boldsymbol{\sigma})_1 = \frac{1}{2\pi\rho^4}&((A-N)(4\pi a_4 + h_2){x_1}^2 + 2(A-N)(4\pi b_4 - h_1)x_1 x_2 - \\
&- (A-N)(4\pi a_4 + h_2){x_2}^2 - 2(2\pi a_1(A+N) + 2\pi(A-N)b_2 - Nk_3 + \\
&+ 2\pi(F+L)l_3){x_1}^3 - 4\pi((3N-A)a_2 - (A-N)b_1){x_1}^2 x_2 + \\
&+ (A-N)(4\pi a_3 - k_1){x_1}^2 x_3 + 2(2\pi(N-3A)a_1 + 2\pi(A-N)b_2 + Nk_3 - \\
&- 2\pi(F+L)l_3)x_1 {x_2}^2 + 2(A-N)(4\pi b_3 - k_2)x_1 x_2 x_3 - 4\pi((A+N)a_2 + \\
&+ (A-N)b_1){x_2}^3 - (A-N)(4\pi a_3 - k_1){x_2}^2 x_3) = 0,
\end{aligned} \tag{14}$$

$$(\operatorname{div}\boldsymbol{\sigma})_2 = \frac{1}{2\pi\rho^4}((A-N)(h_1-4\pi b_4){x_1}^2 + 2(A-N)(h_2+4\pi a_4)x_1x_2 + (A-N)(4\pi b_4 -$$

$$- h_1){x_2}^2 - 4\pi((A-N)a_2 + (A+N)b_1){x_1}^3 + 2(2\pi(A-N)a_1 + 2\pi(-3A +$$

$$+ N)b_2 + Nk_3 - 2\pi(F+L)l_3){x_1}^2x_2 + (A-N)(k_2-4\pi b_3){x_1}^2x_3 +$$

$$+ 4\pi((A-N)a_2 + (A-3N)b_1)x_1{x_2}^2 + 2(A-N)(4\pi a_3 - k_1)x_1x_2x_3 -$$

$$- 2(2\pi(A-N)a_1 + 2\pi(A+N)b_2 - Nk_3 + 2\pi(F+L)l_3){x_2}^3 -$$

$$- (A-N)(4\pi b_3 - k_2){x_2}^2x_3) = 0, \tag{15}$$

$$(\operatorname{div}\boldsymbol{\sigma})_3 = -\frac{1}{2\pi\rho^2}((4\pi(F+L)a_3 + (L-F)k_1 + 8\pi Ll_1)x_1 + (4\pi(F+L)b_3 +$$

$$+ (L-F)k_2 + 8\pi Ll_2)x_2) = 0. \tag{16}$$

If we make the coefficients of the various monomials $(x_1)^j(x_2)^k(x_3)^l$ equal to zero, we arrive at a system of linear equations, not all independent, for the unknowns a_h, b_h, l_h, $h = 1,2,3,4$. By annulling the coefficients of the monomials $x_1^2x_3$ (or $x_2^2x_3$), x_1^2 (or x_2^2), $x_1x_2x_3$, x_1x_2, in (14) (or in (15)) we find respectively (in the aforementioned hypothesis $A \neq N$)

$$a_3 = \frac{k_1}{4\pi}, \quad a_4 = -\frac{h_2}{4\pi}, \quad b_3 = \frac{k_2}{4\pi}, \quad b_4 = \frac{h_1}{4\pi}. \tag{17}$$

If we annul the coefficients of the monomials $x_1^2x_2$, x_2^3 in (14) (or the coefficients of the monomials $x_1x_2^2$, x_1^3 in (15)), we obtain the homogeneous system

$$(3N-A)a_2 + (N-A)b_1 = 0,$$

$$(N+A)a_2 + (N-A)b_1 = 0$$

where the only solution is $a_2 = 0,\ b_1 = 0$ (in the same hypothesis $A \neq N$). If we annul the coefficient of x_1 in (16), we obtain the equation

$$4\pi(F+L)a_3 - (F-L)k_1 + 8\pi Ll_1 = 0.$$

And if we take into account the calculated value of a_3, we find $l_1 = -\dfrac{k_1}{4\pi}$. By annulling the coefficient of x_2 in (16), we obtain the equation

$$4\pi(F+L)b_3 - (F-L)k_2 + 8\pi Ll_2 = 0.$$

And by taking into account the calculated value of b_3, we find $l_2 = -\dfrac{k_2}{4\pi}$.

Finally, by annulling the coefficients of the monomials x_1^3, $x_1x_2^2$ in (14) and the coefficients of the monomials x_2^3, $x_1^2x_2$ in (15), we obtain the following subsystem of four linear equations in the three unknowns a_1, b_2, l_3:

$$\begin{aligned}
2\pi[(A+N)a_1 + (A-N)b_2 + (F+L)l_3] &= Nk_3, \\
2\pi(3A-N)a_1 - 2\pi(A-N)b_2 + 2\pi(F+L)l_3 &= Nk_3, \\
2\pi(A-N)a_1 + 2\pi(A+N)b_2 + 2\pi(F+L)l_3 &= Nk_3, \\
2\pi(N-A)a_1 + 2\pi(3A-N)b_2 + 2\pi(F+L)l_3 &= Nk_3
\end{aligned} \tag{18}$$

which has the ∞ solutions[12]

$$b_2 = a_1, \qquad l_3 = \frac{Nk_3 - 4\pi A a_1}{2\pi(F+L)}. \tag{19}$$

In summary we can say that the displacement (3) satisfies the indefinite equations of elastic equilibrium, and therefore can be considered as an elastic displacement in an equilibrium problem if the following conditions are verified (and $A \neq N$):

$$\begin{aligned}
&a_2 = 0, \quad a_3 = \frac{k_1}{4\pi}, \quad a_4 = -\frac{h_2}{4\pi}, \\
&b_1 = 0, \quad b_2 = a_1, \quad b_3 = \frac{k_2}{4\pi}, \quad b_4 = \frac{h_1}{4\pi}, \\
&l_1 = -a_3 = -\frac{k_1}{4\pi}, \quad l_2 = -b_3 = -\frac{k_2}{4\pi}, \quad l_3 = \frac{Nk_3 - 4\pi A a_1}{2\pi(F+L)},
\end{aligned} \tag{20}$$

while the constants a_1 and l_4 are still undetermined.

So the Cartesian components of the displacement (3), which satisfy the indefinite equations (12), take the form

$$\begin{aligned}
u_1 &= \frac{1}{2\pi}(h_1 - k_3x_2 + k_2x_3)\,\theta + \left(-\frac{h_2}{4\pi} + \frac{k_1}{4\pi}x_3 + a_1x_1\right)\log\rho^2, \\
u_2 &= \frac{1}{2\pi}(h_2 + k_3x_1 - k_1x_3)\,\theta + \left(\frac{h_1}{4\pi} + \frac{k_2}{4\pi}x_3 + a_1x_2\right)\log\rho^2, \\
u_3 &= \frac{1}{2\pi}(h_3 - k_2x_1 + k_1x_2)\,\theta - \left(\frac{k_1}{4\pi}x_1 + \frac{k_2}{4\pi}x_2 - l_3x_3 - l_4\right)\log\rho^2
\end{aligned} \tag{21}$$

where l_3 has the expression (19).

[12] It is sufficient to consider any two of (18) to obtain $(19)_1$. Therefore (18) reduce to the unique equation $4\pi A a_1 - Nk_3 + 2\pi(F+L)l_3 = 0$.

2.2.1. Deformation, stress and forces on the boundary. If (21) are introduced into (6), we find the following expression for strain tensor, relative to the vector field of displacements (21)

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{2\pi\rho^2}(-h_2x_1 - h_1x_2 + 4\pi a_1 {x_1}^2 + k_3 {x_2}^2 + k_1x_1x_3 - k_2x_2x_3) + a_1 \log\rho^2, \\
\epsilon_{22} &= \frac{1}{2\pi\rho^2}(h_2x_1 + h_1x_2 + 4\pi a_1 {x_2}^2 + k_3 {x_1}^2 - k_1x_1x_3 + k_2x_2x_3) + a_1 \log\rho^2, \\
\epsilon_{33} &= l_3 \log\rho^2, \\
\epsilon_{12} &= \frac{1}{2\pi\rho^2}(h_1x_1 - h_2x_2 - (k_3 - 4\pi a_1)x_1x_2 + k_2x_1x_3 + k_1x_2x_3, \\
\epsilon_{13} &= \frac{1}{4\pi\rho^2}(4\pi l_4x_1 - h_3x_2 + 4\pi l_3x_1x_3) - \frac{k_1}{4\pi}, \\
\epsilon_{23} &= \frac{1}{4\pi\rho^2}(4\pi l_4x_2 + h_3x_1 + 4\pi l_3x_2x_3) - \frac{k_2}{4\pi}.
\end{aligned} \tag{22}$$

If (22) are substituted into (10), which express the components of the stress tensor by those of the tensor of deformation, we obtain the stress which satisfies Cauchy's equations of equilibrium (12):

$$\begin{aligned}
\sigma_{11}^{(\tau)} &= \frac{N}{\pi\rho^2}\left(h_2x_1 + h_1x_2 + k_3{x_1}^2 + 4\pi a_1{x_2}^2 - k_1x_1x_3 + k_2x_2x_3\right) - \\
&\quad - \frac{A}{2\pi}(k_3 + 4\pi a_1) + [2a_1(N-A) - Fl_3]\log\rho^2, \\
\sigma_{22}^{(\tau)} &= \frac{N}{\pi\rho^2}\left(-h_2x_1 - h_1x_2 + k_3{x_2}^2 + 4\pi a_1{x_1}^2 + k_1x_1x_3 - k_2x_2x_3\right) - \\
&\quad - \frac{A}{2\pi}(k_3 + 4\pi a_1) + [2a_1(N-A) - Fl_3]\log\rho^2, \\
\sigma_{33}^{(\tau)} &= -\frac{F}{2\pi}(k_3 + 4\pi a_1) - (2Fa_1 - Cl_3)\log\rho^2, \\
\sigma_{12}^{(\tau)} &= \frac{N}{\pi\rho^2}\left[-h_1x_1 + h_2x_2 + (k_3 - 4\pi a_1)x_1x_2 - k_2x_1x_3 - k_1x_2x_3\right], \\
\sigma_{13}^{(\tau)} &= \frac{L}{2\pi\rho^2}(h_3x_2 - 4\pi l_4x_1 - 4\pi l_3x_1x_3) + \frac{Lk_1}{2\pi}, \\
\sigma_{23}^{(\tau)} &= \frac{L}{2\pi\rho^2}(-h_3x_1 - 4\pi l_4x_2 - 4\pi l_3x_2x_3) + \frac{Lk_2}{2\pi}.
\end{aligned} \tag{23}$$

2.2.2. Boundary conditions. Finally, the boundary conditions must be verified,

$$\boldsymbol{\sigma}\cdot\mathbf{n}-\mathbf{f}=\mathbf{0} \quad \Longleftrightarrow \quad \sigma_{h_1}n_1+\sigma_{h_2}n_2+\sigma_{h_3}n_3=f_h, \quad h=1,2,3, \qquad \forall Q^*\in\partial\mathcal{C}^*_\tau, \tag{24}$$

where $\mathbf{f}(\mathbf{Q}^*)$ are the vectorial surface forces and $\mathbf{n}\equiv(n_1,n_2,n_3)$ is the unitary vector normal to $\partial\mathcal{C}^*_\tau$, with an internal orientation.

To calculate the forces which we must apply on the boundary of the cylinder to determine the principal displacement (21), we divide the boundary $\partial\mathcal{C}^*_\tau$ into the parts $\Sigma^*{}_1$, $\Sigma^*{}_2$, $\alpha^*{}_1$ and $\alpha^*{}_2$. By taking into account (23) and projecting (24) onto the axes, we obtain four groups of equations.

First we consider the boundary conditions on the two lateral surfaces:

1) on $\Sigma^*{}_1 \quad (\,\mathbf{n}=\mathbf{c_1}\cos\theta+\mathbf{c_2}\sin\theta, \quad x_1=r\cos\theta,\ x_2=r\sin\theta,\ 0\le x_3\le d\,)$

$$\begin{aligned}
\left[\sigma^{(\tau)}_{11}\cos\theta+\sigma^{(\tau)}_{12}\sin\theta\right]_{\Sigma^*{}_1} &= \frac{1}{2\pi r}[2Nh_2+((2N-A)k_3-4\pi Aa_1)x_1-2Nk_1x_3+\\
&\quad+4\pi(N-A)a_1x_1\log\rho^2-2\pi Fl_3x_1\log\rho^2]_{\Sigma^*{}_1}=(f_1)_{\Sigma^*{}_1},\\
\left[\sigma^{(\tau)}_{12}\cos\theta+\sigma^{(\tau)}_{22}\sin\theta\right]_{\Sigma^*{}_1} &= \frac{1}{2\pi r}[-2Nh_1+((2N-A)k_3-4\pi Aa_1)x_2-2Nk_2x_3+\\
&\quad+4\pi(N-A)a_1x_2\log\rho^2-2\pi Fl_3x_2\log\rho^2]_{\Sigma^*{}_1}=(f_2)_{\Sigma^*{}_1},\\
\left[\sigma^{(\tau)}_{13}\cos\theta+\sigma^{(\tau)}_{23}\sin\theta\right]_{\Sigma^*{}_1} &= \frac{L}{2\pi r}[-4\pi l_4+k_1x_1+k_2x_2-4\pi l_3x_3]=(f_3)_{\Sigma^*{}_1};
\end{aligned}\tag{25}$$

2) on $\Sigma^*{}_2 \quad (\,\mathbf{n}=-\mathbf{c_1}\cos\theta-\mathbf{c_2}\sin\theta, \quad x_1=R\cos\theta,\ x_2=R\sin\theta,\ 0\le x_3\le d\,)$

$$\begin{aligned}
\left[-\sigma^{(\tau)}_{11}\cos\theta-\sigma^{(\tau)}_{12}\sin\theta\right]_{\Sigma^*{}_2} &= \frac{1}{2\pi R}[-2Nh_2-((2N-A)k_3-4\pi a_1)x_1+2Nk_1x_3-\\
&\quad-4\pi(N-A)a_1x_1\log\rho^2+2\pi Fl_3x_1\log\rho^2]_{\Sigma^*{}_2}=(f_1)_{\Sigma^*{}_2},\\
\left[-\sigma^{(\tau)}_{12}\cos\theta-\sigma^{(\tau)}_{22}\sin\theta\right]_{\Sigma^*{}_2} &= \frac{1}{2\pi R}[2Nh_1-((2N-A)k_3-4\pi a_1)x_2+2Nk_2x_3-\\
&\quad-4\pi(N-A)a_1x_2\log\rho^2+2\pi Fl_3x_2\log\rho^2]_{\Sigma^*{}_2}=(f_2)_{\Sigma^*{}_2},\\
\left[-\sigma^{(\tau)}_{13}\cos\theta-\sigma^{(\tau)}_{23}\sin\theta\right]_{\Sigma^*{}_2} &= \frac{L}{2\pi R}[4\pi l_4-k_1x_1-k_2x_2+4\pi l_3x_3]=(f_3)_{\Sigma^*{}_2}.
\end{aligned}\tag{26}$$

We can divide the surface forces (25), (26) applied on $\Sigma^*{}_1$ and on $\Sigma^*{}_2$ into the following vector fields, all equivalent to zero:

$i)$ $\left\{P^*, \mathbf{f}^{(1)'}(P^*)\, \mathrm{d}\Sigma^*{}_1\right\}_{\Sigma^*{}_1}$, consisting of couples of zero arms and so equivalent to zero, in fact

$$\left[f_1^{(1)'}(x_1,x_2,x_3)\right]_{\Sigma^*{}_1} = \frac{1}{2\pi r}\left[((2N-A)k_3 - 4\pi A a_1)x_1 - 2Nk_1x_3 + 4\pi(N-A)a_1x_1\log\rho^2 - \right.$$

$$\left. -2\pi F l_3 x_1 \log\rho^2\right]_{\Sigma^*{}_1} = -\left[f_1^{(1)'}(-x_1,-x_2,x_3)\right]_{\Sigma^*{}_1},$$

$$\left[f_2^{(1)'}(x_1,x_2,x_3)\right]_{\Sigma^*{}_1} = \frac{1}{2\pi r}\left[((2N-A)k_3 - 4\pi A a_1)x_2 - 2Nk_2x_3 + 4\pi(N-A)a_1x_2\log\rho^2 - \right.$$

$$\left. -2\pi F l_3 x_2 \log\rho^2\right]_{\Sigma^*{}_1} = -\left[f_2^{(1)'}(-x_1,-x_2,x_3)\right]_{\Sigma^*{}_1},$$

$$\left[f_3^{(1)'}(x_1,x_2,x_3)\right]_{\Sigma^*{}_1} = \frac{L}{2\pi r}\left[-4\pi l_4 + k_1x_1 + k_2x_2 - 4\pi l_3x_3\right] = -\left[f_3^{(1)'}(-x_1,-x_2,x_3)\right]_{\Sigma^*{}_1};$$

$ii)$ $\left\{P^*, \mathbf{f}^{(2)'}(P^*) d\,\Sigma^*{}_2\right\}_{\Sigma^*{}_2}$, which, analogously, consists of couples of zero arms;

iii) the pair of two constant vector fields, parallel and opposite,

$$\left\{\mathbf{f}^{(1)''} d\,\Sigma^*{}_1\right\}_{\Sigma^*{}_1} \equiv \left\{\mathbf{f}^{(1)''} 2\pi r dx_3\right\}_{\Sigma^*{}_1} \quad \text{and} \quad \left\{\mathbf{f}^{(2)''} d\,\Sigma^*{}_2\right\}_{\Sigma^*{}_2} \equiv \left\{\mathbf{f}^{(2)''} 2\pi R dx_3\right\}_{\Sigma^*{}_2}$$

$$\begin{gathered}
(f_1^{(1)})''_{\Sigma^*{}_1} d\,\Sigma^*{}_1 = 2Nh_2dx_3, \quad (f_2^{(1)})''_{\Sigma^*{}_1} d\,\Sigma^*{}_1 = -2Nh_1dx_3, \quad (f_3^{(1)})''_{\Sigma^*{}_1} d\,\Sigma^*{}_1 = -4\pi L l_4 dx_3, \\
(f_1^{(2)})''_{\Sigma^*{}_2} d\,\Sigma^*{}_2 = -2Nh_2dx_3, \quad (f_2^{(2)})''_{\Sigma^*{}_2} d\,\Sigma^*{}_2 = 2Nh_1dx_3, \quad (f_3^{(2)})''_{\Sigma^*{}_2} d\,\Sigma^*{}_2 = 4\pi L l_4 dx_3,
\end{gathered} \tag{27}$$

that together have, by symmetry, their center coincident with the center of the cylinder, and are equivalent to their resultant applied to the center. Since their two resultants $\mathbf{r}''_{(1)}$ and $\mathbf{r}''_{(2)}$ are[13]

$$\begin{gathered}
r_1^{(1)''} = 2Nh_2d\,, \quad r_2^{(1)''} = -2Nh_1d\,, \quad r_3^{(1)''} = -4\pi L l_4 d, \\
r_1^{(2)''} = -2Nh_2d\,, \quad r_2^{(2)''} = 2Nh_1d\,, \quad r_3^{(2)''} = 4\pi L l_4 d,
\end{gathered} \tag{28}$$

it is obvious that, being $\mathbf{r}''_{(1)} + \mathbf{r}''_{(2)} = \mathbf{0}$, also the vector field $\{(P^*, \mathbf{f}^{(1)''})_{\Sigma^*{}_1}, (P^*, \mathbf{f}^{(2)''})_{\Sigma^*{}_2}\}$ is equivalent to a couple of zero arm.

Finally we consider the boundary conditions on the two bases:

[13] $\mathbf{r}^{(1)''} = \int_{\Sigma^*{}_1} \mathbf{f}^{(1)''} d\,\Sigma_1^* = \mathbf{f}^{(1)''} 2\pi r d, \qquad \mathbf{r}^{(2)''} = \int_{\Sigma^*{}_2} \mathbf{f}^{(2)''} d\,\Sigma_2^* = \mathbf{f}^{(2)''} 2\pi R d\,.$

3) on $\alpha^*{}_1$ $\quad(\mathbf{n} = \mathbf{c_3} \equiv (0,0,1)\,, x_1 = \rho\cos\theta,\ x_2 = \rho\sin\theta,\ x_3 = 0)$,

$$[\sigma_{13}^{(\tau)}]_{\alpha^*{}_1} = \frac{L}{2\pi}\left[k_1 + \frac{1}{\rho^2}(h_3x_2 - 4\pi l_4 x_1)\right]_{\alpha^*{}_1} = (f_1)_{\alpha^*{}_1},$$

$$[\sigma_{23}^{(\tau)}]_{\alpha^*{}_1} = \frac{L}{2\pi}\left[k_2 - \frac{1}{\rho^2}(h_3x_1 + 4\pi l_4 x_2)\right]_{\alpha^*{}_1} = (f_2)_{\alpha^*{}_1}, \tag{29}$$

$$[\sigma_{33}^{(\tau)}]_{\alpha^*{}_1} = \left[\frac{F}{2\pi}(k_3 + 4\pi a_1) + (2Fa_1 + Cl_3)\log\rho^2\right]_{\alpha^*{}_1} = (f_3)_{\alpha^*{}_1};$$

4) on $\alpha^*{}_2$ $\quad(\mathbf{n} = -\mathbf{c_3} \equiv (0,0,-1),\ x_1 = \rho\cos\theta,\ x_2 = \rho\sin\theta,\ x_3 = d)$

$$[-\sigma_{13}^{(\tau)}]_{\alpha^*{}_2} = -\frac{L}{2\pi}\left[k_1 + \frac{1}{\rho^2}\left(h_3x_2 - 4\pi(l_3d + l_4)x_1\right)\right]_{\alpha^*{}_2} = (f_1)_{\alpha^*{}_2},$$

$$[-\sigma_{23}^{(\tau)}]_{\alpha^*{}_2} = -\frac{L}{2\pi}\left[k_2 - \frac{1}{\rho^2}\left(h_3x_1 + 4\pi(l_3d + l_4)x_2\right)\right]_{\alpha^*{}_2} = (f_2)_{\alpha^*{}_2}, \tag{30}$$

$$[-\sigma_{33}^{(\tau)}]_{\alpha^*{}_2} = -\left[\frac{F}{2\pi}(k_3 + 4\pi a_1) + (2Fa_1 + Cl_3)\log\rho^2\right]_{\alpha^*{}_2} = (f_3)_{\alpha^*{}_2}.$$

Also the surface forces exerting on the two bases must be equivalent to zero. So, taking into consideration (20), (30), we must put

$$l_3 = 0, \tag{31}$$

and consequently we obtain from (19)

$$a_1 = \frac{N}{A}\frac{k_3}{4\pi}. \tag{32}$$

Condition (31) together with (32) must be verified so that the surface forces corresponding to the dislocation are equivalent to zero, therefore l_3 and a_1 are not arbitrary.

2.3. Vector field of displacements. From (31), (32) we can now write the definitive expression for the vector field of displacements (21),

$$u_1 = \frac{1}{2\pi}\left\{(h_1 - k_3x_2 + k_2x_3)\,\theta + \frac{1}{2}\left(-h_2 + k_1x_3 + \frac{N}{A}k_3x_1\right)\,\log\rho^2\right\},$$

$$u_2 = \frac{1}{2\pi}\left\{(h_2 + k_3x_1 - k_1x_3)\,\theta + \frac{1}{2}\left(h_1 + k_2x_3 + \frac{N}{A}k_3x_2\right)\,\log\rho^2\right\}, \tag{33}$$

$$u_3 = \frac{1}{2\pi}\left\{(h_3 - k_2x_1 + k_1x_2)\,\theta - \frac{1}{2}\left(k_1x_1 + k_2x_2 - 4\pi l_4\right)\,\log\rho^2\right\}.$$

These formulas demonstrate that the vector field of displacements depends exclusively on the ratio N/A of only two of the five elastic constants which characterize a transversally isotropic elastic body.

Remark. If we impose the isotropy conditions (11) on the five elastic constants, and put $A = \lambda + 2\mu$, $N = \mu$ and $l_4 = 0$, we once again obtain, from (33), Volterra's formulas (2) relative to the isotropic case.

2.4. Deformation, stress and surface forces corresponding to the established displacement. If we substitute the values $l_3 = 0$ and $a_1 = \frac{N}{A}\frac{k_3}{4\pi}$ in (5), (22), (23), and (25)–(30), we obtain the following explicit definitive expressions (corresponding to the displacement (33)):

1) for the gradient of displacement $\nabla\mathbf{u}$,

$$\frac{\partial u_1}{\partial x_1} = \frac{1}{2\pi\rho^2}(-h_2\,x_1 - h_1\,x_2 + \frac{N}{A}\,k_3\,{x_1}^2 + k_3\,{x_2}^2 + k_1\,x_1\,x_3 - k_2\,x_2\,x_3) + \frac{N}{A}\,\frac{k_3}{4\pi}\,\log\rho^2,$$

$$\frac{\partial u_1}{\partial x_2} = \frac{1}{2\pi\rho^2}\left(h_1\,x_1 - h_2\,x_2 + \left(\frac{N}{A} - 1\right)k_3\,x_1\,x_2 + k_2\,x_1\,x_3 + k_1\,x_2\,x_3\right) - \frac{k_3}{2\,\pi}\,\arctan\theta,$$

$$\frac{\partial u_1}{\partial x_3} = \frac{k_2}{2\pi}\arctan\theta + \frac{k_1}{4\pi}\log\rho^2,$$

$$\frac{\partial u_2}{\partial x_1} = \frac{1}{2\pi\rho^2}\left(h_1\,x_1 - h_2\,x_2 + \left(\frac{N}{A} - 1\right)\,k_3\,x_1\,x_2 + k_2\,x_1\,x_3 + k_1\,x_2\,x_3\right) + \frac{k_3}{2\pi}\,\arctan\theta,$$

$$\frac{\partial u_2}{\partial x_2} = \frac{1}{2\pi\rho^2}(h_2\,x_1 + h_1\,x_2 + \frac{N}{A}\,k_3\,{x_2}^2 + k_3\,{x_1}^2 - k_1\,x_1\,x_3 + k_2\,x_2\,x_3) + \frac{N}{A}\,\frac{k_3}{4\,\pi}\,\log\rho^2, \quad (34)$$

$$\frac{\partial u_2}{\partial x_3} = -\frac{k_1}{2\pi}\arctan\theta + \frac{k_2}{4\,\pi}\,\log\rho^2,$$

$$\frac{\partial u_3}{\partial x_1} = \frac{1}{2\pi\rho^2}(4\,\pi\,l_4\,x_1 - h_3\,x_2 - k_1\,{x_1}^2 - k_1\,{x_2}^2) - \frac{k_2}{2\pi}\arctan\theta + \frac{k_1}{4\pi}\log\rho^2,$$

$$\frac{\partial u_3}{\partial x_2} = \frac{1}{2\pi\rho^2}(4\,\pi\,l_4\,x_2 + h_3\,x_1 - k_2\,{x_1}^2 - k_2\,{x_2}^2) + \frac{k_1}{2\pi}\arctan\theta - \frac{k_2}{4\pi}\log\rho^2,$$

$$\frac{\partial u_3}{\partial x_3} = 0;$$

2) for the tensor field of strain,

$$\epsilon_{11} = \frac{1}{2\pi\rho^2}\left(-h_2x_1 - h_1x_2 + \frac{N}{A}k_3{x_1}^2 + k_3{x_2}^2 + k_1x_1x_3 - k_2x_2x_3\right) + \frac{N}{A}\frac{k_3}{4\pi}\log\rho^2,$$

$$
\begin{aligned}
\epsilon_{22} &= \frac{1}{2\pi\rho^2}\left(h_2x_1 + h_1x_2 + \frac{N}{A}k_3{x_2}^2 + k_3{x_1}^2 - k_1x_1x_3 + k_2x_2x_3\right) + \frac{N}{A}\frac{k_3}{4\pi}\log\rho^2,\\
\epsilon_{33} &= 0,\\
\epsilon_{12} &= \frac{1}{2\pi\rho^2}\left(h_1x_1 - h_2x_2 + \left(\frac{N}{A}-1\right)k_3x_1x_2 + k_2x_1x_3 + k_1x_2x_3\right),\\
\epsilon_{13} &= \frac{1}{4\pi\rho^2}(4\pi l_4x_1 - h_3x_2) - \frac{k_1}{4\pi},\\
\epsilon_{23} &= \frac{1}{4\pi\rho^2}(4\pi l_4x_2 + h_3x_1) - \frac{k_2}{4\pi};
\end{aligned}
\tag{35}
$$

3) for the tensor field of stress,

$$
\begin{aligned}
\sigma_{11}^{(\tau)} &= \frac{N}{\pi\rho^2}\left(h_2x_1 + h_1x_2 + \frac{N}{A}k_3{x_2}^2 + k_3{x_1}^2 - k_1x_1x_3 + k_2x_2x_3\right) -\\
&\quad -(N+A)\frac{k_3}{2\pi} + (N-A)\frac{N}{A}\frac{k_3}{2\pi}\log\rho^2,\\
\sigma_{22}^{(\tau)} &= -\frac{N}{\pi\rho^2}\left(h_2x_1 - h_1x_2 + \frac{N}{A}k_3{x_1}^2 + k_3{x_2}^2 + k_1x_1x_3 - k_2x_2x_3\right) -\\
&\quad -(N+A)\frac{k_3}{2\pi} + (N-A)\frac{N}{A}\frac{k_3}{2\pi}\log\rho^2,\\
\sigma_{33}^{(\tau)} &= -F\left(\frac{N}{A}+1\right)\frac{k_3}{2\pi} - F\frac{N}{A}\frac{k_3}{2\pi}\log\rho^2,\\
\sigma_{12}^{(\tau)} &= \frac{N}{\pi\rho^2}\left(-h_1x_1 + h_2x_2 - \left(\frac{N}{A}-1\right)k_3x_1x_2 - k_2x_1x_3 - k_1x_2x_3\right),\\
\sigma_{13}^{(\tau)} &= -\frac{L}{2\pi\rho^2}(4\pi l_4x_1 - h_3x_2) + \frac{Lk_1}{2\pi},\\
\sigma_{23}^{(\tau)} &= -\frac{L}{2\pi\rho^2}(4\pi l_4x_2 + h_3x_1) + \frac{Lk_2}{2\pi};
\end{aligned}
\tag{36}
$$

4) for the surface forces, respectively, on

$$
{\Sigma^*}_1 = \begin{cases}
(f_1)_{{\Sigma^*}_1} = \dfrac{1}{2\pi r}\left[2Nh_2 + \left(\dfrac{N}{A}-1\right)(A + N\log\rho^2)k_3x_1 - 2Nk_1x_3\right]_{{\Sigma^*}_1},\\
(f_2)_{{\Sigma^*}_1} = \dfrac{1}{2\pi r}\left[-2Nh_1 + \left(\dfrac{N}{A}-1\right)(A + N\log\rho^2)k_3x_2 - 2Nk_2x_3\right]_{{\Sigma^*}_1},\\
(f_3)_{{\Sigma^*}_1} = \dfrac{L}{2\pi r}[-4\pi l_4 + k_1x_1 + k_2x_2]_{{\Sigma^*}_1},
\end{cases}
\tag{37}
$$

$$\Sigma^*{}_2 = \begin{cases} (f_1)_{\Sigma^*{}_2} = & \dfrac{1}{2\pi R}\left[-2Nh_2 - \left(\dfrac{N}{A}-1\right)(A+N\log\rho^2)k_3x_1 + 2Nk_1x_3\right]_{\Sigma^*{}_2}, \\ (f_2)_{\Sigma^*{}_2} = & \dfrac{1}{2\pi R}\left[2Nh_1 - \left(\dfrac{N}{A}-1\right)(A+N\log\rho^2)k_3x_2 + 2Nk_2x_3\right]_{\Sigma^*{}_2}, \\ (f_3)_{\Sigma^*{}_2} = & \dfrac{L}{2\pi R}[4\pi l_4 - k_1x_1 - k_2x_2]_{\Sigma^*{}_2}, \end{cases} \tag{38}$$

$$\alpha^*{}_1 = \begin{cases} (f_1)_{\alpha^*{}_1} = & \dfrac{L}{2\pi}\left[k_1 + \dfrac{1}{\rho^2}(h_3x_2 - 4\pi l_4x_1)\right]_{\alpha^*{}_1}, \\ (f_2)_{\alpha^*{}_1} = & \dfrac{L}{2\pi}\left[k_2 - \dfrac{1}{\rho^2}(h_3x_1 + 4\pi l_4x_2)\right]_{\alpha^*{}_1}, \\ (f_3)_{\alpha^*{}_1} = & -\dfrac{Fk_3}{2\pi}\left[1 + \left(1+\log\rho^2\right)\dfrac{N}{A}\right]_{\alpha^*{}_1}, \end{cases} \tag{39}$$

$$\alpha^*{}_2 = \begin{cases} (f_1)_{\alpha^*{}_2} = & \dfrac{L}{2\pi}\left[-k_1 - \dfrac{1}{\rho^2}(h_3x_2 - 4\pi l_4x_1)\right]_{\alpha^*{}_2}, \\ (f_2)_{\alpha^*{}_2} = & \dfrac{L}{2\pi}\left[-k_2 + \dfrac{1}{\rho^2}(h_3x_1 + 4\pi l_4x_2)\right]_{\alpha^*{}_2}, \\ (f_3)_{\alpha^*{}_2} = & \dfrac{Fk_3}{2\pi}\left[1 + \left(1+\log\rho^2\right)\dfrac{N}{A}\right]_{\alpha^*{}_2}. \end{cases} \tag{40}$$

Conclusion. If a hollow elastic cylinder, homogeneous and transversally isotropic $\mathcal{C}$, is initially found in a natural state $\mathcal{C}^*_\tau$ and experiences a many-valued isotermic displacement $\mathcal{C}^*_\tau \to \mathcal{C}_\tau$ as in (21) (and consequently a regular deformation), then, in the equilibrium configuration $\mathcal{C}_\tau$, stress (36) and congruent deformation (35) are present in every internal point; and surface forces (37) – (40) equivalent to zero are exerted on the boundary.

Remark. While displacement (33) and tensor field of strain (35)) depend only on the ratio N/A, the tensor field of stress (36) depends on four of the five elastic constants.

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