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ON THE LAGRANGIAN AND HAMILTONIAN ASPECTS OF INFINITE-DIMENSIONAL DYNAMICAL SYSTEMS AND THEIR FINITE-DIMENSIONAL REDUCTIONS

ПРО ЛАГРАНЖЕВІ ТА ГАМІЛЬТОНОВІ АСПЕКТИ НЕСКІНЧЕННОВИМІРНИХ ДИНАМІЧНИХ СИСТЕМ ТА ЇХ СКІНЧЕННОВИМІРНУ РЕДУКЦІЮ

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Some aspects of the description of Lagrangian and Hamiltonian formalisms naturally arising from the invariance structure of given nonlinear dynamical systems on the infinite-dimensional functional manifold is presented. The basic ideas used to formulate the canonical symplectic structure are borrowed from the Cartan's theory of differential systems on the associated jet-manifolds. The symmetry structure reduced on the invariant submanifolds of critical points of some nonlocal Euler – Lagrange functional is described thoroughly for both differential and differential discrete dynamical systems. The Hamiltonian representation for a hierarchy of Lax-type equations on a dual space to the Lie algebra of integral-differential operators with matrix coefficients, extended by evolutions for eigenfunctions and adjoint eigenfunctions of the corresponding spectral problems, is obtained via some special Backlund transformation. The connection of this hierarchy with integrable by Lax spatially two-dimensional systems is studied.

Наведено деякі аспекти опису лагранжевого та гамільтонового формалізму, який природно виникає із структури інваріантності заданих нелінійних динамічних систем на нескінченновимірному функціональному многовиді. Основні ідеї, які використовуються для формування канонічної симплектичної структури, взято з теорії Картана диференціальних систем на відповідних многовидах струмів. Для диференціальних та диференціальних дискретних динамічних систем наведено детальний опис структури симетрій, які редуковані на інваріантні підмноговиди критичних точок деяких нелокальних ейлерово-лагранжевих функціоналів. За допомогою деякого перетворення Беклунда отримано гамільтонове зображення для ієрархії рівнянь лаксового типу на двоїстому до алгебри Лі просторі інтегрально-диференціальних операторів з матричними коефіцієнтами, яке продовжено еволюціями власних функцій та спряжених власних функцій відповідних спектральних задач. Вивчено зв'язок між цією ієрархією та інтегровними за Лаксом просторово-двовимірними системами.

1. Introduction. One of the fundamental problems in modern theory of infinite-dimensional dynamical systems is that of their invariant reduction on some invariant submanifolds with enough rich mathematical structures as to treat their properties analytically. The first approaches to these problems were suggested already at the end of the preceding century, in the classical works of S. Lie, J. Liouville, J. Lagrange, V. R. Hamilton, J. Poisson and E. Cartan. They introduced at first the important concepts of symmetry, conservation law, symplectic, Poisson and Hamiltonian structures as well as invariant reduction procedure, which appeared to be extremely useful for the proceeding studies. These notions were widely generalized further by Souriau [1], Marsden and Weinstein [2, 3], Lax [4], Bogoyavlensky and Novikov [5], as well as by many other researchers [6 – 10]. It seems worthwhile to mention here also the recent enough studies in [11 – 18], where special reduction methods were proposed for integrable nonlinear dynamical systems on both functional and operator manifolds. In the present paper we describe in detail the reduction procedure for infinite dimensional dynamical systems on an invariant set of critical points of some global invariant functional. The method uses the Cartan's differential-geometric treating of differential ideals in Grassmann algebra over the associated jet-manifold. As one of the main results, we show also that both the reduced dynamical systems and their symmetries generate Hamiltonian flows on the invariant critical submanifolds of local and nonlocal functionals with respect to the canonical symplectic structure on it. These results are generalized to the case of differential-difference dynamical systems that are given on discrete infinite-dimensional manifolds. The direct procedure to construct the invariant Lagrangian functionals for a given a priori Lax-type integrable dynamical system is presented for both the differential and the differential-difference cases of the manifold *M*. Some remarks on the Lagrangian and Hamiltonian formalisms, concerned with infinite-dimensional dynamical systems with symmetries are given. The Hamiltonian representation for a hierarchy of Lax-type equations on a space dual to the Lie algebra of integral-differential operators with matrix coefficients, extended by evolutions for eigenfunctions and adjoint eigenfunctions of the corresponding spectral problems, is obtained via some special Backlund transformation. The connection of this hierarchy with Lax integrable spatially two-dimensional systems is studied.

2. General setting. We are interested in treating a given nonlinear dynamical system

$$\frac{du}{dt} = K[u], \tag{2.1}$$

with respect to an evolution parameter $t \in \mathbb{R}$ on an infinite-dimensional functional manifold $M \subset C^{(\infty)}(\mathbb{R};\mathbb{R}^m)$, possessing two additional ingredients: a homogeneous and autonomous conservation law $\mathcal{L} \in D(M)$ and a number of homogeneous autonomous symmetries $du/dt_j = K_j[u]$, $j = \overline{1,k}$, with evolution parameters $t_j \in \mathbb{R}$. The dynamical system (2.1) is not-supposed to be in general Hamiltonian, all the maps $K, K_j : M \to T(M)$, $j = \overline{1,k}$, are considered to be smooth and well-defined on M.

To pose the problem to be discussed further more definitely, let us use the jet-manifold $J^{(\infty)}(\mathbb{R};\mathbb{R}^m)$, locally isomorphic to the functional manifold M mentioned above. This means the following: the vector field (2.1) on M is completely equivalent to that on the jet-manifold $J^{(\infty)}(\mathbb{R};\mathbb{R}^m)$ via the representation [19, 20]

$$(M \ni u \to K[u]) \xrightarrow{\text{jet}} \Big(K(u, u^{(1)}, \ldots, u^{(n+1)}) \leftarrow (x;\, u, u^{(1)}, \ldots, u^{(\infty)}) \in J^{(\infty)}(\mathbb{R};\mathbb{R}^m)\Big), \tag{2.2}$$

where $n \in \mathbf{Z}_+$ is fixed, $x \in \mathbb{R}$ is the function parameter of the jet-bundle $J^{(\infty)}(\mathbb{R};\mathbb{R}^m) \xrightarrow{\pi} \mathbb{R}$, and π is the usual projection on the base $\mathbb{R}$. Let us allow also that the smooth functional $\mathcal{L} \in D(M)$ is a conservation law of the dynamical system (2.1), that is, $d\mathcal{L}/dt = 0$ along orbits of (2.1) for all $t \in \mathbb{R}$. Due to the jet-representation (2.2) we can write the density of the functional $\mathcal{L} \in D(M)$ in the following form:

$$\mathcal{L} = \int\limits_{\mathbb{R}} dx \mathcal{L}[u], \tag{2.3}$$

with $\mathbb{R}\times\mathbb{R}^m \ni [x;u] \xrightarrow{\text{jet}} (x;u,u^{(1)},\ldots,u^{(N+1)}) \in J^{(N+1)}(\mathbb{R};\mathbb{R}^m)$ being the standard jet-mapping and the number $N \in \mathbf{Z}_+$ fixed. Besides, the functional (2.3) will be assumed to be nondegenerate in the sense that the Hessian of $\mathcal{L} : J^{(N+1)}(\mathbb{R};\mathbb{R}^m) \to \mathbb{R}$ has nonvanishing determinant, $\det \left\| \dfrac{\partial^2 \mathcal{L}\left(u,u^{(1)},\ldots,u^{(N+1)}\right)}{\partial u^{(N+1)}\partial u^{(N+1)}} \right\| \neq 0$.

3. Lagrangian reduction. Consider now the set of critical points $M_n \subset M$ of the functional $\mathcal{L} \in D(M)$,

$$M_N = \{u \in M : \operatorname{grad} \ \mathcal{L}[u] = 0\}, \tag{3.1}$$

where, due to (2.2), $\operatorname{grad} \mathcal{L}[u] := \delta\mathcal{L}(u,\ldots,u^{(N+1)})/\delta u$ is the Euler variational derivative. As proved by Lax [4], the manifold $M_N \subset M$ is smoothly imbedded and well-defined, due to the condition $\operatorname{Hess}\mathcal{L} \neq 0$. Besides, the manifold M_N appears to be invariant with respect to the dynamical system (2.1). This means in particular that the Lie derivative of any vector field $X : M \to T(M)$, tangent to the manifold M_N, with respect to the vector field (2.1) is again tangent to M_N, that is, the implication

$$X[u] \in T_u(M_N) \Rightarrow [K,X][u] \in T_u(M_N) \tag{3.2}$$

holds for all $u \in M_N$. Here we are in a position to begin with a study of the intrinsic structure of the manifold $M_N \subset M$ within the geometric Cartan's theory developed on the jet-manifold $J^{(\infty)}(\mathbb{R};\mathbb{R}^m)$ [4, 20–22]. Let us define an ideal $I(\xi) \subset \Lambda(J^{(\infty)})$, generated by the vector one-forms $\xi_j^{(1)} = du^{(j)} - u^{(j+1)}dx$, $j \in \mathbf{Z}_+$, which vanish on the vector field d/dx on the jet-manifold $J^{(\infty)}(\mathbb{R};\mathbb{R}^m)$,

$$i_{\frac{d}{dx}}\xi_j^{(1)} = 0, \quad j \in \mathbf{Z}_+, \tag{3.3}$$

where $x \in \mathbb{R}$ belongs to the jet-bundle base, $i_{\frac{d}{dx}}$ is the intrinsic derivative along the vector field

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \sum_{j\in\mathbf{Z}_+} \left\langle u^{(j+1)}, \frac{\partial}{\partial u^{(j)}} \right\rangle,$$

where $\langle .,. \rangle$ is the standard scalar product in $\mathbb{R}^m$. The vector field (2.1) on the jet-manifold $J^{(\infty)}(\mathbb{R};\mathbb{R})$ has an analogous representation,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{j=\mathbf{Z}_+} \left\langle K^{(j)}, \frac{\partial}{\partial u^{(j)}} \right\rangle, \tag{3.4}$$

where, by definition, $K^{(j)} := \frac{d^j}{dx^j}K$, $j = \mathbf{Z}_+$. There is the following problem: how to build the intrinsic variables on the manifold $M_N \subset M$ from the jet-manifold coordinates on $J^{(\infty)}(\mathbb{R};\mathbb{R}^m))$?

To proceed with the solution of the problem above, let us study the 1-form

$$d\mathcal{L} = \Lambda^1\left(J^{(\infty)}(\mathbb{R};\mathbb{R})\right)$$

as one defined on the submanifold $M_N \subset M$. We have the following chain of identities in the Grassmann subalgebra $\Lambda(J^{(2N+2)}(\mathbb{R};\mathbb{R}^m))$:

$$d\mathcal{L} = d\left(i_{\frac{d}{dx}}\mathcal{L}dx\right) = di_{\frac{d}{dx}}\left(\mathcal{L}dx + \sum_{j=0}^{N}\langle p_j, \mathbb{R}\rangle\right) =$$

$$= \left(di_{\frac{d}{dx}} + i_{\frac{d}{dx}}d\right)\left(\mathcal{L}\,dx + \sum_{j=0}^{N}\left\langle p_j, \xi_j^{(1)}\right\rangle\right) - i_{\frac{d}{dx}}d\left(\mathcal{L}\,dx + \sum_{j=0}^{N}\left\langle p_j, \xi_j^{(1)}\right\rangle\right), \tag{3.5}$$

where $p_j : J^{(2N+2)}(\mathbb{R};\mathbb{R}^m) \to \mathbb{R}^m$, $j = \overline{0,N}$, are some unknown vector-functions. Requiring now that the 2-form $d(\mathcal{L}dx + \sum_{j=0}^{N}\langle p_j, \xi_j^{(1)}\rangle)$ do not depend on the differentials $du^{(j)}$, $j = \overline{1, N+1}$, that is

$$i_{\frac{\partial}{\partial u^{(j)}}}\left(d\mathcal{L}\wedge dx + \sum_{k=0}^{N}\left\langle dp_k \wedge \xi_j^{(1)}\right\rangle\right) = 0, \tag{3.6}$$

one can thus determine the vector-functions $p_j = \mathbb{R}^m$, $j = \overline{0,N}$. As a result we obtain the following simple recurrence relations:

$$\frac{dp_j}{dx} + p_{j-1} = \frac{\partial\mathcal{L}}{\partial u^{(j)}} \tag{3.7}$$

for $j = \overline{1, N+1}$, setting $p_{-1} = 0 = p_{N+1}$ by definition. The unique solution to (3.7) is given by the following expressions, $j = \overline{0,N}$:

$$p_j = \sum_{k=0}^{N}(-1)^k\frac{d^k}{dx^k}\frac{\partial\mathcal{L}}{\partial u^{(j+k+1)}}. \tag{3.8}$$

Thereby we have got, owing to (3.5) and (3.6), the following final representation for the differential $d\mathcal{L}$:

$$d\mathcal{L} = \frac{d}{dx}\left[\mathcal{L} - \sum_{j=0}^{N}\left\langle p_j, u^{(j+1)}\right\rangle\right]dx - \left\langle \operatorname{grad}\mathcal{L}[u], u^{(1)}\right\rangle dx +$$

$$+ \frac{d}{dx}\left(\sum_{j=0}^{N}\left\langle p_j, du^{(j)}\right\rangle\right) + \langle \operatorname{grad}\mathcal{L}[u], du\rangle, \tag{3.9}$$

with $\dfrac{d}{dx} := di_{\frac{d}{dx}} + i_{\frac{d}{dx}}d$ being the Lie derivative along the vector field $\dfrac{d}{dx}$, and $\operatorname{grad}\mathcal{L}[u] := \delta\mathcal{L}/\delta u$, as it was mentioned above in Section 2. Below we intend to treat the representation (3.9) using the symplectic structure that arises from the above analysis on the invariant submanifold $M_N \subset M$.

4. Symplectic analysis and Hamiltonian formulation. Let us put, into the expression (3.9), the condition $\operatorname{grad}\mathcal{L}[u] = 0$ for all $u = M_N$. Then the following equality is satisfied:

$$d\mathcal{L} = \frac{d}{dx}\alpha^{(1)}, \quad \alpha^{(1)} = \sum_{j=0}^{N}\left\langle p_j, du^{(j)}\right\rangle, \tag{4.1}$$

since the function $h^{(x)} := \sum_{j=0}^{N}\langle p_j, u^{(j+1)}\rangle - \mathcal{L}(u,\ldots,u^{(N+1)})$ satisfies the condition $dh^{(x)}/dx = -\langle\operatorname{grad}\mathcal{L}[u], u^{(1)}\rangle$ for all $x = \mathbb{R}$, owing to the relations (3.7). Taking now the external derivative of (4.1), we obtain that

$$\frac{d}{dx}\Omega^{(2)} = 0, \quad \Omega^{(2)} = d\alpha^{(1)}, \tag{4.2}$$

where we have used the well known identity $d\cdot\dfrac{d}{dx} = \dfrac{d}{dx}\cdot d$. From (4.2) we can conclude that the vector field d/dx on the submanifold $M_N \subset M$ is Hamiltonian with respect to the canonical symplectic structure $\Omega^{(2)} = \sum_{j=0}^{N}\langle dp_j \wedge du^{(j)}\rangle$. It is a very simple exercise to see that the function $h^{(x)} : J^{(2N+2)}(\mathbb{R};\mathbb{R}^m) \to \mathbb{R}$ defined above plays a role of the corresponding Hamiltonian function for the vector field d/dx on M_N, i.e., the equation

$$dh^{(x)} = -i_{\frac{d}{dx}}\Omega^{(2)} \tag{4.3}$$

holds on M_N. Therefore, we have got the following theorem.

Theorem 4.1. *The critical submanifold $M_N \subset M$ defined by (3.1) for a given non degenerate smooth functional $L = D(M) \subset D(J^{(N+1)}(\mathbf{R};\mathbf{R}^m))$, being imbedded into the jet-manifold $J^{(\infty)}(\mathbf{R};\mathbf{R}^m)$, carries a canonical symplectic structure such that the induced vector field d/dx on M_N is Hamiltonian.*

The theorem analogous to the above was stated before in different terms by many authors [6, 23]. Our derivation presented here is much simpler and constructive, giving rise to all ingredients of symplectic theory, stemming from imbedding the invariant submanifold M_N into the jet-manifold.

Now we are going to proceed further to studying the vector field (2.1) on the manifold $M_N \subset M$ endowed with the symplectic structure $\Omega^{(2)} = \Lambda^2(J^{(N+1)}(\mathbb{R};\mathbb{R}^m))$, constructed via the formula (4.2).

We have the following implications:

$$\begin{aligned} \frac{d\mathcal{L}}{dt} = 0 \Rightarrow \langle\operatorname{grad}\mathcal{L}[u], K[u]\rangle = -\frac{dh^{(t)}}{dx}, \\ \frac{d\mathcal{L}}{dx} = 0 \Rightarrow \left\langle\operatorname{grad}\mathcal{L}[u], \frac{du}{dx}\right\rangle = -\frac{dh^{(x)}}{dx}, \end{aligned} \tag{4.4}$$

where the functions $h^{(t)}$ and $h^{(x)}$ serve as corresponding Hamiltonian ones for the vector fields d/dt and d/dx. This means in part that the following equations hold:

$$dh^{(x)} = -i_{\frac{d}{dx}}\Omega^{(2)}, \quad dh^{(t)} = -i_{\frac{d}{dt}}\Omega^{(2)}. \tag{4.5}$$

To prove the above statement (4.5), we shall build the following quantities (for the vector field d/dx at first):

$$d \cdot i_{\frac{d}{dx}} \langle \operatorname{grad} \mathcal{L}[u], du \rangle = -\frac{d}{dx}\left(dh^{(x)}\right) \tag{4.6}$$

stemming from (4.4), and

$$i_{\frac{d}{dx}} \cdot d \langle \operatorname{grad} \mathcal{L}[u], du \rangle = -\frac{d}{dx}\left(i_{\frac{d}{dx}}\Omega^{(2)}\right) \tag{4.7}$$

stemming from (3.9), where we used the above mentioned evident identity $\left[i_{\frac{d}{dx}}, \frac{d}{dx}\right] = 0$. Adding now the expressions (4.6) and (4.7) entails the following one:

$$\frac{d}{dx}\langle \operatorname{grad} \mathcal{L}[u], du \rangle = -\frac{d}{dx}(dh^{(x)} + i_{\frac{d}{dx}}\Omega^{(2)}) \tag{4.8}$$

for all $x = \mathbb{R}$ and $u = M$. Since $\operatorname{grad} \mathcal{L}[u] = 0$ for all $u = M_N$, we obtain from (4.8) that the first equality in (4.5) is valid in the case of the vector field d/dx reduced on M_N. The analogous procedure fits also for the vector field d/dt reduced on the manifold $M_N \subset M$. The even difference of the procedure above stems from the condition on the vector fields d/dt and d/dx to be commutative, $[d/dt, d/dx] = 0$, which gives the needed identity $\left[i_{\frac{d}{dt}}, \frac{d}{dx}\right] \equiv i_{[\frac{d}{dt},\frac{d}{dx}]} = 0$ as a simple consequence of the procedure considered above. This completes the proof of equations (4.5).

Theorem 4.2. *Dynamical systems d/dt and d/dx reduced on the invariant submanifold $M_N \subset M$ (3.1) are Hamiltonian ones with the corresponding Hamiltonian functions built from the equations (4.4) in a unique way.*

By the way, we have stated also that the Hamiltonian functions $h^{(x)}$ and $h^{(t)}$ on the submanifold $M_N \subset M$ commute with each other, that is, $\{h^{(t)}, h^{(x)}\} = 0$, where $\{\cdot,\cdot\}$ is the Poisson structure on $D(M_N)$ corresponding to the symplectic structure (4.2). This indeed follows from the equalities (4.4), since $\{h^{(t)}, h^{(x)}\} = \frac{dh^{(x)}}{dt} = -\frac{dh^{(t)}}{dx} \equiv 0$ on the manifold $M_N \subset M$.

5. Symmetry invariance. Let us consider now any vector field $K_j : M \to T(M)$, $j = \overline{1,k}$, that are symmetry fields related to the given vector field (2.1), i.e., $[K, K_j] = 0$, $j = \overline{1,k}$. Since the conservation law $\mathcal{L} \in D(M)$ for the vector field (2.1) need not to be such for the vector fields K_j, $j = \overline{1,k}$, the submanifold $M_N \subset M$ need not to be invariant also with respect to these vector fields. Therefore, if a vector field $X \in T(M_N)$, the vector field $[K_j, X] \notin T(M_N)$ in general, if $\frac{d}{dt_j}$, $j = \overline{1,k}$, are chosen as symmetries of (2.1). Let us consider the following identity for some functions $\tilde{h}_j : J^{(2N+2)}(\mathbb{R};\mathbb{R}^m) \to \mathbb{R}$, $j = \overline{1,k}$, which follow from the conditions $\left[\frac{d}{dt}, \frac{d}{dt_j}\right] = 0$, $j = \overline{1,k}$, on M :

$$\frac{d}{dt} i_{\frac{d}{dt_j}} \langle \operatorname{grad} \mathcal{L}[u], du\rangle = -\frac{d\tilde{h}_j[u]}{dx}. \tag{5.1}$$

Lemma 5.1. *The functions $\widetilde{h}_j[u]$, $j = \overline{1,k}$, reduced on the invariant submanifold $M_N \subset M$ turn into constant ones. These constants can be chosen obviously as zeros.*

Proof. We have $\left[\frac{d}{dt}, i_{\frac{d}{dt_j}}\right] = 0$, $j = \overline{1,k}$, hence

$$i_{\frac{d}{dt_j}}\Big(i_{\frac{d}{dt}} d + d i_{\frac{d}{dt}}\Big) \langle \operatorname{grad} \mathcal{L}[u], du\rangle = -\frac{d\tilde{h}_j}{dx} \Rightarrow$$

$$\Rightarrow i_{\frac{d}{dt_j}}\Big(i_{\frac{d}{dt}} d \langle \operatorname{grad} \mathcal{L}[u], du\rangle + d i_{\frac{d}{dt}} \langle \operatorname{grad} \mathcal{L}[u], du\rangle\Big) =$$

$$= -i_{\frac{d}{dt_j}} i_{\frac{d}{dt}} \frac{d}{dx}\Omega^{(2)} - i_{\frac{d}{dt_j}} \frac{d}{dx}\Big(dh^{(+)}\Big) = -\frac{d}{dx} i_{\frac{d}{dt_j}}\Big(i_{\frac{d}{dt}}\Omega^{(2)} + dh^{(t)}\Big) = -\frac{d\tilde{h}_j}{dx}.$$

Whence we obtain that on the whole jet-manifold $M \subset J^{\infty}(\mathbb{R};\mathbb{R}^m)$ the following identities hold:

$$i_{\frac{d}{dt_j}}\Big(dh^{(t)} + i_{\frac{d}{dt}}\Omega^{(2)}\Big) = \widetilde{h}_j \tag{5.2}$$

for all $j = \overline{1,k}$. Since on the submanifold $M_N \subset M$ one has $i_{\frac{d}{dt}}\Omega^{(2)} = -dh^{(t)}$, we find that $\widetilde{h}_j \equiv 0$, $j = \overline{1,k}$, which proves the lemma.

Note 5.1. The result above could be stated also using the standard functional operator calculus of [6]. Indeed,

$$\frac{d}{dt}\left(i_{\frac{d}{dt_j}} \langle \operatorname{grad} \mathcal{L}[u], du\rangle\right) = \frac{d}{dt} \langle \operatorname{grad} \mathcal{L}[u], K_j[u]\rangle =$$

$$= \left\langle \frac{d}{dt}\operatorname{grad} \mathcal{L}[u], K_j[u]\right\rangle + \left\langle \operatorname{grad} \mathcal{L}[u], \frac{d}{dt} K_j[u]\right\rangle =$$

$$= \big\langle -K'^{*}\operatorname{grad} \mathcal{L}[u], K_j[u]\big\rangle + \big\langle \operatorname{grad} \mathcal{L}[u], K_j' \cdot K[u]\big\rangle =$$

$$= -\big\langle K'^{*}\operatorname{grad} \mathcal{L}[u], K_j[u]\big\rangle + \big\langle \operatorname{grad} \mathcal{L}[u], K' \cdot K_j[u]\big\rangle =$$

$$= -\frac{d}{dx}\mathcal{H}\left[\operatorname{grad} \mathcal{L}[u], K_j[u]\right] = -\frac{d\tilde{h}_j[u]}{dx}. \tag{5.3}$$

Here the bilinear form $\mathcal{H}\{\cdot,\cdot\}$ is found via the usual definition of the adjoined operator K'^{*} for a given operator $K' : L_2 \to L_2$ with respect to the natural scalar product $(\cdot,\cdot)$,

$$\big(K'^{*}a, b\big) := (a, K'b), \quad (a,b) := \int\limits_{\mathbb{R}} dx \langle a, b\rangle, \tag{5.4}$$

whence we simply obtain

$$\big\langle K'^{*}a, b\big\rangle - \big\langle a, K'b\big\rangle = \frac{d\,\mathcal{H}[a,b]}{dx} \tag{5.5}$$

for all $a, b \in L_2$. Therefore, we can identify $\widetilde{h}_j[u] = \mathcal{H}[\operatorname{grad}\mathcal{L}[u], K_j[u]]$ for all $u \in M$, $j = \overline{1,k}$. If $u \in M_N$, we thereupon obtain that $\widetilde{h}_j[u] \equiv 0$, $j = \overline{1,k}$, that was needed to prove.

As a result of the lemma proved above one gets the following: the functions $\tilde{h}_j[u]$, $j = \overline{1,k}$, can not serve as nontrivial Hamiltonian functions for the dynamical systems d/dt_j, $j = \overline{1,k}$, on the submanifold $M_N \subset M$. To overcome this difficulty we assume the invariant submanifold $M_N \subset M$ to possess some additional symmetries d/dt_j, $j = \overline{1,k}$, which satisfy the following characteristic criterion: $L_{\frac{d}{dt_j}} \operatorname{grad}\mathcal{L}[u] = 0$, $j = \overline{1,k}$, for all $u \in M_N$. This means that for $j = \overline{1,k}$,

$$L_{\frac{d}{dt_j}} \operatorname{grad}\mathcal{L}[u] = G_j(\operatorname{grad}\mathcal{L}[u]), \tag{5.6}$$

where $G_j(\cdot)$, $j = \overline{1,k}$, are some linear vector-valued functionals on $T^*(M)$. Otherwise, equations (5.6) are equivalent to the following:

$$i_{\frac{d}{dt_j}} \langle \operatorname{grad}\mathcal{L}[u], du\rangle = \frac{-dh_j[u]}{dx} + g_j(\operatorname{grad}\mathcal{L}[u]), \tag{5.7}$$

where $g_j(\cdot)$, $j = \overline{1,k}$, are some scalar linear functionals on $T^*(M)$. From (3.9) and (5.7) we hence find that for all $j = \overline{1,k}$

$$L_{\frac{d}{dt_j}} \langle \operatorname{grad}\mathcal{L}[u], du\rangle = -\frac{d}{dx}\left(dh_j[u] + i_{\frac{d}{dt_j}}\Omega^{(2)}\right) + dg_j(\operatorname{grad}\mathcal{L}[u]). \tag{5.8}$$

If we put now $u \in M_N$, that is, $\operatorname{grad}\mathcal{L}[u] = 0$, we will immediately find the following: for all $j = \overline{1,k}$,

$$dh_j[u] + i_{\frac{d}{dt_j}}\Omega^{(2)} = 0. \tag{5.9}$$

Whence we can make a conclusion that the vector fields d/dt_j, $j = \overline{1,k}$, are Hamiltonian too on the submanifold $M_N \subset M$. Since $dh_j/dx = \{h^{(x)}, h_j\} = 0$, $j = \overline{1,k}$, on the manifold M_N, we obtain that $dh^{(x)}/dt_j = 0$, $j = \overline{1,k}$. This is also an obvious corollary of the commutativity $[d/dt_j, d/dx] = 0$, $j = \overline{1,k}$, for all $x, t_j \in \mathbb{R}$ on the whole manifold M. Indeed, in the general case we have the identity $\{h^{(x)}, h_j\} = i_{[\frac{d}{dt_j},\frac{d}{dx}]}\Omega^{(2)}$, whence the equalities $\{h^{(x)}, h_j\} \equiv 0$ hold for all $j = \overline{1,k}$ on the submanifold $M_N \subset M$, since $\left[\frac{d}{dt_j}, \frac{d}{dx}\right] = 0$ on M_N due to (5.8). The analysis carried out above makes it possible to treat given vector fields d/dt_j, $j = \overline{1,k}$, satisfying either conditions (5.6) or conditions (5.7) on the canonical symplectic jet-submanifold $M_N \subset M$ analytically as Hamiltonian systems.

6. Liouville integrability. Now we suppose that the vector field d/dt_j, $j = \overline{1,k}$, are independent and commute both with each other on the jet-submanifold $M_N \subset M$ and with the vector fields d/dt and d/dx on the manifold M. Besides, the submanifold $M_N \subset M$ is assumed to be compact and smoothly imbedded into the jet-manifold $J^{(\infty)}(\mathbb{R};\mathbb{R}^m)$. If the dimension $\dim M_N = 2k+4$, Liouville theorem [6, 23] implies that the dynamical systems d/dx and d/dt are Hamiltonian and integrable by quadratures on the submanifold $M_N \subset M$. This is the case for all Lax integrable nonlinear dynamical systems of the Korteweg – de Vries-type [4 – 6, 23] on spatially one-dimensional functional manifolds.

7. Discrete dynamical systems. Let there be given a differential discrete smooth dynamical system

$$\frac{du_n}{dt} = K_n[u] \tag{7.1}$$

with respect to a continuous evolution parameter $t \in \mathbb{R}$ on an infinite-dimensional discrete manifold $M \subset L_2(\mathbf{Z};\mathbb{R}^m)$ of infinite vector-sequences under the condition of rapid decrease in $n \in \mathbf{Z}$, $\sup_{n\in\mathbf{Z}} |n|^k \|u_n\|_{\mathbb{R}^m} < \infty$ for all $k \in \mathbf{Z}_+$ at each point $u = (\ldots, u_n, u_{n+1}, \ldots) \in M$, where $u_n \in \mathbb{R}^m$, $n \in \mathbf{Z}$.

Assume further that the dynamical system (7.1) possesses a conservation law $\mathcal{L} \in D(M)$, that is, $d\mathcal{L}/dt = 0$ along orbits of (7.1). Via a standard operator analysis one gets, from (3.5), the variational derivative of the functional $\mathcal{L} := \sum_{n\in\mathbf{Z}} \mathcal{L}_n[u]$:

$$\operatorname{grad}\mathcal{L}_n := \frac{\delta\mathcal{L}[u]}{\delta u_n} = \mathcal{L}_n'^{\,*}[u]\cdot 1, \tag{7.2}$$

where the last right-hand operation of multiplying by unity is to be done by component.

Lemma 7.1. *Let $\Lambda(M)$ be the infinite-dimensional Grassmannian algebra on the manifold M. Then the differential $dL_n[u] \in \Lambda^1(M)$ enjoys the following reduced representation:*

$$d\mathcal{L}_n[u] = \langle \operatorname{grad}\mathcal{L}_n, du_n\rangle + \frac{d\alpha_n^{(1)}[u]}{dn}, \tag{7.3}$$

where the one-form $\alpha_n^{(1)}[u] \in \Lambda^1(M)$ is determined in a unique way, $\langle\cdot,\cdot\rangle$ is the usual scalar product in R^m and $d/dn = \Delta - 1$, Δ is the usual shift operator.

Proof. By definition we obtain for the external differential $d\mathcal{L}_n[u]$ the following chain of representations for each $n \in \mathbf{Z}$:

$$
\begin{aligned}
d\mathcal{L}_n[u] &= \sum_{k=0}^{N} \left\langle \frac{\partial \mathcal{L}_n[u]}{\partial u_{n+k}}, du_{n+k} \right\rangle = \\
&= \sum_{k=0}^{N} \sum_{s=0}^{k} \frac{d}{dn} \left\langle \frac{\partial \mathcal{L}_{n-s}[u]}{\partial u_{n+k-s}}, du_{n+k-s} \right\rangle + \sum_{k=0}^{N} \left\langle \frac{\partial \mathcal{L}_{n-k}[u]}{\partial u_n}, du_n \right\rangle = \\
&= \frac{d}{dn} \sum_{k=0}^{N} \sum_{s=0}^{k} \left\langle \frac{\partial \mathcal{L}_{n-s}[u]}{\partial u_{n+k-s}}, du_{n+k-s} \right\rangle + \sum_{k=-N}^{N} \triangle^{-k} \left\langle \frac{\partial \mathcal{L}_n[u]}{\partial u_{n+k}}, du_n \right\rangle = \\
&= \frac{d}{dn} \sum_{k=0}^{N} \sum_{s=0}^{k} \left\langle \frac{\partial \mathcal{L}_{n-s}[u]}{\partial u_{n+k-s}}, du_{n+k-s} \right\rangle + \left\langle \mathcal{L}_n'^{\,*}.1, du_n \right\rangle = \\
&= \frac{d}{dn} \alpha_n^{(1)}[u] + \left\langle \operatorname{grad} \mathcal{L}_n[u], du_n \right\rangle,
\end{aligned}
\tag{7.4}
$$

where $N \in \mathbf{Z}_+$ is a fixed number depending on the jet-form of the functional $\mathcal{L} \in D(M)$,

$$
\alpha_n^{(1)}[u] = \sum_{k=0}^{N} \sum_{s=0}^{k} \left\langle \frac{\partial \mathcal{L}_{n-s}[u]}{\partial u_{n+k-s}}, du_{n+k-s} \right\rangle = \sum_{k=0}^{N} \sum_{j=0}^{k} \left\langle \frac{\partial \mathcal{L}_{n+j-k}[u]}{\partial u_{n+j}}, du_{n+j} \right\rangle, \tag{7.5}
$$

and

$$
\operatorname{grad} \mathcal{L}_n[u] = \mathcal{L}_n'^{\,*} \cdot 1 = \sum_{k=0}^{N} \frac{\partial \mathcal{L}_{n-k}[u]}{\partial u_n}.
$$

The latter equality in (7.4) proves Lemma 7.1 completely.

The proved above representation (7.3) gives rise to the following stationary problem being posed on the manifold M :

$$
M_N = \{ u \in M : \operatorname{grad} \mathcal{L}_n = 0 \} \tag{7.6}
$$

for all $n \in \mathbf{Z}$, where by definition $\det \left\| \dfrac{\partial^2 \mathcal{L}n[u]}{\partial u_{N+1} \partial u_{N+1}} \right\| = 0$. In virtue of (7.3) we obtain the validity of the following theorem.

Theorem 7.1. *The finite-dimensional Lagrangian submanifold $M_N \subset M$ defined by (7.6), is a symplectic one with the canonical symplectic structure $\Omega_n^{(2)} = d\alpha_n^{(1)}$ that is independent of the discrete variable $n \in \mathbf{Z}$.*

Proof. From (7.3) we have that on the manifold $M_N \subset M$, $d\mathcal{L}_n[u] = d\alpha_n^{(1)}[u]/dn$, whence for all $n \in \mathbf{Z}$, $d\Omega_n^{(2)}/dn = 0$. This obviously means that $\Omega_{n+1}^{(2)} = \Omega_n^{(2)}$ for all $n \in \mathbf{Z}$, or, equivalently, the 2-form $\Omega_n^{(2)}$ does not depend on the discrete variable $n \in \mathbf{Z}$. As the 2-form $\Omega_n^{(2)} := d\alpha_n^{(1)}$ by definition, this form is chosen to be symplectic on the manifold $M_N \subset M$. For this 2-form to be nondegenerate on M_N, we assume that the Hessian of $\mathcal{L}_n$ equals $\det\left\|\dfrac{\partial^2\mathcal{L}_n[u]}{\partial u_{n+N+1}\partial u_{n+N+1}}\right\| \neq 0$ on M_N. The latter proves the theorem.

Let us consider now the dynamical system (7.1) reduced on the submanifold $M_N \subset M$. To present it as the vector field d/dt on M_N, we need at first to represent it as a Hamiltonian flow on M_N. To do this, let us write the following identities on M :

$$\begin{aligned} i_{\frac{d}{dt}} d\,\langle \operatorname{grad}\mathcal{L}_n, du_n\rangle &= -\frac{d}{dn} i_{\frac{d}{dt}}\Omega_n^{(2)}[u], \\ d i_{\frac{d}{dt}}\,\langle \operatorname{grad}\mathcal{L}_n, du_n\rangle &= -\frac{d}{dn}(dh_n^{(t)}[u]), \end{aligned} \tag{7.7}$$

which are valid for all $n \in \mathbf{Z}$. Adding the last identities in (7.7), we come to the following one for all $n \in \mathbf{Z}$:

$$\frac{d}{dt}\langle \operatorname{grad}\mathcal{L}_n, du_n\rangle = -\frac{d}{dn}\left(i_{\frac{d}{dt}}\Omega_n^{(2)}[u] + h_n^{(t)}[u]\right). \tag{7.8}$$

Having reduced the identity (7.8) on the manifold $M_N \subset M$, we obtain the needed expression for all $u \in M_N$, $N \in \mathbf{Z}$,

$$i_{\frac{d}{dt}}\Omega_n^{(2)}[u] + h_n^{(t)}[u] = 0. \tag{7.9}$$

The latter means that the dynamical system (7.1) on the manifold M_N is a Hamiltonian one, with the function $h_n^{(t)}[u]$ being a Hamiltonian function defined explicitly by the second identity in (7.7).

We assume now that the symplectic structure $\Omega_n^{(2)}[u]$ on M_N can be represented as follows:

$$\Omega_n^{(2)}[u] = \sum_{j=0}^{N}\langle dp_{j+n}\wedge du_{j+n}\rangle, \tag{7.10}$$

where the generalized coordinates $p_{j+n} \in \mathbb{R}^m$, $j = \overline{0,N}$, are determined from the following discrete jet-expression $\mathcal{L}_n[u] := \mathcal{L}(u_n, u_{n+1}, \ldots, u_{n+N+1})$, $n \in \mathbf{Z}$,

$$\begin{aligned} \alpha_n^{(1)}[u] := \sum_{j=0}^{N}\langle p_{j+n}, du_{j+n}\rangle &= \sum_{k=0}^{N}\sum_{j=0}^{k}\left\langle \frac{\partial\mathcal{L}_{n+j-k}}{\partial u_{n+j}}, du_{n+j}\right\rangle = \\ &= \sum_{j=0}^{N}\sum_{k=j}^{N}\left\langle \frac{\partial\mathcal{L}_{n+j-k}}{\partial u_{n+j}}, du_{n+j}\right\rangle, \end{aligned}$$

whence we get the final expression

$$p_{j+n} := \sum_{k=j}^{N}\frac{\partial\mathcal{L}_{n+j-k}[u]}{\partial u_{n+j}}, \tag{7.11}$$

where $j = \overline{0, N}$, $u \in M_N \subset M$.

Now we are in a position to reformulate the given dynamical system (7.1) as that on the reduced manifold $M_N \subset M$,

$$\frac{du_{n+j}}{dt} = \{h_n^{(t)}, u_{n+j}\} = \frac{\partial h_n^{(t)}}{\partial p_{n+j}}, \quad \frac{dp_{n+j}}{dt} = \{h_n^{(t)}, p_{n+j}\} = -\frac{\partial h_n^{(t)}}{\partial u_{n+j}} \tag{7.12}$$

for all $n \in \mathbf{Z}$, $j = \overline{0, N}$. Thereby the problem of embedding a given discrete dynamical system (7.1) into a vector field flow on the manifold $M_N \subset M$ is solved completely with the final result (7.12).

8. Invariant Lagrangians construction: functional manifold case. In the case where the given nonlinear dynamical system (2.1) is integrable one of Lax-type, we can proceed effectively to find a commuting infinite hierarchy of conservation laws that can serve as the invariant Lagrangians considered above.

At first we have to use the important property [4] of the complexified gradient functional $\varphi = \operatorname{grad} \gamma \in T^*(M) \otimes \mathbb{C}$ generated by an arbitrary conservation law $\gamma \in D(M)$, i.e., the following Lax-type equation:

$$\frac{d\varphi}{dt} + K'^{*}\varphi = 0, \tag{8.1}$$

where the prime sign denotes the usual Frechet derivative of the local functional $K : M \to$ $\to T(M)$ on the manifold M, the star "*" denotes its conjugation operator with respect to the nondegenerate standard convolution functional $(\cdot, \cdot) = \int_{\mathbb{R}} dx \langle \cdot, \cdot \rangle$ on $T^*(M) \times T(M)$. The equation (8.1) admits, which follows from [24–26], the special asymptotic solution,

$$\varphi(x, t; \lambda) \cong (1, a(x, t; \lambda))^{\tau} \exp[\omega(x, t; \lambda) + \partial^{-1}\sigma(x, t; \lambda)], \tag{8.2}$$

where $a(x, t; \lambda) \in \mathbb{C}^{m-1}$, $\sigma(x, t; \lambda) \in \mathbb{C}$, $\omega(x, t; \lambda)$ is some dispersive function. The sign "τ " denotes here the transposition used in the matrix analysis. For any complex parameter $\lambda \in \mathbb{C}$, at $|\lambda| \to \infty$, the following expansions take place:

$$a(x, t; \lambda) \simeq \sum_{j \in \mathbf{Z}_+} a_j[x, t; u]\lambda^{-j+s(a)}, \quad \sigma(x, t; \lambda) \simeq \sum_{j \in \mathbf{Z}_+} \sigma_j[x, t; u]\lambda^{-j+s(\sigma)}.$$

Here $s(a)$ and $s(\sigma) \in \mathbf{Z}_+$ are some appropriate nonnegative integers, the operation ∂^{-1} means the inverse to the differentiation d/dx, that is, $d/dx \cdot \partial^{-1} = 1$ for all $x \in \mathbb{R}$.

To find an explicit form of the representation (8.2) in the case when the associated Lax-type representation [6] depends parametrically on the spectral parameter $\lambda(t; z) \in \mathbb{C}$, satisfying the following nonisospectral condition:

$$\frac{d\lambda(t; z)}{dt} = g(t; \lambda(t; z)), \quad \lambda(t; z)|_{t=0^+} = z \in \mathbb{C}, \tag{8.3}$$

for some meromorphic function $g(t; \cdot) : \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{R}_+$, we must analyze more carefully the asymptotic solutions to the Lax equation (8.1). Namely, we are going to treat more exactly the case when the solution $\varphi \in T^*(M)$ to (8.1) is represented as an appropriate trace-functional of a Lax spectral problem at the moment $\tau = t \in \mathbb{R}_+$ with the spectral parameter $\lambda(t;\lambda) \in \mathbb{C}$ satisfying the condition (8.3), the evolution of the given dynamical system (2.1) is considered with respect to the introduced above parameter $\tau \in \mathbb{R}$, that is,

$$\frac{du}{d\tau} = K[x,\tau;u], \tag{8.4}$$

$u|_{\tau=0} = \bar{u} \in M$ is some Cauchy data on M. This means that the functional

$$\tilde{\varphi}(x,\tau;\tilde{\lambda}) := \operatorname{reg}\operatorname{grad}\operatorname{Sp} S(x,\tau;\tilde{\lambda}), \quad \tilde{\lambda} = \tilde{\lambda}(\tau;\lambda(t;z)) \in \mathbb{C}, \tag{8.5}$$

where $S(x,\tau;\tilde{\lambda})$ is the monodrony matrix corresponding to a Lax-type spectral problem assumed to exist, has to satisfy the corresponding Lax equation at any point $u \in M$ subject to (8.4),

$$\frac{d\widetilde{\varphi}}{d\tau} + K'^{*}[u] \cdot \widetilde{\varphi} = 0 \tag{8.6}$$

for all $\tau \in \mathbb{R}_+$. Under the above assumption it is obvious that the spectral parameter $\widetilde{\lambda} = \widetilde{\lambda}(\tau;\lambda(t;z))$, where

$$\frac{d\widetilde{\lambda}}{d\tau} = \widetilde{g}(\tau;\widetilde{\lambda}), \quad \widetilde{\lambda}\Big|_{\tau=0} = \lambda(t;z) \in \mathbb{C}, \tag{8.7}$$

$\tilde{g}(t;\cdot) : \mathbb{C} \to \mathbb{C}$ is some meromorphic function found simply from (8.6) for instance at $u = 0$, the Cauchy data $\lambda(t;z) \in \mathbb{C}$, for all $t \in \mathbb{R}_+$, corresponds to (8.3), the parameter $z \in \mathbb{C}$ is a spectrum value of the associate Lax-type spectral problem at a moment $t \in \mathbb{R}_+$.

Now we are in a position to formulate the following lemma.

Lemma 8.1. *The Lax equation (8.6), as the parameter $\tau = t \in \mathbb{R}_+$, admits an asymptotic solution in the form*

$$\tilde{\varphi}\left(x,\tau;\tilde{\lambda}\right) \cong \left(1,\tilde{a}\left(x,\tau;\tilde{\lambda}\right)\right)^{\tau} \exp\left[\tilde{\omega}\left(x,\tau;\tilde{\lambda}\right) + \partial^{-1}\tilde{\sigma}\left(x,\tau;\tilde{\lambda}\right)\right], \tag{8.8}$$

where $\tilde{a}(x,\tau;\tilde{\lambda}) \in \mathbb{C}^{m-1}$, $\tilde{\sigma}(x,\tau;\tilde{\lambda}) \in \mathbb{C}$, are some local functionals on $M, \tilde{\omega}(x,\tau;\tilde{\lambda}) \in \mathbb{C}$ is some dispersion function for all $x \in \mathbb{R}, \tau \in \mathbb{R}_+$, and if for $|\lambda| \to \infty$ the property $|\tilde{\lambda}| \to \infty$ as $\tau = t \in \mathbb{R}_+$ holds, the following expansions follows:

$$\tilde{a}(x,\tau;\tilde{\lambda}) \simeq \sum_{j\in\mathbf{Z}_+} \tilde{a}_j[x,\tau;u]\tilde{\lambda}^{-j+s(\tilde{a})}, \quad \tilde{\sigma}(x,\tau;\tilde{\lambda}) \simeq \sum_{j\in\mathbf{Z}_+} \tilde{\sigma}_j[x,\tau;u]\tilde{\lambda}^{-j+s(\tilde{\sigma})}, \tag{8.9}$$

with $s(\tilde{a})$ and $s(\tilde{\sigma}) \in \mathbf{Z}_+$ being some integers.

Proof. In virtue of the theory of asymptotic expansions for arbitrary differential spectral problems, the result (8.8) will hold provided the representation (8.5) is valid and the spectral parameter $\lambda(t;z) \in \mathbb{C}$ is taken subject to (8.7). But this is the case because of the Lax-type integrability of the dynamical system (8.4). Further, due to the mentioned above integrability of (8.4), as well as to the well known Stokes property of asymptotic solutions to linear equations like (8.1), the condition (8.3) holds for some meromorphic function $g(t;\cdot) : \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{R}_+$, enjoing the determining property $\dfrac{d}{dt}\displaystyle\int_{\mathbb{R}} \tilde{\sigma}(x,t;\tilde{\lambda}(t;\lambda(t;z)))dx = 0$ for all $t \in \mathbb{R}_+$. The latter proves the lemma completely.

As a result of Lemma 8.1 one can formulate the following important theorem.

Theorem 8.1. *The Lax integrable parametrically isospectral dynamical system (8.4), as* $\tau = t \in \mathbb{R}_+$, *admits an infinite hierarchy of conservation laws, in general nonuniform with respect to the variables* $x \in \mathbb{R}$, $\tau \in \mathbb{R}_+$, *which can be represented in an exact form in virtue of the asymptotic expansion (8.8) and (8.9).*

Proof. Indeed, due to the expansion (8.8), we can obtain right away that the functional

$$\tilde{\gamma}(t;\lambda(t;z)) = \int\limits_{\mathbb{R}} dx\tilde{\sigma}\left(x,t;\tilde{\lambda}\left(t;\lambda(t;z)\right)\right) \tag{8.10}$$

does not depend on the parameter $t \in \mathbb{R}_+$ at $\tau = t \in \mathbb{R}_+$, that is,

$$\left.\frac{d\tilde{\gamma}}{d\tau}\right|_{\tau=t\in\mathbb{R}_+} = 0 \tag{8.11}$$

for all $t \in \mathbb{R}_+$. If we also make the parameter $\tau \in \mathbb{R}_+$ tend to $t \in \mathbb{R}_+$, due to (8.5) we obtain that $\tilde{\varphi}(x,\tau;\tilde{\lambda})\big|_{\tau=t\in\mathbb{R}_+} \to \varphi(x,t;\lambda)$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, and $\lambda(t;z) \in \mathbb{C}$. This means that a complexified local functional $\varphi(x,t;z) \in T^*(M) \otimes \mathbb{C}$ satisfies the equation (8.1) at each point $u \in M$. As an obvious result, the following identifications hold:

$$\tilde{\omega}(x,\tau;\tilde{\lambda})\big|_{\tau=t\in\mathbb{R}_+} \to \omega(x,t;z), \quad \tilde{\sigma}(x,\tau;\tilde{\lambda})\big|_{\tau=t\in\mathbb{R}_+} \to \sigma(x,t;z)$$

for all $z \in \mathbb{C}$. Hence, the functional $\gamma(z) := \tilde{\gamma}(\tau;\lambda(t;z))|_{\tau=t\in\mathbb{R}_+} = \displaystyle\int_{\mathbb{R}} dx\sigma(x,t;z) \in D(M)$ doesn't depend on the evolution parameter $t \in \mathbb{R}_+$ and, due to equation (8.1), is a conserved quantity for the nonlinear dynamical system (2.1) under consideration, i.e.,

$$\frac{d\gamma(t;z)}{dt} = 0 \tag{8.12}$$

for all $t \in \mathbb{R}_+$ and $z \in \mathbb{C}$. Therefore, this makes it possible to use the equation (8.12) jointly with (8.7) to find the asymptotic expansions (8.9) and (8.3) in an exact form. To do this, we first need to substitute the asymptotic expansion (8.8) in the determining equation (8.6) for the asymptotic expansions (8.9) to be found explicitly at the moment $\tau = t \in \mathbb{R}_+$. Keeping in mind that, at $\tau = t \in \mathbb{R}_+$, $|\lambda| \to \infty$ if $|\tilde{\lambda}| \to \infty$, and solving step by step the resulting recurrence relations for the coefficients in (8.9), we will get the functional $\gamma(z) := \tilde{\gamma}(\tau;\lambda(t;z))|_{\tau=t\in\mathbb{R}_+}$, $z \in \mathbb{C}$, in the form fitting for the use the criteria equation (8.12). As the second step, we need to use the differential equation (8.7) as to satisfy the criteria equation (8.12) pointwise for all

$t \in \mathbb{R}_+$. This means, in particular, that

$$\frac{d\gamma(z)}{dt} = \frac{d}{dt}\left(\sum_{j\in\mathbf{Z}_+}\int_{\mathbb{R}} dx\tilde{\sigma}_j[x,\tau;u]\tilde{\lambda}^{-j+s(\tilde{\sigma})}\Big|_{\tau=t\in\mathbb{R}_+}\right) =$$

$$= \int\limits_{\mathbb{R}} dx \sum_{j\in\mathbf{Z}_+}\left[\frac{d\tilde{\sigma}_j[x,\tau;u]}{dt}\tilde{\lambda}^{-j+s(\tilde{\sigma})} + \tilde{\sigma}_j[x,\tau;u]\tilde{\lambda}^{-j+s(\tilde{\sigma})-1}(s(\tilde{\sigma})-j)\frac{d\tilde{\lambda}}{dt}\right]\Bigg|_{\tau=t\in\mathbb{R}_+} \Rightarrow$$

$$\Rightarrow \int\limits_{\mathbb{R}} dx \sum_{j\in\mathbf{Z}_+}\left[\Big(\frac{d\tilde{\sigma}_j}{dt}\Big)\tilde{\lambda}^{-j+s(\tilde{\sigma})} + \sum_{k>>-\infty}(s(\tilde{\sigma})-k)\tilde{\sigma}_k\tilde{g}_{j-k-1}(t)\tilde{\lambda}^{-j+s(\tilde{\sigma})}\Big|_{\tau=t\in\mathbb{R}_+} + \right.$$

$$\left. + \sum_{j\in\mathbf{Z}_+}\tilde{\sigma}_j[x,t;u]\tilde{\lambda}^{-j+s(\tilde{\sigma})-1}(s(\tilde{\sigma})-j)\frac{\partial\tilde{\lambda}}{\partial\lambda}g(t;\lambda)\Big|_{\tau=t\in\mathbb{R}_+}\right] \equiv 0, \tag{8.13}$$

where we have put by definition $\tilde{g}(\tau;\tilde{\lambda}) :\simeq \sum_{k>>-\infty}\tilde{g}_k(\tau)\tilde{\lambda}^{-k}$ for $\tau \in \mathbb{R}_+$ and $|\tilde{\lambda}| \to \infty$. Since the spectral parameter $\lambda = \lambda(t;z)$, at the moment $t = 0^+$, coincides with an arbitrary complex value $z \in \mathbb{C}$, the condition $|z| \to \infty$ together with (8.13) at the moment $t = 0^+$ gives rise to the following recurrent relations:

$$\sum_{j\in\mathbf{Z}_+}\left[\frac{\partial\tilde{\sigma}_j}{dt} + \tilde{\sigma}_j'\cdot K[t;u] + \sum_{k>>-\infty}(s(\tilde{\sigma})-k)\tilde{\sigma}_k\cdot\tilde{g}_{j-k-1}\right]\tilde{\lambda}^{-j+s(\tilde{\sigma})}\Big|_{\tau=t\in\mathbb{R}_+} =$$

$$= \sum_{j\in\mathbf{Z}_+}\tilde{\sigma}_j(s(\tilde{\sigma})-j)\frac{\partial\tilde{\lambda}}{\partial\lambda}g(t;\lambda)\tilde{\lambda}^{-j+s(\tilde{\sigma})-1}\Big|_{\tau=t\in\mathbb{R}_+} \equiv 0 \pmod{d/dx} \tag{8.14}$$

for all $j \in \mathbf{Z}_+$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, and $u \in M$. Having solved the algebraic relations (8.14) for the unknown function $g(t;\lambda)$, $t \in \mathbb{R}_+$, we will obtain the generating functional $\gamma(z)$, $z \in \mathbb{C}$, of conservation laws for (2.1) in an exact form. This completes the constructive part of the proof of the theorem above.

From the practial point of view we need first to get the differential equation (8.7) in an exact, maybe in an asymptotic form, and find further the dispersive function $\tilde{\omega}(x,t;\tilde{\lambda})$ and the local generating functional $\tilde{\sigma}(x,\tau;\tilde{\lambda})$ defined via (8.8) and (8.9) for all $x \in \mathbb{R}, \tau \in \mathbb{R}_+$ and $|\tilde{\lambda}| \to \infty$, and next one can find the equation (8.3) using the algorithm based on the relations (8.14). This, together with the possibility of applying the general scheme of the gradient-holonomic algorithm [27], gives rise to determining in many cases the above mentioned Lax-type representation completely in an exact form, which successfully solves the pretty complex direct problem of the integrability theory of nonlinear dynamical systems on functional manifolds.

Having obtained the generation function $\gamma(z) \in D(M)$, $z \in \mathbb{C}$, of an infinite hierarchy of conservation laws of the dynamical system (2.1) on the manifold M, we can build appropriately a general Lagrangian functional $\mathcal{L}_N \in D(M)$ as follows:

$$\mathcal{L}_N = -\gamma_{N+1} + \sum_{j=0}^{N} c_j \gamma_j, \tag{8.15}$$

where, by definition, $\gamma(z) = \int_{\mathbb{R}} dx \sigma(x,t;z)$ and, for $|z| \to \infty$, the functionals

$$\gamma_j = \int_{\mathbb{R}} dx \sigma_j[x,t;z], \quad j \in \mathbf{Z}_+,$$

are conservation laws due to expansion (8.2), with $c_j \in \mathbb{R}$, $j = \overline{0,N}$, are some arbitrary constants and $N \in \mathbf{Z}_+$ is an arbitrary but fixed nonnegative integer. If the differential order of the functional $\gamma_{N+1} \in D(M)$ is the highest among the orders of the functionals $\gamma_j \in D(M)$, $j = \overline{0,N}$, and additionally, this Lagrangian is not degenerate, that is, $\det(\mathrm{Hess}\, \gamma_{N+1}) \neq 0$, we can apply in general amost all the theory developed before for proving that the critical submanifold $M_N = \{u \in M : \mathrm{grad}\, \mathcal{L}_N = 0\}$ is a finite-dimensional invariant manifold inserted into the standard jet-manifold $J^{(\infty)}(\mathbb{R};\mathbb{R}^m)$ with the canonical symplectic structure subject to which our dynamical system is a finite-dimensional Hamiltonian flow on the invariant submanifold M_N.

9. Invariant Lagrangian construction : discrete manifold case. Let us consider the discrete Lax integrable dynamical system on a discrete manifold M without an a priory given Lax-type representation. The problem arises of how to get the corresponding conservation laws via the gradient-holonomic algorithm [6]. To realize this, let us study solutions to the Lax equation

$$\frac{d\varphi_n}{dt} + K_n'[\tau,u] \cdot \varphi_n = 0, \tag{9.1}$$

where the local functionals $\varphi_n[u] \in T^*_{u_n}(M)$ at the point $u_n \in M$, $n \in \mathbf{Z}$. In analogy with the approach of Section 7, we assert that equation (9.1) admits a comlexified generating solution $\varphi_n = \varphi_n(t;\lambda) \in T^*_{u_n}(M) \otimes \mathbb{C}$, $n \in \mathbf{Z}$, with $z \in \mathbb{C}$ a complex parameter in the form

$$\varphi_n(t;z) \cong (1, a_n(t;z))^\tau \exp\left[\omega(t;z)\right] \left(\prod_{j=-\infty}^{n} \sigma_j(t;z) \right), \tag{9.2}$$

where $\omega(t;z)$ is some dispersive function for $t \in \mathbb{R}_+$, $a_n(t;z) \in \mathbb{C}^{m-1}$, $\sigma_n(t;z) \in \mathbb{R}$ are local functionals on M with the following asymptotic expansions at $|z| \to \infty$:

$$a_n(t;z) \simeq \sum_{j\in\mathbf{Z}_+} a_n[t;u] z^{-j+s(a)}, \quad \sigma_n^{(j)}(t;z) \simeq \sum_{j\in\mathbf{Z}_+} \sigma_n[t;u] z^{-j+s(\sigma)}. \tag{9.3}$$

To find an explicit form of the asymptotic representation (9.2), we need to additionally study the asymptotic solutions to the following attached Lax equation with respect to the new evolution parameter $\tau \in \mathbb{R}_+$,

$$\frac{d\tilde{\varphi}_n}{d\tau} + K_n'^{\,*}[\tau,u] \cdot \tilde{\varphi}_n = 0, \tag{9.4}$$

where $\tilde{\varphi}_n \in T^*_{u_n}(M)\otimes\mathbb{C}$, and the point $u \in M$ evolves subject to the following dynamical system:

$$\frac{du_n}{d\tau} = K_n[\tau; u] \tag{9.5}$$

for all $n \in \mathbf{Z}$. Having made the assumption above we can assert, based on the general theory of asymptotic solutions to linear equations like (9.4), that it also admits, in general, another asymptotic solution in a similar form,

$$\tilde{\varphi}_n(\tau;\tilde{\lambda}) \cong (1, \tilde{a}_n(\tau;\tilde{\lambda}))^\tau \exp[\tilde{\omega}(\tau;\tilde{\lambda})] \prod_{j=-\infty}^{n} \tilde{\sigma}_j(\tau;\tilde{\lambda}), \tag{9.6}$$

where for all $n \in \mathbf{Z}$ and at $\tau \in \mathbb{R}_+$, the asymptotic expansions

$$\begin{gathered} \tilde{a}_n(\tau;\tilde{\lambda}) \simeq \sum_{j\in\mathbf{Z}_+} \tilde{a}_n^{(j)}[x,\tau;u]\tilde{\lambda}^{-j+s(\tilde{a})}, \\ \tilde{\sigma}_n(\tau;\tilde{\lambda}) \simeq \sum_{j\in\mathbf{Z}_+} \tilde{\sigma}_n^{(j)}[\tau;u]\tilde{\lambda}^{-j+s(\tilde{\sigma})} \end{gathered} \tag{9.7}$$

hold. The expansions above are valid if $|\tilde{\lambda}| \to \infty$ as $|\lambda(t;z)| \to \infty$, $z \in \mathbb{C}$. The latter is the case because of the Lax integrability of the dynamical system (9.5). The evolution

$$\frac{d\tilde{\lambda}}{d\tau} = \tilde{g}(\tau;\tilde{\lambda}), \quad \tilde{\lambda}\big|_{t=0} = \lambda(t;z) \in \mathbb{C}, \tag{9.8}$$

where $\tilde{g}(\tau;\cdot) : \mathbb{C} \to \mathbb{C}$ is some meromorphic mapping for all $\tau \in \mathbb{R}_+$, is, in general, found by making use of the corresponding solution to (9.4) at $u = 0$.

Substituting the expansions (9.6) and (9.7) into (9.4), we obtain some recurrence relations that give rise to a possibility of finding exact expressions for local functionals $\tilde{\sigma}_j[t;u_n]$, $j \in \mathbf{Z}_+$. Having done this, we assert that the functional

$$\gamma(t;z) = \sum_{n\in\mathbf{Z}} \ln\tilde{\sigma}_n(\tau;\tilde{\lambda})\big|_{\tau=t\in\mathbb{R}_+} \Rightarrow \sum_{n\in\mathbf{Z}} \ln\sigma_n(t;z), \tag{9.9}$$

where $\tilde{\lambda} = \tilde{\lambda}(\tau;\lambda)$, $\tau \in \mathbb{R}_+$, and $\lambda(t;z) \in \mathbb{C}$, is a meromorphic solution to the equation

$$\frac{d\lambda}{dt} = g(t;\lambda), \quad \lambda|_{t=0^+} = z \in \mathbb{C} \tag{9.10}$$

with still independent meromorphic function $g(t;\cdot)$ for almost all $t \in \mathbb{R}_+$. The latter can be found by making use of the following determining condition: the local functional

$$\tilde{\varphi}_n(\tau;\tilde{\lambda})\big|_{\tau=t\in\mathbb{R}_+} \to \varphi_n(t;z) \in T^*(M)\otimes\mathbb{C}$$

for all $t \in \mathbb{R}_+$ and $z \in \mathbb{C}$. Hence, the following equality holds immediately:

$$\frac{d}{dt}\left(\sum_{n\in\mathbf{Z}_+} \ln\tilde{\sigma}_n(\tau;\tilde{\lambda})\big|_{\tau=t\in\mathbb{R}_+}\right) =$$

$$= \sum_{n\in\mathbf{Z}_+} \tilde{\sigma}_n^{-1}(t;\tilde{\lambda})\left[\frac{\partial\tilde{\sigma}_n}{\partial t} + \tilde{\sigma}_n' \cdot K_n[u] + \frac{\partial\tilde{\sigma}_n}{\partial\tilde{\lambda}}\, g(t;\tilde{\lambda})\big|_{\tau=t\in\mathbb{R}_+} + \right.$$

$$\left. + \frac{\partial\tilde{\sigma}_n}{\partial\tilde{\lambda}}\,\frac{\partial\tilde{\lambda}}{\partial\lambda}\, g(t;\lambda)\big|_{\tau=t\in\mathbb{R}_+}\right] = 0 \tag{9.11}$$

for all $t \in \mathbb{R}_+$. Equating the coefficients of (9.11) at all powers of the spectral parameter $\lambda(t;z) \in \mathbb{C}$ to zero modulus d/dn, $n \in \mathbf{Z}$, we will find recurrence relations for the function $g(t;\lambda)$ of (9.8). Thereby, using the equation (9.10) and the expansion $\sigma(t;z) \simeq \sum_{j\in\mathbf{Z}_+} \sigma_j[t;u_n]\times$ $\times z^{-j+s(\gamma)}$ for $|z| \to \infty$, where $s(\sigma) \in \mathbf{Z}_+$ is some integer, we obtain an infinite hierarchy of discrete conservation laws of the initially given nonlinear dynamical system (2.1) on the manifold M. But because the conservation laws built above parametrically depend on the evolution parameter $t \in \mathbb{R}_+$, we cannot use right now the theory developed before to prove the Hamiltonian properties of the corresponding vector fields on the invariant submanifolds. To do this in an appropriate way, it is necessary to augment the theory developed before with some important details.

10. The reduction procedure on nonlocal Lagrangian submanifolds. ***10.1. The general algebraic scheme.*** Let $\tilde{\mathcal{G}} := C^\infty(\mathbb{S}^1;\mathcal{G})$ be a Lie algebra of loops, taking values in a matrix Lie algebra $\mathcal{G}$. By means of $\tilde{\mathcal{G}}$ one constructs the Lie algebra $\hat{\mathcal{G}}$ of matrix integral-differential operators [28],

$$\hat{a} := \sum_{j\ll\infty} a_j\xi^j, \tag{10.1}$$

where the symbol $\xi := \partial/\partial x$ denotes the differentiation with respect to the independent variable $x \in \mathbb{R}/2\pi\mathbf{Z} \simeq \mathbb{S}^1$. The usual Lie commutator on $\hat{\mathcal{G}}$ is defined as:

$$\left[\hat{a},\hat{b}\right] := \hat{a}\circ\hat{b} - \hat{b}\circ\hat{a} \tag{10.2}$$

for all $\hat{a},\hat{b} \in \hat{\mathcal{G}}$, where "$\circ$"is the product of integral-differential operators and takes the form

$$\hat{a}\circ\hat{b} := \sum_{\alpha\in\mathbf{Z}_+} \frac{1}{\alpha!}\,\frac{\partial^\alpha\hat{a}}{\partial\xi^\alpha}\,\frac{\partial^\alpha\hat{b}}{\partial x^\alpha}. \tag{10.3}$$

On the Lie algebra $\hat{\mathcal{G}}$ there exists the ad-invariant nondegenerate symmetric bilinear form

$$\left(\hat{a},\hat{b}\right) := \int_0^{2\pi} \mathrm{Tr}\,(\hat{a}\circ\hat{b})\,dx, \tag{10.4}$$

where Tr-operation for all $\hat{a} \in \hat{\mathcal{G}}$ is given by the expression

$$\operatorname{Tr} \hat{a} := \operatorname{res}_{\xi} \operatorname{Sp} \hat{a} = \operatorname{Sp} a_{-1}, \tag{10.5}$$

with Sp being the usual matrix trace. With the scalar product (10.4) the Lie algebra $\hat{\mathcal{G}}$ is transformed into a metrizable one. As a consequence, the linear space, dual to $\hat{\mathcal{G}}$, of the matrix integral-differential operators $\hat{\mathcal{G}}^*$ is naturally identified with the Lie algebra $\hat{\mathcal{G}}$, that is $\hat{\mathcal{G}}^* \simeq \hat{\mathcal{G}}$.

The linear subspaces $\hat{\mathcal{G}}_+ \subset \hat{\mathcal{G}}$ and $\hat{\mathcal{G}}_- \subset \hat{\mathcal{G}}$ such as

$$\begin{gathered}\hat{\mathcal{G}}_+ := \left\{ \hat{a} := \sum_{j=0}^{n(\hat{a})\ll\infty} a_j \xi^j \ : \ a_j \in \tilde{\mathcal{G}}, \ j = \overline{0, n(\hat{a})} \right\}, \\ \hat{\mathcal{G}}_- := \left\{ \hat{b} := \sum_{j=0}^{\infty} \xi^{-(j+1)} b_j \ : \ b_j \in \tilde{\mathcal{G}}, \ j \in \mathbf{Z}_+ \right\},\end{gathered} \tag{10.6}$$

are Lie subalgebras in $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-$. Because of the splitting of $\hat{\mathcal{G}}$ into the direct sum of its Lie subalgebras one can construct the so-called Lie – Poisson structure [14, 27, 29, 30] on $\hat{\mathcal{G}}^*$, using a special linear endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$:

$$\mathcal{R} := \frac{P_+ - P_-}{2}, \quad P_\pm \hat{\mathcal{G}} := \hat{\mathcal{G}}_\pm, \quad P_\pm \hat{\mathcal{G}}_\mp = 0. \tag{10.7}$$

For any Frechet smooth functionals $\gamma, \mu \in \mathcal{D}(\hat{\mathcal{G}}^*)$, the Lie – Poisson bracket on $\hat{\mathcal{G}}^*$ is given by the expression

$$\{\gamma, \mu\}_{\mathcal{R}} (\hat{l}) = \left(\hat{l}, [\nabla\gamma(\hat{l}), \nabla\mu(\hat{l})]_{\mathcal{R}}\right), \tag{10.8}$$

where $\hat{l} \in \hat{\mathcal{G}}^*$ and for all $\hat{a}, \hat{b} \in \hat{\mathcal{G}}$ the $\mathcal{R}$-commutator in (10.8) has the form [14, 27]

$$\left[\hat{a}, \hat{b}\right]_{\mathcal{R}} := \left[\mathcal{R}\hat{a}, \hat{b}\right] + \left[\hat{a}, \mathcal{R}\hat{b}\right], \tag{10.9}$$

subject to which the linear space $\hat{\mathcal{G}}$ becomes a Lie algebra too. The gradient $\nabla\gamma(\hat{l}) \in \hat{\mathcal{G}}$ of a functional $\gamma \in \mathcal{D}(\hat{\mathcal{G}}^*)$ at a point $\hat{l} \in \hat{\mathcal{G}}^*$ with respect to the scalar product (10.4) is defined as

$$\delta\gamma(\hat{l}) := \left(\nabla\gamma(\hat{l}), \delta\hat{l}\right), \tag{10.10}$$

where the linear space isomorphism $\hat{\mathcal{G}} \simeq \hat{\mathcal{G}}^*$ is taken into account.

The Lie – Poisson bracket (10.8) generates Hamiltonian dynamical systems on $\hat{\mathcal{G}}^*$ related to Casimir invariants $\gamma \in I(\mathcal{G}^*)$ and satisfying the condition

$$\left[\nabla\gamma(\hat{l}), \hat{l}\right] = 0, \tag{10.11}$$

as the corresponding Hamiltonian functions. Due to the expressions (10.8) and (10.11) the Hamiltonian system mentioned above takes the form

$$\frac{d\hat{l}}{dt} := \left[\mathcal{R}\nabla\gamma(\hat{l}), \hat{l}\right] = \left[\nabla\gamma_+(\hat{l}), \hat{l}\right], \tag{10.12}$$

being equivalent to the usual commutator Lax-type representation [27, 31]. The relation (10.12) is a compatibility condition for the linear integral-differential equations

$$\begin{aligned} \hat{l} f &= \lambda f, \\ \frac{df}{dt} &= \nabla\gamma_+(\hat{l}) f, \end{aligned} \tag{10.13}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and the vector-function $f \in W(\mathbb{S}^1; \mathbf{H})$ is an element of some matrix representation for the Lie algebra $\hat{\mathcal{G}}$ in some functional Banach space $\mathbf{H}$.

Algebraic properties of the equation (10.12) together with (10.14) and the associated dynamical system on the space of adjoint functions $f^* \in W^*(\mathbb{S}^1; \mathbf{H})$,

$$\frac{df^*}{dt} = -\left(\nabla\gamma(\hat{l})\right)_+^* f^*, \tag{10.14}$$

where $f^* \in W^*$ is a solution to the adjoint spectral problem

$$\hat{l}^* f^* = \nu f^*, \tag{10.15}$$

considered as some coupled evolution equations on the space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$, is an object of our further investigation.

10.2. The tensor product of Poisson structures and its Backlund transformation. To compactify the description below we will use the following notation for the gradient vector:

$$\nabla\gamma\left(\tilde{l}, \tilde{f}, \tilde{f}^*\right) := \left(\frac{\delta\gamma}{\delta\tilde{l}}, \frac{\delta\gamma}{\delta\tilde{f}}, \frac{\delta\gamma}{\delta\tilde{f}^*}\right)^T$$

for any smooth functional $\gamma \in \mathcal{D}(\hat{\mathcal{G}}^* \oplus W \oplus W^*)$. On the spaces $\hat{\mathcal{G}}^*$ and $W \oplus W^*$ there exist canonical Poisson structures [27, 30, 32]

$$\frac{\delta\gamma}{\delta\tilde{l}} \overset{\tilde{\theta}}{:\to} \left[\left(\frac{\delta\gamma}{\delta\tilde{l}}\right)_+, \tilde{l}\right] - \left[\frac{\delta\gamma}{\delta\tilde{l}}, \tilde{l}\right]_+ \tag{10.16}$$

at a point $\tilde{l} \in \hat{\mathcal{G}}^*$ and

$$\left(\frac{\delta\gamma}{\delta\tilde{f}}, \frac{\delta\gamma}{\delta\tilde{f}^*}\right)^T \overset{\tilde{J}}{:\to} \left(\frac{\delta\gamma}{\delta\tilde{f}^*}, -\frac{\delta\gamma}{\delta\tilde{f}}\right)^T \tag{10.17}$$

at a point $(\tilde{f}, \tilde{f}^*) \in W \oplus W^*$ correspondingly. It should be noted that the Poisson structure (10.17) is transformed into (10.12) for any Casimir functional $\gamma \in I(\hat{\mathcal{G}}^*)$. Thus, on the extended space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ one can obtain a Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of the structures (10.17) and (10.18).

Let us consider the following Backlund transformation [27, 32, 33]:

$$\left(\hat{l}, f, f^*\right) :\xrightarrow{B} \left(\tilde{l}(\hat{l}, f, f^*), \tilde{f} = f, \tilde{f}^* = f^*\right), \tag{10.18}$$

generating on $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ a Poisson structure Θ with respect to the variables $(\hat{l}, f, f^*)$ of the coupled evolution equations (10.12), (10.14), (10.15).

The main condition for the mapping (10.19) to be defined is the coincidence of the dynamical system

$$\left(\frac{d\hat{l}}{dt}, \frac{df}{dt}, \frac{df^*}{dt}\right)^T := -\Theta\nabla\gamma(\hat{l}, f, f^*) \tag{10.19}$$

with (10.12), (10.14), (10.15) in the case of $\gamma \in I(\hat{\mathcal{G}}^*)$, i.e., if this functional is taken to be not dependent of the variables $(f, f^*) \in W \oplus W^*$. To satisfy this condition, one has to find a variation of any smooth Casimir functional $\gamma \in I(\hat{\mathcal{G}}^*)$ as $\delta\tilde{l} = 0$, considered as a functional on $\hat{\mathcal{G}}^* \oplus W \oplus W^*$, taking into account flows (10.14), (10.15) and the Backlund transformation (10.19),

$$\begin{aligned}
\delta\gamma(\tilde{l}, \tilde{f}, \tilde{f}^*)\big|_{\delta\tilde{l}=0} &= \left(\left\langle \frac{\delta\gamma}{\delta\tilde{f}}, \delta\tilde{f}\right\rangle\right) + \left(\left\langle \frac{\delta\gamma}{\delta\tilde{f}^*}, \delta\tilde{f}^*\right\rangle\right) = \\
&= \left(\left\langle -\frac{d\tilde{f}^*}{dt}, \delta\tilde{f}\right\rangle\right) + \left(\left\langle \frac{d\tilde{f}}{dt}, \delta\tilde{f}^*\right\rangle\right)\bigg|_{\tilde{f}=f,\, \tilde{f}^*=f^*} = \\
&= \left(\left\langle \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+^* f^*, \delta f\right\rangle\right) + \left(\left\langle \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ f, \delta f^*\right\rangle\right) = \\
&= \left(\left\langle f^*, \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ \delta f\right\rangle\right) + \left(\left\langle \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ f, \delta f^*\right\rangle\right) = \\
&= \left(\frac{\delta\gamma}{\delta\hat{l}}, \delta f\xi^{-1} \otimes f^*\right) + \left(\frac{\delta\gamma}{\delta\hat{l}}, f\xi^{-1} \otimes \delta f^*\right) = \\
&= \left(\frac{\delta\gamma}{\delta\hat{l}}, \delta(f\xi^{-1} \otimes f^*)\right) := \left(\frac{\delta\gamma}{\delta\hat{l}}, \delta\hat{l}\right).
\end{aligned} \tag{10.20}$$

As a result of the expression (10.21) one obtains the relations

$$\delta\hat{l}\big|_{\delta\tilde{l}=0} = \delta(f\xi^{-1} \otimes f^*), \tag{10.21}$$

or, having assumed the linear dependence of $\hat{l}$ and $\tilde{l} \in \hat{\mathcal{G}}^*$, one gets right away that

$$\hat{l} = \tilde{l} + f\xi^{-1} \otimes f^*. \tag{10.22}$$

Thus, the Backlund transformation (10.19) can now be written as

$$(\hat{l}, f, f^*) :\xrightarrow{B} \left(\tilde{l} = \hat{l} - f\xi^{-1} \otimes f^*,\ \tilde{f} = f,\ \tilde{f}^* = \tilde{f}^*\right). \tag{10.23}$$

The expression (10.24) generalizes the result obtained in the papers [27, 33] for the Lie algebra $\hat{\mathcal{G}}$ of integral-differential operators with scalar coefficients. The existence of the Backlund transformation (10.19) makes it possible to formulate the following theorem.

Theorem 10.1. *A dynamical system on $\hat{\mathcal{G}}^* \oplus W \oplus W^*$, being Hamiltonian with respect to the canonical Poisson structure $\tilde{\Theta} : T^*(\hat{\mathcal{G}}^* \oplus W \oplus W^*) \to T(\hat{\mathcal{G}}^* \oplus W \oplus W^*)$, and generated by the evolution equations:*

$$\frac{d\tilde{l}}{dt} = \left[\nabla\gamma_+(\tilde{l}), \tilde{l}\right] - \left[\nabla\gamma(\tilde{l}), \tilde{l}\right]_+, \quad \frac{d\tilde{f}}{dt} = \frac{\delta\gamma}{\delta\tilde{f}^*}, \quad \frac{d\tilde{f}^*}{dt} = -\frac{\delta\gamma}{\delta\tilde{f}}, \tag{10.24}$$

with $\gamma \in I(\mathcal{G}^*)$ being the Casimir functional at $\hat{l} \in \hat{\mathcal{G}}^*$ connected with $\tilde{l} \in \hat{\mathcal{G}}^*$ by (10.23), is equivalent to the system (10.12), (10.14) and (10.15) via the constructed above Backlund transformation (10.24).

By means of simple calculations via the formula (see e. g. [27, 30])

$$\tilde{\Theta} = B'\Theta B'^*,$$

where $B' : T(\hat{\mathcal{G}}^* \oplus W \oplus W^*) \to T(\hat{\mathcal{G}}^* \oplus W \oplus W^*)$ is the Frechet derivative of (10.24), one brings about the following form of the Poisson structure Θ on $\hat{\mathcal{G}}^* \oplus W \oplus W^* \ni (\hat{l}, f, f^*)$:

$$\nabla\gamma(\hat{l}, f, f^*) :\xrightarrow{\Theta} \left(\begin{array}{c} \left[\hat{l}, \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+\right] - \left[\hat{l}, \frac{\delta\gamma}{\delta\hat{l}}\right]_+ - \\ -f\xi^{-1} \otimes \frac{\delta\gamma}{\delta f} + \frac{\delta\gamma}{\delta f^*}\xi^{-1} \otimes f^* \;\; \frac{\delta\gamma}{\delta f^*} - \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ f \;\; -\frac{\delta\gamma}{\delta f} + \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+^* f \end{array} \right).$$

This permits to formulate the next theorem.

Theorem 10.2. *The dynamical system (10.20), being Hamiltonian with respect to the Poisson structure Θ in the form (10.26) and a functional $\gamma \in I(\hat{\mathcal{G}}^*)$, gives the inherited Hamiltonian representation for the coupled evolution equations (10.12), (10.14), (10.15).*

By means of the expression (10.23) one can construct Hamiltonian evolution equations describe commutative flows on the extended space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ at a fixed element $\tilde{l} \in \hat{\mathcal{G}}^*$.

Due to (10.24) every equation of such a type is equivalent to the system

$$\frac{d\hat{l}}{d\tau_n} = \left[\hat{l}^n_+, \hat{l}\right],$$

$$\frac{df}{d\tau_n} = \hat{l}^n_+ f, \tag{10.25}$$

$$\frac{df^*}{d\tau_n} = -(\hat{l}^*)^n_+ f^*,$$

generated by the Casimir invariants $\gamma_n \in I(\hat{\mathcal{G}}^*)$, $n \in \mathbf{N}$, involutive with respect to the Poisson bracket (10.17) and taking here the standard form $\gamma_n = 1/(n+1)(\hat{l}^n, \hat{l})$ at $\hat{l} \in \hat{\mathcal{G}}^*$.

The compatibility conditions for the Hamiltonian systems (10.25) for different $n \in \mathbb{Z}_+$ can be used for obtaining Lax integrable equations on usual spaces of smooth 2π-periodic multivariable functions that will be done in the next section.

10.3. The Lax-type integrable Davey – Stewartson equation and its triple linear representation. Choose the element $\tilde{l} \in \hat{\mathcal{G}}^*$ in an exact form such as

$$\tilde{l} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xi - \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix},$$

where $u, \bar{u} \in C^\infty(\mathbb{S}^1; \mathbb{C})$ and $\mathcal{G} = \mathrm{gl}\,(2; \mathbb{C})$. Then

$$\hat{l} = \tilde{l} + \begin{pmatrix} f_1\xi^{-1}f_1^* & f_1\xi^{-1}f_2^* + u \\ \bar{u} + f_2\xi^{-1}f_1^* & f_2\xi^{-1}f_2^* \end{pmatrix}, \tag{10.26}$$

where $f = (f_1, f_2)^T$ and $f^* = (f_1^*, f_2^*)^T$, "$^-$"can denote the complex conjugation. Below we will study the evolutions (10.25) of vector-functions $(f, f^*) \in W(\mathbb{S}^1; \mathbb{C}^2) \oplus W^*(\mathbb{S}^1; \mathbb{C}^2)$ with respect to the variables $y = \tau_1$ and $t = \tau_2$ at the point (10.26). They can be obtained from the second and the third equations in (10.25) by putting $n = 1$ and $n = 2$, as well as from the first one. The latter is the compatibility condition of the spectral problem

$$\hat{l}\Phi = \lambda\Phi, \tag{10.27}$$

where $\Phi = (\Phi_1, \Phi_2)^T \in W(\mathbb{S}^1; \mathbb{C}^2)$, $\lambda \in \mathbb{C}$ is some parameter, with the following linear equations:

$$\frac{d\Phi}{dy} = \hat{l}_+\Phi, \tag{10.28}$$

$$\frac{d\Phi}{dt} = \hat{l}^2_+\Phi, \tag{10.29}$$

arising from (10.26) at $n = 1$ and $n = 2$ correspondingly. The compatibility of equations (10.28) and (10.29) leads to the relations

$$\begin{gathered}
\frac{\partial u}{\partial y} = -2f_1f_2^*, \quad \frac{\partial \bar{u}}{\partial y} = -2f_1^*f_2, \\
\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial x} - uf_2, \quad \frac{\partial f_1^*}{\partial y} = \frac{\partial f_1^*}{\partial x} - \bar{u}f_2^*, \\
\frac{\partial f_2}{\partial y} = -\frac{\partial f_2}{\partial x} + \bar{u}f_1, \quad \frac{\partial f_2^*}{\partial y} = -\frac{\partial f_2^*}{\partial x} + uf_1^*.
\end{gathered} \tag{10.30}$$

Analogously, replacing $t \in \mathbb{C}$ by $it \in i\mathbb{R}$, $i^2 = -1$, one gets from (10.29) and (10.30):

$$\begin{gathered}
\frac{du}{dt} = i\left(\frac{\partial^2 u}{\partial x \partial y} + 2u(f_1f_1^* + f_2f_2^*)\right), \quad \frac{d\bar{u}}{dt} = -i\left(\frac{\partial^2 \bar{u}}{\partial x \partial y} + 2\bar{u}(f_1f_1^* + f_2f_2^*)\right), \\
\frac{\partial(f_1f_1^*)}{\partial y} - \frac{\partial(f_1f_1^*)}{\partial x} = \frac{1}{2}\frac{\partial(u\bar{u})}{\partial y} = -\left(\frac{\partial(f_2f_2^*)}{\partial x} + \frac{\partial(f_2f_2^*)}{\partial y}\right), \\
\frac{df_1}{dt} = i\left(\frac{\partial^2 f_1}{\partial x^2} + (2f_1f_1^* - u\bar{u})f_1 - \frac{\partial u}{\partial x}f_2\right), \\
\frac{df_1^*}{dt} = -i\left(\frac{\partial^2 f_1^*}{\partial x^2} + (2f_1f_1^* - u\bar{u})f_1^* - \frac{\partial \bar{u}}{\partial x}f_2^*\right), \\
\frac{df_2}{dt} = i\left(\frac{\partial^2 f_2}{\partial x^2} - (2f_2f_2^* + u\bar{u})f_2 - \frac{\partial \bar{u}}{\partial x}f_1\right), \\
\frac{df_2^*}{dt} = -i\left(\frac{\partial^2 f_2^*}{\partial x^2} - (2f_2f_2^* + u\bar{u})f_2^* - \frac{\partial u}{\partial x}f_1^*\right).
\end{gathered} \tag{10.31}$$

The relations (10.31), (10.32) take the well known form of the Davey – Stewartson equation [30, 34] at $\bar{u} \in C^\infty(\mathbb{S}^1;\mathbb{C})$, which is the complex conjugate of $u \in C^\infty(\mathbb{S}^1;\mathbb{C})$. The compatibility for every pair of equations (10.28), (10.29) and (10.30), which can be rewritten as the first order ordinary linear differential equations as

$$\frac{d\Phi}{dx} = \begin{pmatrix} \lambda & u & -f_1 \\ \bar{u} & -\lambda & f_2 \\ f_1^* & f_2^* & 0 \end{pmatrix} \Phi, \tag{10.32}$$

$$\frac{d\Phi}{dy} = \begin{pmatrix} \lambda & 0 & -f_1 \\ 0 & \lambda & -f_2 \\ f_1^* & f_2^* & 0 \end{pmatrix} \Phi, \tag{10.33}$$

$$\frac{d\Phi}{dt} = i \begin{pmatrix} \lambda^2 + f_1 f_1^* & \dfrac{1}{2}\dfrac{\partial u}{\partial y} & -\lambda f_1 - \dfrac{\partial f_1}{\partial y} \\ -\dfrac{1}{2}\dfrac{\partial \bar{u}}{\partial y} & \lambda^2 - f_2 f_2^* & -\lambda f_2 - \dfrac{\partial f_1}{\partial y} \\ \lambda f_1^* + \dfrac{\partial f_1^*}{\partial y} & \lambda f_2^* + \dfrac{\partial f_2^*}{\partial y} & 0 \end{pmatrix} \Phi, \tag{10.34}$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T \in W(\mathbb{S}^1; \mathbb{C}^3)$, providing its Lax-type integrability. Thus, the following theorem holds.

Theorem 10.3. *The Davey – Stewartson equation (10.32), (10.33) possesses the Lax representation as a compatibility condition for equations (10.34) under the additional natural constraint (10.27).*

In fact, one has found above a triple linearization for a (2+1)-dimensional dynamical system, that is a new important ingredient of the Lie algebraic approach to Lax-type integrable flows, based on the Backlund-type transformation (10.23) developed in this work. It is clear that a similar construction of a triple linearization like (10.4) can be done for many other both old and new $(2+1)$-dimensional dynamical systems, on what we plant to stop in detail in another work under preparation.

11. Conclusion. The developed above theory of parametrically Lax-type integrable dynamical systems allows to widen to a great extent the class of exactly treated nonlinear models in many fields of science. It is to be noted here the following important mathematical fact obtained in the paper: almost every nonlinear dynamical system admits a parametrically isospectral Lax-type representation but a given dynamical system is the Lax-type integrable if an evolution of the spectrum parameter doesn't depend on a point $u \in M$ at all Cauchy data. This result has allowed us to develop a very effective direct criterion for the following problem: whether a given nonlinear dynamical system on the functional manifold M is parametrically Lax-type integrable or not. Having the problem above solved, we have suggested the reduction procedure for the associated nonlinear dynamical systems to be descended on the invariant submanifold $M_N \subset M$ built before inheriting the canonical Hamiltonian structure and the Liouville complete integrability. Thereby, the powerful techniques of perturbation theory can be successfully used for dynamical systems under consideration, as well as the relationships between the full Hamiltonian theory and various Hamiltonian truncations could be now got understandable more deeply.

The imbedding problem for infinite-dimensional dynamical systems with additional structures such as invariants and symmetries is as old as the Newton – Lagrange mechanics, having been treated by many researches, using both analytical and algebraic methods. The powerful differential-geometric tools used here were created mainly in works by E. Cartan at the beginning of the twentieth century. The great impact in the development of imbedding the methods was done in recent time, especially owing to the theory of isospectral deformations for some linear structures built on special vector bundles over the space M as the base of a given nonlinear dynamical system. Among them there are such structures as the moment map $l : M \to \mathcal{G}^*$ into the adjoint space to the Lie algebra $\mathcal{G}$ of symmetries, acting on the symplectic phase space M equivariantly [6, 7], the connection of the Cartan – Eresman structures appearing via the Wahlquist – Estabrook approach [8], and many others.

For the last years the general structure of Lagrangian and Hamiltonian formalisms was studied thoroughly using both geometrical and algebraical methods [9, 10]. The special attention was paid to the theory of differential-difference dynamical systems on the infinite-dimensional manifolds [10, 35]. Some number of articles was devoted to the theory of pure discrete dynamical systems [36 – 39], as well treating the interesting examples [39] appeared to be important for applications.

In future work we intend to treat further imbedding problems for infinite-dimensional both continuous and discrete dynamical systems basing on the differential-geometric Cartan's theory of differential ideals in Grassmann algebras over jet-manifolds, intimately connected with the problem under consideration. As it is well known, there existed by now only two regular enough algorithmic approaches [27, 28, 33] to constructing integrable multidimensional (mainly 2 + 1) dynamical systems on infinite-dimensional functional spaces. Our approach, devised in this work, is substantially based on the results previously done in [27, 33], explains completely the computational properties of multidimensional flows before delivered in works [30, 34]. As the key points of our approach there used the canonical Hamiltonian structures naturally existing on the extended phase space and the related with them Backlund transformation which saves Casimir invariants of a chosen matrix integral-differential Lie algebra. The latter gives rise to some additional Hamiltonian properties of the considered extended evolution flows studied before in [27, 30] making use of the standard inverse scattering transform [27, 30, 31] and the formal symmetry reduction for the KP-hierarchy [32, 33] of commuting operator flows.

As one can convince ourselves analyzing the structure of the Backlund-type transformation (10.24), it strongly depends on the type of an ad-invariant scalar product chosen on an operator Lie algebra $\hat{\mathcal{G}}$ and its Lie algebra decomposition like (10.6). Since there exist in general other possibilities of choosing such decompositions and ad-invariant scalar products on $\hat{\mathcal{G}}$, they give rise naturally to another resulting types of the corresponding Backlund transformations, which can be a subject of another special investigation. Let us here only mention the choice of a scalar product related with the operator Lie algebra $\hat{\mathcal{G}}$ centrally extended by means of the standard Maurer – Cartan two-cocycle [14, 27, 28], bringing about new types of multidimensional integrable flows.

The last aspect of the Backlund approach to constructing Lax-type integrable flows and their partial solutions which is worth of mention is related with Darboux – Backlund-type transformations [30, 40] and their new generalization recently developed in [33, 41]. They give rise to very effective procedures of constructing multidimensional integrable flows on functional spaces with arbitrary number of independent variables simultaniously delivering a wide class of their exact analytical solutions, depending on many constant parameters, which can appear to be useful for diverse applications in applied sciences.

All the mentioned above Backlund-type transformations aspects can be studied as special investigations, giving rise to new directions in the theory of multidimensional evolution flows and their integrability.

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