UDC 517.9

POSITIVE SOLUTIONS OF LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

ДОДАТНІ РОЗВ'ЯЗКИ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ІМПУЛЬСНОЮ ДІЄЮ

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The paper deals with existence of positive (nonnegative) solutions of linear homogeneous impulsive differential equations. The main result is also applied to investigate the similar problem for higher order linear homogeneous impulsive differential equations. All results are formulated in terms of coefficients of the equations.

Розглядається існування додатних (невід'ємних) розв'язків лінійних однорідних диференціальних рівнянь з імпульсною дією. Основний результат також використовується для вивчення подібної проблеми для лінійних однорідних рівнянь вищого порядку з імпульсною дією. Всі результати сформульовано в термінах коефіцієнтів рівнянь.

1. Introduction and preliminaries. The problem of positive solutions for different type of differential equations has attracted the attention of many researchers [1–4]. They considered positiveness of solutions defined on the positive half line, whole line, and solutions of boundary-value problems.

In the last several decades, theory of impulsive differential equations has been developed very intensively to keep up with demands of disciplines such as biology, mechanics, medicine, etc. Many results of the theory can be found in profoundly written books [5] and [6] and in the references cited in these books. Naturally, the problem of positive solutions for the impulsive differential equations has become important. One can mention the paper [7] in this subject.

Our article concerns with existence of positive solutions of linear impulsive homogeneous systems of the first order and of higher order linear equations. Apparently, this work is one of the first in the subject.

To obtain the result we shall develop, for impulsive differential equations, a method which was first proposed in [8]. We intent to find conditions on the impulsive part of the systems, which provide, together with conditions on differential equations, existence of nonnegative solutions on positive half line.

Let us denote by $\mathbb{R}$, $\mathbb{N}$, the sets of all real numbers and positive integers respectively. Throughout the paper, some abbreviations are used to simplify the notation: If $x = (x_1, \ldots, x_n)$ is a vector, then $x \geq 0$ means that $x_k \geq 0$ for $k = 1, \ldots, n$. Similarly, if $A = (a_{ik})$ is an $n \times n$ matrix, then $A \geq 0$ means that $a_{ik} \geq 0$ for $i, k = 1, \ldots, n$.

Consider the system of impulsive linear differential equations,

$$x'(t) = -A(t)x, \quad t \neq \theta_i, \tag{1}$$

$$\Delta x \mid_{t=\theta_k} = -B_k x, \tag{2}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+ = [0, \infty)$, $A(t)$ is an $n \times n$ matrix of continuous functions and B_k are constant $n \times n$ matrices.

We assume that throughout the paper, the following conditions on the system (1), (2) are fulfilled:

C$_1$) $A(t) \in C(\mathbb{R}_+)$, $A(t) \geq 0$ for $t \in \mathbb{R}_+$;

C$_2$) $\{\theta_k\} \subset \mathbb{R}_+ \backslash \{0\}$, $k \in \mathbb{N}$, is a strictly ordered sequence such that $\theta_k \to \infty$ as $k \to \infty$;

C$_3$) $B_k \geq 0$, $\det(I - B_k)^{-1} \neq 0$ and $(I - B_k)^{-1} \geq 0$ for $k \in \mathbb{N}$.

From the theory of impulsive differential equations [5, 6], it is known that the system (1), (2) satisfies the conditions for existence and uniqueness of solution. Moreover, every solution $x(t)$ of the system can be continued to $+\infty$.

We also consider the linear impulsive differential equation of n-th order,

$$a_0(t)y^{(n)} + \sum_{k=1}^{n}(-1)^{k+1}a_k(t)y^{(n-k)} = 0, \quad t \neq \theta_i, \tag{3}$$

$$\Delta \widehat{y} \mid_{t=\theta_i} = B\widehat{y}, \tag{4}$$

where $\widehat{y}(t) = [y(t), \ldots, y^{(n-1)}(t)]^T$, $y^{(i)} = \dfrac{d^i y}{dt^i}$ and

$$B_k = \begin{bmatrix} b_{11}^k & b_{12}^k & \ldots & b_{1n}^k \\ b_{21}^k & b_{22}^k & \ldots & b_{2n}^k \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1}^k & b_{n2}^k & \ldots & b_{nn}^k \end{bmatrix}.$$

For the n-th order system, following conditions are imposed:

D$_1$) the coefficient functions $a_k(t) \in C(\mathbb{R}_+)$ for $k = 1, \ldots, n$ satisfy the following:

$$a_0 > 0, \quad a_k \geq 0, \quad k = 2, \ldots, n;$$

D$_2$) $\{\theta_k\} \subset \mathbb{R}_+ \backslash \{0\}$, $k \in \mathbb{N}$, is a strictly ordered sequence such that $\theta_k \to \infty$ as $k \to \infty$;

D$_3$) $(-1)^{i+j} b_{ij} \leq 0$ for $i, j = 1, \ldots, n$, $\det(I - B_k)^{-1} \neq 0$ and $(I - B_k)^{-1} \geq 0$, $k \in \mathbb{N}$.

One should emphasize that the idea of considering higher order impulsive differential equations with discontinuity in all derivatives is not new [9–13]. It is understood that the solutions of the systems (1), (2) and (3), (4) are left continuous functions with discontinuities of the first type at the points θ_i.

2. Main results.

Theorem 1. *Assume that assertions $C_1) - C_3)$ are fulfilled. Then there exists a solution $x = x(t)$ of (1), (2), which is not identically zero, satisfying*

$$x(t) \geq 0, \quad -x'(t) \geq 0, \quad t \in \mathbb{R}_+, \tag{5}$$

$$-\Delta x(\theta_i) \geq 0, \quad \Delta x'(\theta_i) \geq 0, \quad i \in \mathbb{N}. \tag{6}$$

Proof. We intend to prove existence of positive valued solution $x^0(t)$ constructively as a limit of solutions which are positive on sections.

a) Fix an integer $r > 0$ and $x_0 > 0$. We shall show that for a given initial condition (r, x_0), there exists a positive valued solution $x^r(t) = x(t, r, x_0)$ of (1) and (2) on $[0, r]$. We assume, without loss of generality, that $\theta_l < r \leq \theta_{l+1}$ for some $l \geq 1$. Let us show that on $(\theta_l, r]$, $x^r(t)$ is decreasing and positive valued. Since $x^r(r) = x_0 > 0$, it follows that $x^r(t) > 0$ for t near r, hence $\dfrac{x^r(t)}{dt} \leq 0$ for t near r. Thus $x^r(t) \geq x^r(r) > 0$ for t less than and close to r. This argument shows that $x^r(t) > 0$ and $\dfrac{x^r(t)}{dt} \leq 0$ for $t \in (\theta_l, r]$. In order to define a value $x^r(\theta_l^-)$, we, by using (2), obtain,

$$x^r(\theta_l^+) = (I - B_l)x^r(\theta_l^-)$$

and, hence,

$$x^r(\theta_l^-) = (I - B_l)^{-1}x^r(\theta_l^+).$$

Then, we could continue the solution on $(\theta_{l-1}, \theta_l]$. Proceeding in this way, we construct the solution $x^r(\theta_i)$ on interval $[0, r]$. In the case when $0 < r < \theta_1$ one can proceed in the same manner as it has been done for the interval $(\theta_l, r]$ above.

b) In stage a) we construct a solution $x^r(t)$ for every $r \geq 1$. It is clear that $z^r(t) = \dfrac{x^r(t)}{\|x^r(0)\|}$ is a solution of (1) and (2) and $\|z^r(0\| = 1$. There exists a subsequence of $z^r(0)$, $r \geq 1$, which converges to z^0 with $\|z^0\| = 1$ (we assume without loss of generality that the convergent subsequence is the sequence $z^r(0)$ itself). Fix an arbitrary integer $i \geq 1$. Using Theorem 5 from [5], one can show that $z^r(t)$ is convergent in sup-norm on $[0, i]$ to the function $x^0(t)$, which is a solution of (1) and (2) on the interval $[0, i]$ and it is nonnegative since all solutions $z^r(t)$ are positive on $[0, i]$ if $r \geq i$. Since i is arbitrary, $x^0(t)$ is a nonnegative solution of (1) and (2) on $\mathbb{R}_+$.

c) Substituting $x^0(t)$ in (1) we obtain the second inequality in (5),

$$\frac{dx^0(t)}{dt} = -A(t)x^0(t) \leq 0, \quad t \neq \theta_i.$$

Similarly using $x^0(\theta_i) \geq 0$, condition $C_3)$ and equation (2), we have the first inequality in (6) as follows:

$$-\Delta x^0(\theta_i) = B_i\, x^0(\theta_i) \geq 0.$$

While for the second inequality in (6), we use (2) again,

$$\Delta x^{0'}(\theta_i) = x^{0'}(\theta_i^+) - x^{0'}(\theta_i^-) = -A(\theta_i^+)\,x(\theta_i^+) + A(\theta_i^-)\,x(\theta_i^-) = -A(\theta_i)\,\Delta x(\theta_i) \geq 0.$$

This concludes the proof.

In the following theorem, we generalize the above result to the n-th order case.

Theorem 2. *Assume that the linear impulsive differential equation of n-th order satisfies the conditions $D_1) - D_3)$. Then the system (3), (4) has a solution $y = y(t)$ which is positive for $t \in \mathbb{R}_+$ and, what is more,*

$$(-1)^j\, y^{(j)} \geq 0, \quad j = 0, 1, \ldots, n-1, \tag{7}$$

$$\Delta y \leq 0, \Delta y' \geq 0, \ldots, (-1)^{n-1}\Delta y^{(n-1)} \leq 0. \tag{8}$$

Proof. Let

$$g(t) = \exp\left[\int\limits_1^t a_1(s)ds\right] > 0.$$

Next, we change the variables,

$$y = x_1, y^{'} = -x_2, \ldots, y^{(n-2)} = (-1)^{n-2}x_n, \quad y^{(n-1)} = (-1)^{n-1}\frac{x_n}{g}.$$

Then, the system becomes

$$x_1' = -x_2, \quad t \neq \theta_i,$$

$$x_2' = -x_3, \quad t \neq \theta_i,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_{n-1}' = -\frac{x_n}{g}, \quad t \neq \theta_i,$$

$$x_n' = -gx_{n-1}\frac{a_2(t)}{a_0(t)} - gx_{n-2}\frac{a_3(t)}{a_0(t)} - \ldots - gx_1\frac{a_n(t)}{a_0(t)}, \quad t \neq \theta_i,$$

$$\Delta\widehat{x}\mid_{t=\theta_i} = \widehat{B}_i\widehat{x},$$

where $\widehat{x}(t) = [x_1(t), \ldots, x_n(t)]^T$ and

$$\widehat{B}_i = \begin{bmatrix} b_{11}^i & -b_{12}^i & \cdots & (-1)^{n+1}b_{1n}^i \\ -b_{21}^i & b_{22}^i & \cdots & (-1)^{n+2}b_{2n}^i \\ \ldots & \ldots & \ldots & \ldots \\ (-1)^{n+1}b_{n1}^i & (-1)^{n+2}b_{n2}^i & \ldots & (-1)^{n+n}b_{nn}^i \end{bmatrix}.$$

If the above transformed system is identified with the one in Theorem 1, we have

$$\widehat{x}(t) \geq 0, \quad -\widehat{x}'^{(}t) \geq 0, \quad \text{and} \quad -\Delta\widehat{x}'(t) \geq 0, \quad \Delta\widehat{x}'(t) \geq 0.$$

By retaining the original variables we have

$$\begin{pmatrix} y \\ -y' \\ \vdots \\ (-)^{n-1}y^{n-1} \end{pmatrix} \geq 0 \quad \text{and} \quad -\begin{pmatrix} \Delta y \\ -\Delta y' \\ \vdots \\ (-1)^{n-1}\Delta y^{n-1} \end{pmatrix} \geq 0.$$

This concludes the proof.

3. Examples.

Example 1. Consider the following coupled system:

$$\begin{aligned} mx'' &= 2kx + ky, \quad t \neq \theta_i, \\ 2my'' &= kx + 2ky, \quad t \neq \theta_i, \\ \Delta x|_{t=\theta_i} &= -\frac{1}{2}\,x + \frac{1}{4}\,x', \\ \Delta x'|_{t=\theta_i} &= \frac{1}{4}\,x - \frac{1}{2}\,x', \\ \Delta y|_{t=\theta_i} &= -\frac{1}{2}\,y, \\ \Delta y'|_{t=\theta_i} &= -\frac{1}{2}\,y', \end{aligned} \tag{9}$$

where m and k are positive real numbers and $\theta_i = 2i$, $i = 1, 2, \ldots$. Theorem 2 is not applicable for this example. But by changing the variables in the above system, Theorem 1 will be applicable,

$$x = z_1, \quad y = z_3, \quad x' = -z_2, \quad y' = -z_4$$

so, the system becomes,

$$z' = -Az, \quad t \neq \theta_i,$$

$$\Delta z|_{t=\theta_i} = Bz,$$

where $z = [z_1, z_2, z_3, z_4]^T$. If the above system is identified with (1), (2), the matrices A and B

become

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2k}{m} & 0 & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{2m} & 0 & \frac{k}{2m} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

and we can evaluate that

$$(I - B_i)^{-1} = \begin{bmatrix} \frac{8}{3} & \frac{4}{3} & 0 & 0 \\ \frac{4}{3} & \frac{8}{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The system (9) satisfies the conditions of Theorem 1, so there exists a solution which satisfies

$$z \geq 0, \quad -z' \geq 0 \quad \text{and} \quad -\Delta z \geq 0, \quad \Delta z' \geq 0.$$

By retaining the original variables,

$$\begin{pmatrix} x \\ y \end{pmatrix} \geq 0, \quad -\begin{pmatrix} x' \\ y' \end{pmatrix} \geq 0 \quad \text{and} \quad -\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \geq 0, \quad \begin{pmatrix} \Delta x' \\ \Delta y' \end{pmatrix} \geq 0.$$

Example 2. Consider the following second order system:

$$\begin{aligned} y'' + y' - (\sin^2 t)x &= 0, \quad t \neq \theta_i, \\ \Delta y|_{t=\theta_i} &= -2y + 0{,}001y', \\ \Delta y'|_{t=\theta_i} &= 0{,}2y - 3y', \end{aligned} \tag{10}$$

where $\theta_i = i, i = 1, 2, \ldots$. If the above system is identified with (3), (4), the matrix B becomes

$$B = \begin{bmatrix} -2 & 0{,}001 \\ 0{,}2 & -3 \end{bmatrix},$$

and one can evaluate that

$$(I - B)^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{25}{30} \cdot 10^{-5} \\ \frac{1}{60} & \frac{1}{4} \end{bmatrix},$$

and $a_0 = 1$, $a_2 = \sin^2 t \geq 0$. The system (10) satisfies the conditions of Theorem 2, so, there is at least one solution $y = y(t)$ which satisfies

$$y \geq 0, \quad -y' \geq 0,$$

$$-\Delta y \geq 0, \quad \Delta y' \geq 0.$$

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Received 13.02.2005,
after revision — 23.05.2005