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OSCILLATION OF NONLINEAR HYPERBOLIC DIFFERENTIAL EQUATIONS WITH IMPULSES *

КОЛИВАННЯ НЕЛІНІЙНИХ ГІПЕРБОЛІЧНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ІМПУЛЬСАМИ

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We study oscillatory properties of solutions for nonlinear impulsive hyperbolic differential equations and find new necessary and sufficient conditions for oscillations are established.

Вивчено властивості коливань розв'язків нелінійних гіперболічних диференціальних рівнянь з імпульсами і знайдено нові необхідні та достатні умови для існування коливань.

1. Introduction. The theory of differential equations can be applied to many fields, such as biology, population growth, engineering, medicine, physics and chemistry. In the last few years, a few of papers have been published on oscillation theory of partial differential equations. Many have been done under the assumption that the state variables and the system parameters change continuously. However, one may easily visualize situations in nature where an abrupt change such as a shock or disasters may occur. These phenomena are short-time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modelling these problems, that these perturbations act instantaneously, that is, in the form of impulses. In 1991, the first paper on this class of equations [1] was published. The qualitative theory of this class of equations, however, is still in an initial stage of development. For instance, on oscillation theory of impulsive partial differential equations only a few papers have been published. Recently, Bainov, Minchev, Deng, Fu and Luo [2 – 5] investigated the oscillation of solutions of impulsive partial differential equations with or without deviating argument. But there is a scarcity in the study of oscillation theory of nonlinear impulsive hyperbolic partial differential equations.

In this paper, we shall discuss the oscillatory properties of solutions for a class of nonlinear hyperbolic differential equations with impulses (1), under the boundary condition (4). It should be noted that the equation we discuss here is nonlinear. Up to now, we did not find a work on oscillations for is kind of the problem. We consider the following:

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$$\frac{\partial^2 u}{\partial t^2} = a(t)h(u)\Delta u - q(t,x)f(u(t,x)),$$
$$t \neq t_k, \quad (t,x) \in R_+ \times \Omega = G, \tag{1}$$

$$u(t_k^+, x) - u(t_k^-, x) = q_k u(t_k, x),\ t = t_k,\ k = 1, 2, \ldots, \tag{2}$$

$$u_t(t_k^+, x)dx - u_t(t_k^-, x)dx = b_k u_t(t_k, x)dx \tag{3}$$

with the boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad (t,x) \in R_+ \times \partial\Omega. \tag{4}$$

Here $\Omega \subset R^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and n is a unit exterior normal vector of $\partial\Omega$.

Assume that the following conditions are fulfilled:

H_1) $a(t) \in PC(R_+, R_+), q(t,x) \in C(R_+ \times \overline{\Omega}, (0, \infty))$; where PC denotes the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k, k = 1, 2, \ldots$ and left continuous at $t = t_k, k = 1, 2, \ldots$;

H_2) $h'(u), f(u) \in C(R, R)$; $f(u)/u \geq C = \text{const} > 0$, for $u \neq 0$; $uh'(u) \geq 0$, and $q_k > -1, b_k > -1, 0 < t_1 < t_2 < \ldots < t_k < \ldots, \lim\limits_{t\to\infty} t_k = \infty$;

H_3) $u(t,x)$ and their derivatives $u_t(t,x)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k, k = 1, 2, \ldots,$ and left continuous at $t = t_k, u(t_k, x) = u(t_k^-, x), u_t(t_k, x) = u_t(t_k^-, x), k = 1, 2, \ldots$.

Definition 1. *By a solution of problem (1), (4), we mean that any function $u(t,x)$ which satisfies the condition H_3) and coincides with the solution of the problem (1), (2), (3) and (4).*

We introduce the notations: $\Gamma_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \Omega\}, \Gamma = \bigcup_{k=0}^{\infty} \Gamma_k$, $\overline{\Gamma}_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \overline{\Omega}\}, \overline{\Gamma} = \bigcup_{k=0}^{\infty} \overline{\Gamma}_k$, $v(t) = \int\limits_{\Omega} u(t,x)dx$, and $p(t) = \min q(t,x)$, $x \in \overline{\Omega}$.

Definition 2. *The solution $u \in C^2(\Gamma) \bigcap C^1(\overline{\Gamma})$ of problem (1), (4) is called nonoscillatory in the domain G if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.*

2. Oscillation properties of the problem (1), (4). The following is the main theorem of this paper. The proof of the theorem needs the following lemmas.

Lemma 1. *Let $u \in C^2(\Gamma) \bigcap C^1(\overline{\Gamma})$ be a positive solution of the problem (1), (4) in G, then the function $v(t)$ satisfies the impulsive differential inequality*

$$v''(t) + Cp(t)v(t) \leq 0,\ t \neq t_k, \tag{5}$$

$$v(t_k^+) = (1 + q_k)v(t_k), \;\; k = 1, 2, \ldots, \tag{6}$$

$$v'(t_k^+) = (1 + b_k)v'(t_k), \;\; k = 1, 2, \ldots. \tag{7}$$

Proof. Let $u(t, x)$ be a positive solution of the problem (1), (4) in G. Without loss of generality, we may assume that $u(t, x) > 0$ for any $(t, x) \in [t_0, \infty) \times \Omega$.

For $t \geq t_0, t \neq t_k, k = 1, 2, \ldots,$ integrating (1) with respect to x over Ω yields

$$\frac{d^2}{dt^2}\left[\int\limits_{\Omega} u dx\right] = a(t)\int\limits_{\Omega} h(u)\Delta u dx - \int\limits_{\Omega} q(t, x)f(u(t, x))dx, \;\; t \geq t_0, \;\; t \neq t_k.$$

By Green's formula and the boundary condition, we have

$$\int\limits_{\Omega} h(u)\Delta u dx = \int\limits_{\partial\Omega} h(u)\frac{\partial u}{\partial n}ds - \int\limits_{\Omega} h'(u)|\mathrm{grad}\, u|^2 dx \leq -\int\limits_{\Omega} h'(u)|\mathrm{grad}\, u|^2 dx \leq 0.$$

From condition H_2), we can easily obtain

$$\int\limits_{\Omega} q(t, x)f(u(t, x))dx \geq Cp(t)\int\limits_{\Omega} u(t, x)dx.$$

It follows from the above that

$$v'' + Cp(t)v(t) \leq 0, \;\; t \geq t_0, \;\; t \neq t_k, \tag{8}$$

where $v(t) > 0$.

For $t > t_0, t = t_k, k = 1, 2, \ldots,$ we have

$$\int\limits_{\Omega} u(t_k^+, x)dx - \int\limits_{\Omega} u(t_k^-, x)dx = q_k\int\limits_{\Omega} u(t_k, x)dx,$$

$$\int\limits_{\Omega} u_t(t_k^+, x)dx - \int\limits_{\Omega} u_t(t_k^-, x)dx = b_k\int\limits_{\Omega} u_t(t_k, x)dx.$$

This implies

$$v(t_k^+) = (1 + q_k)v(t_k), \tag{9}$$

$$v'(t_k^+) = (1 + b_k)v'(t_k), \;\; k = 1, 2, \ldots. \tag{10}$$

Hence we obtain that $v(t) > 0$ is a positive solution of differential inequality (5) – (7). This ends the proof of the lemma.

Definition 3. *The solution $v(t)$ of differential inequality (5) – (7) is called eventually positive (negative) if there exists a number t^* such that $v(t) > 0$ ($v(t) < 0$) for $t \geq t^*$.*

Lemma 2 (Theorem 1.4.1). *Assume that*
(i) $m(t) \in PC^1[R^+, R]$ is left continuous at t_k for $k = 1, 2, \ldots$;
(ii) for $k = 1, 2, \ldots, t \geq t_0$,

$$m'(t) \leq p(t)m(t) + q(t),\ t \neq t_k,$$

$$m(t_k^+) \leq d_k m(t_k) + e_k,\ t \neq t_k,$$

where $p(t), q(t) \in C(R^+, R), d_k \geq 0$ and e_k are real constants, $PC^1[R^+, R] = \{x : R^+ \to \to R; x(t)$ is continuous and continuously differentiable everywhere except for some t_k at which $x(t_k^+), x(t_k^-), x'(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k) = x(t_k^-), x'(t_k) = x'(t_k^-)\}$.
Then

$$m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(r)dr\right) q(s)ds +$$

$$+ \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right) e_k.$$

From Lemma 2 we can obtain the following lemma (see also [5]).

Lemma 3. *Let $v(t)$ be an eventually positive (negative) solution of differential inequality (5) – (7). Assume that there exists $T \geq t_0$ such that $v(t) > 0$ ($v(t) < 0$) for $t \geq T$. If the following condition holds,*

$$\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{1 + b_k}{1 + q_k} ds = +\infty, \tag{11}$$

then $v'(t) \geq 0$ ($v'(t) \leq 0$) for $t \in [T, t_l] \bigcup (\bigcup_{k=l}^{+\infty} (t_k, t_{k+1}))$, where $l = \min\{k : t_k \geq T\}$.

Theorem 1. *Let condition (11) and the following condition hold:*

$$\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{1 + q_k}{1 + b_k} p(s)ds = +\infty. \tag{12}$$

Then each solution of the problem (1) – (4) oscillates in G.

Proof. Let $u(t, x)$ be a nonoscillatory solution of (1), (4). Without loss of generality, we can assume that $u(t, x) > 0$ for any $(t, x) \in [t_0, \infty) \times \Omega$. From Lemma 1, we know that $v(t)$ is a positive solution of (5) – (7). Thus from Lemma 4, we can find that $v'(t) \geq 0$ for $t \geq t_0$.

For $t \geq t_0, t \neq t_k, k = 1, 2, \ldots$, define

$$w(t) = \frac{v'(t)}{v(t)},\ t \geq t_0.$$

Then we have $w(t) > 0, t \geq t_0, v'(t) - w(t)v(t) = 0$. We may assume that $v(t_0) = 1$. Thus in view of (5)–(7) we have that, for $t \geq t_0$,

$$v(t) = \exp\left(\int_{t_0}^{t} w(s)ds\right), \tag{13}$$

$$v'(t) = w(t)\exp\left(\int_{t_0}^{t} w(s)ds\right), \tag{14}$$

$$v''(t) = w^2(t)\exp\left(\int_{t_0}^{t} w(s)ds\right) + w'(t)\exp\left(\int_{t_0}^{t} w(s)ds\right). \tag{15}$$

We substitute (13)–(15) into (5) and can obtain the following inequality:

$$w^2(t) + w'(t) + Cp(t) \leq 0.$$

From (6), (7) we get

$$w(t_k^+) = \frac{v'(t_k^+)}{v(t_k^+)} = \frac{1+b_k}{1+q_k} w(t_k),\ k = 1, 2, \ldots .$$

It follows that

$$w'(t) \leq -Cp(t),\ t \neq t_k,$$

$$w(t_k^+) = \frac{1+b_k}{1+q_k} w(t_k),\ t = t_k.$$

By using Lemma 3, we obtain

$$w(t) \leq w(t_0) \prod_{t_0 < t_k < t} \frac{1+b_k}{1+q_k} + \int_{t_0}^{t} \prod_{s < t_k < t} \frac{1+b_k}{1+q_k}(-Cp(s))ds =$$

$$= \prod_{t_0 < t_k < t} \frac{1+b_k}{1+q_k} \left\{ w(t_0) - \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{1+b_k}{1+q_k} Cp(s)ds \right\}.$$

Since $w(t) > 0$, the last inequality contradicts (12).

The proof of Theorem 1 is completed.

It should be noted that obviously all solutions of problem (1), (4) are oscillatory if there exists a subsequence n_k of n such that $q_{n_k} < -1$, for $k = 1, 2, \ldots$. So we only discuss the case $q_k > -1$.

3. Necessary and sufficient conditions. In this section, we will establish necessary and sufficient conditions for oscillation of an impulsive hyperbolic partial differential equation. We consider the following linear problem:

$$u_{tt} = a(t)\Delta u + p(t)u(t,x),$$

$$t \neq t_k,\ (t,x) \in R_+ \times \Omega = G, \tag{16}$$

$$u(t_k^+) - u(t_k^-) = q_k u(t_k, x),\ t = t_k,\ k = 1, 2, \ldots, \tag{17}$$

$$u_t(t_k^+) - u_t(t_k^-) = b_k u_t(t_k, x),\ t = t_k,\ k = 1, 2, \ldots, \tag{18}$$

wiht boundary condition (4).

Theorem 2. *A necessary and sufficient condition of oscillations in domain G for all solutions of the problems (16) – (18), (4) is that all solutions of the following impulsive differential equation (19) – (21) be oscillatory:*

$$\frac{d^2v}{dt^2} + p(t)v(t) = 0, \tag{19}$$

$$v(t_k^+) - v(t_k^-) = q_k v(t_k),\ k = 1, 2, \ldots, \tag{20}$$

$$v'(t_k^+) - v'(t_k^-) = b_k v'(t_k),\ k = 1, 2, \ldots. \tag{21}$$

Proof. *Sufficiency.* We argue by contradiction. Let $u(t,x)$ be a nonoscillatory solution of the problem (16) – (18), (4). Without loss of generality, we may assume that there exists a $t_0 \geq T$ such that $u(t,x) > 0$ for any $(t,x) \in [t_0, +\infty) \times \Omega$.

For $t \geq t_0, t \neq t_k, k = 1, 2, \ldots$, integrating (16) with respect to x over Ω yields

$$\frac{d^2}{dt^2}\int\limits_{\Omega} u dx = a(t)\int\limits_{\Omega} \Delta u dx - \int\limits_{\Omega} p(t)u(t,x)dx.$$

By Green's formula, we have

$$\int\limits_{\Omega} \Delta u dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} ds = 0.$$

Denote $v(t) = \int\limits_{\Omega} u(t,x)dx$. Then $v(t) > 0$. It follows that

$$\frac{d^2v}{dt^2} + p(t)v(t) = 0. \tag{22}$$

For $t \geq t_0, t = t_k, k = 1, 2, \ldots$, similarly to (9), (10) we have

$$v(t_k^+) - v(t_k^-) = q_k v(t_k),$$
$$v'(t_k^+) - v'(t_k^-) = b_k v'(t_k), \ k = 1, 2, \ldots . \tag{23}$$

Hence we obtain that $v(t) > 0$ satisfies equation (19) – (21). This means that the impulsive differential equation (19) – (21) has a nonoscillatory solution. A contradiction. This ends the proof of the sufficient condition.

Necessity. We still argue by contradiction. Let $v(t)$ be a nonoscillatory solution of equation (19) – (21). Without loss of generality, we may assume that there exists a t_1 large enough such that $v(t) > 0$ for any $t \in [t_1, +\infty)$.

For $t \geq t_1, t \neq t_k, k = 1, 2, \ldots$, set $u(t,x) = v(t)$, we have $u(t,x) > 0$ and then easily obtain

$$\Delta u(t,x) = \Delta v(t) = 0.$$

Making use of this result, from equation (19), we obtain

$$\frac{d^2 v(t)}{dt^2} + a(t)\Delta v(t) + p(t)v(t) = 0.$$

This means that $u(t,x) = v(t)$ satisfies equation (16).

For $t \geq t_1, t = t_k, k = 1, 2, \ldots$, from the conditions (22), (23), it is easy to see that the function $u(t,x) = v(t)$ satisfies (17), (18).

And because $\dfrac{\partial v}{\partial x} = 0, x \in \partial\Omega, u(t,x) = v(t)$ also satisfies boundary condition (4). This indicates that problem (16) – (18), (4) has a nonoscillatory solution. This is a contradiction. This ends the proof of Theorem 2.

4. Remark. The results of this paper, from the practical standpoint, is very convenient because these criterions depend only on the coefficients of the equations and the impulsive term. The necessary and sufficient condition of oscillations reveals a relation between the impulsive partial differential equation and the impulsive differential equation. It should be noted that the equations we discuss here are nonlinear.

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