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GINZBURG – LANDAU SYSTEM OF COMPLEX MODULATION EQUATIONS FOR A DISTRIBUTED NONLINEAR-DISSIPATIVE TRANSMISSION LINES

СИСТЕМА РІВНЯНЬ ГІНЗБУРГА – ЛАНДАУ НА КОМПЛЕКСНУ МОДУЛЯЦІЮ ДЛЯ РОЗПОДІЛЕНОЇ НЕЛІНІЙНО ДИСИПАТИВНОЇ ЛІНІЇ ПЕРЕДАЧ

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The envelope modulation of a monoinductance transmission line is reduced to generalized coupled Ginzburg – Landau equations from which is deduced a single cubic-quintic Ginzburg – Landau equation containing derivatives with respect to the spatial variable in the cubic terms. We investigate the modulational instability of the spatial wave solutions of both the system and the single equation. For the generalized coupled Ginzburg – Landau system we consider only the zero wavenumbers of the perturbations whose modulational instability conditions depend only on the system's coefficients and the wavenumbers of the carriers. In this case, a modulational instability criterion is established which depends both on the perturbation wavenumbers and the carrier. We also study the coherent structures of the generalized coupled Ginzburg – Landau system and present some numerical studies.

Огортуючу модуляцію моноіндуктивної лінії передач зведено до узагальнених пов'язаних між собою рівнянь Гінзбурга – Ландау, звідки отримано одне рівняння Гінзбурга – Ландау третього – п'ятого порядку, яке містить похідні відносно просторової змінної в кубічних членах. Для системи та рівняння досліджено модуляційну нестійкість розв'язків у формі просторової хвилі. Для системи Гінзбурга – Ландау розглянуто лише збурення з нульовими хвильовими числами, для яких умови модуляційної нестійкості залежать тільки від коефіцієнтів системи та хвильових чисел носіїв. У цьому випадку отримано критерій для модуляційної нестійкості, який залежить як від хвильових чисел збурень, так і від носія. Також вивчаються когерентні структури системи Гінзбурга – Ландау та проведено деякий числовий аналіз.

1. Introduction. Nonlinear transmission lines (NTL's) are networks which consist of nonlinear capacitors periodically placed between sections of the transmission line to guide and contain electromagnetic radiation. Recently, the terahertz region of the electromagnetic spectrum loosely defined as the range between 100 GHz and 10 THz has become very attractive for future wide-band systems. In this regard, electronic engineers are now exploring different possibilities to develop signal sources operating at these frequencies [1]. There are various kinds of transmission lines, such as twin lead, coaxial cable and wave guides. Transmission line theory is quite relevant to engineering physics, especially in the area of optics and wave theory. Propagating waves in transmission lines are reflected from abrupt discontinuities just as optical waves are partially reflected off boundaries between two materials of different refraction indexes. Also, standing waves can occur in transmission lines if there are two boundaries present, just as on a vibrating string with two fixed points.

Different physico-chemical systems driven out of equilibrium may undergo Hopf bifurcations leading to rich spatio-temporal behavior. When these bifurcations occur with broken spatial symmetries, they induce the formation of wave patterns described by order parameters of the form:

$$\Psi = A\, e^{i(k_c x - \omega_c t)} + B\, e^{i(-k_c x - \omega_c t)} + c.c, \tag{1.1}$$

where $c.c$ stands for the complex conjugate of the preceding terms. The slow dynamics of the complex-valued wave amplitudes A and B obey complex Ginzburg – Landau (GL) equations, and k_c and ω_c are the critical wavenumber and the critical frequency, respectively. This is the case, for example, for Rayleigh – Bénard convection in binary fluids, Taylor – Couette instabilities between co-rotating cylinders, electro-convection in nematic liquid crystals, or the transverse field of high Fresnel number lasers. The main purpose of this paper is to study the dynamics of modulated wave trains in a distributed nonlinear electrical transmission line. As primary modes, we consider traveling waves. When these primary modes are essentially one-dimensional and the system possesses left-right reflection symmetry, weakly nonlinear patterns are of the form of Eq. (1.1) where A and B are the complex-valued amplitudes of the right- and left-traveling waves. Using a perturbation method and passing to the continuum limit, we show that A and B obey the generalized Ginzburg – Landau (GGL) systems.

In the next section we write down the circuit equations governing small-amplitude pulses on systems of dissipative NTL's. After scaling the coordinates and taking a continuum limit, the multiple-scales method and the perturbation method are used in Section 3 to reduce the circuit equations to a new nonlinear system of partial differential equations (PDEs) that we call generalized coupled Ginzburg – Landau (GCGL) equations. We undertake an analytic study of the stability of the particular solutions of these equations in Section 4. The coherent structures are studied in Section 5. The results are summarized in Section 6.

2. Electrical models. Let us consider a distributed transmission line with the simplest periodical structure consisting of the elements shown in Fig. 1, where the capacitance C is a function of voltage [2]. Here, the transmission lines are studied as circuit models in terms of circuit theory where voltages and currents, instead of fields, are the variables. Figure 1 shows an infinitesimal segment of a physical dissipative transmission line. In the figure, the distributed parameters of the line, R, L, C, and G, are the per-unit-length resistance, inductance, capacitance, and conductance, respectively.

For the circuit in Fig. 1, we no longer have the familiar linear charge-voltage relation $CQ_n = Q$, but rather the nonlinear differential relationship

$$C(V_n) = \frac{dQ_n}{dV_n}. \tag{2.1}$$

To solve (2.1) for the voltage as a function of time, we use the familiar relations at the nodes

$$V_{n-1} - V_n = \frac{d\Phi_n}{dt} + RI_n, \qquad I_n - I_{n+1} = \frac{dQ_n}{dt} + GV_n. \tag{2.2}$$

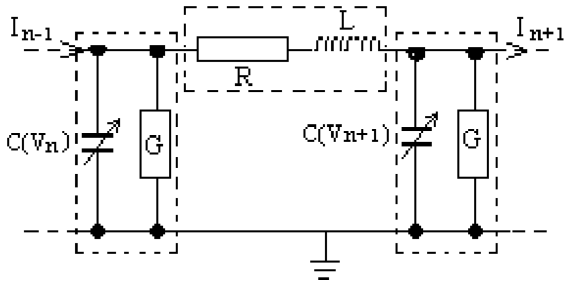


Fig. 1. A typical section of a nonlinear transmission line.

To obtain an equation in terms of V only, one notes that the magnetic flux can be expressed in terms of the current using $\Phi_n = LI_n$, while the charge can be eliminated using (2.1). Using these relations in (2.2) leads to the following formulae:

$$V_{n-1} - V_n = L\,\frac{dI_n}{dt} + RI_n, \qquad V_n - V_{n+1} = L\,\frac{dI_{n+1}}{dt} + RI_{n+1},$$
$$I_n - I_{n+1} = C(V_n)\,\frac{dV_n}{dt} + GV_n, \qquad \frac{dI_n}{dt} - \frac{dI_{n+1}}{dt} = \frac{d}{dt}\left[C(V_n)\,\frac{dV_n}{dt}\right] + G\,\frac{dV_n}{dt}. \tag{2.3}$$

Eliminating I_n and I_{n+1}, we obtain the following second-order difference-differential equation:

$$\frac{d}{dt}\left[C(V_n)\,\frac{dV_n}{dt}\right] + G\,\frac{dV_n}{dt} + \frac{R}{L}\,C(V_n)\,\frac{dV_n}{dt} = \frac{1}{L}\,(V_{n-1} - 2V_n + V_{n+1}) - \frac{RG}{L}\,V_n. \tag{2.4}$$

This is the circuit equation describing the voltage $V_n(t)$ on a single line.

In this analysis, the nonlinear capacitance $C(V_{n,m})$ is of the form

$$C(V) = \frac{C_0}{1 + (V/V_0)^p}, \tag{2.5}$$

where C_0 and V_0 are arbitrary capacitance and voltage scales, respectively, and $p > 0$. Similar forms for $C(V)$ have been used in the past to model the capacitance of certain varactor diodes as part of comparisons with experimental measurements of solitary waves in NTL's [3]. Substituting (2.5) into (2.4), we find

$$\frac{d^2Q_n}{dt^2} + G\,\frac{dV_n}{dt} + \frac{R}{L}\,\frac{dQ_n}{dt} = \frac{1}{L}\,(V_{n-1} - 2V_n + V_{n+1}) - \frac{RG}{L}\,V_n. \tag{2.6}$$

To obtain an approximate solution of discretized Eq. (2.6), we can invoke the continuum limit by a Taylor expansion of the voltage at $V_{n\pm1}$ as follows:

$$V_{n\pm1} = V_n \pm \frac{\partial V_n}{\partial n} + \frac{1}{2!}\frac{\partial^2 V_n}{\partial n^2} \pm \frac{1}{3!}\frac{\partial^3 V_n}{\partial n^3} + \ldots. \tag{2.7}$$

By defining the quantity $\delta \equiv x'/n$, eliminating terms of order higher than δ^2 from (2.7) and assuming that $V \ll V_0$, we obtain the following wave equation valid in a weakly dispersive and nonlinear regime:

$$C_0 \frac{\partial^2}{\partial t^2}\left(V - \frac{V^{p+1}}{(p+1)\,V_0^p}\right) + G\,\frac{\partial V}{\partial t} + \frac{RC_0}{L}\,\frac{\partial}{\partial t}\left(V - \frac{V^{p+1}}{(p+1)\,V_0^p}\right) - \frac{\delta^2}{L}\,\frac{\partial^2 V}{\partial x'^2} + \frac{RG}{L}\,V_n = 0. \quad (2.6')$$

In what follows, we study the case where $p = 2$ and use the transformation $x = x'/\delta$. Then Eq. (2.6′) becomes

$$C_0 \frac{\partial^2}{\partial t^2}\left(V + bV^3\right) - \frac{1}{L}\,\frac{\partial^2 V}{\partial x^2} + \left(G + \frac{RC_0}{L}\right)\frac{\partial V}{\partial t} + \frac{bRC_0}{L}\,\frac{\partial V^3}{\partial t} + \frac{RG}{L}V = 0, \qquad (2.8)$$

where $b = -1/\left(3V_0^p\right)$.

3. Derivation of the GCGL equations. In this section we use the voltage Eq. (2.8) to derive a second-order partial differential system that will be called GCGL equations. To construct these equations we use the method of multiple-scale by introducing two slow time scales $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$ in addition to the initial time $T_0 = t$ and one large length scale $X_1 = \epsilon x$ in addition to the initial spatial variable $X_0 = x$:

$$V = \sum_{j=1}^{3} \epsilon^{j/2}\left(u_{jj}\,e^{ji\theta_1} + v_{jj}\,e^{ji\theta_2}\right) + \epsilon^{5/2}\left(u_{42}e^{2i\theta_1} + v_{42}\,e^{2i\theta_2}\right) + c.c + \ldots, \qquad (3.1)$$

where

$$\theta_1 = kX_0 - \omega T_0, \quad \theta_2 = -kX_0 - \omega T_0, \quad u_{jk} = u_{jk}(X_1, T_1, T_2), \quad v_{jk} = v_{jk}(X_1, T_1, T_2),$$

and u_{jk} and v_{jk} are complex-valued amplitudes of the right- and left-traveling waves. We then order the damping coefficient in (2.8) so that the damping and nonlinearity effects appear in the same perturbation equations. Thus, we set $G + RC_0/L = \epsilon^2\mu_1$.

Inserting the perturbation expansion (3.1) into the nonlinear Eq. (2.8) we obtain a series of nonhomogeneous equations at different orders of $\left(\epsilon, e^{i\theta_1}, e^{i\theta_2}\right)$.

At order $\left(\epsilon^{1/2}, e^{i\theta_1}, e^{i\theta_2}\right)$ we have

$$\left(-C_0\omega^2 + \frac{k^2}{L} + \frac{RG}{L}\right)\left(u_{11}\,e^{i\theta_1} + v_{11}\,e^{i\theta_2}\right) = 0,$$

and the nontriviality condition of u_{11} and v_{11} gives the following linear dispersion relation, which, for further use, we solve for k and differentiate with respect to k:

$$-C_0\omega^2 + \frac{k^2}{L} + \frac{RG}{L} = 0, \qquad \omega = \sqrt{\frac{k^2 + RG}{C_0 L}}, \qquad \frac{\partial\omega}{\partial k} = \frac{k}{\sqrt{C_0 L\left(k^2 + RG\right)}} = \frac{1}{C_0 L}\,\frac{k}{\omega}. \quad (3.2)$$

At order $\left(\epsilon, e^{i\theta_1}, e^{i\theta_2}\right)$ we have

$$\left(-4C_0\omega^2 + 4\frac{k^2}{L} + \frac{RG}{L}\right)\left(u_{22}\,e^{2i\theta_1} + v_{22}\,e^{2i\theta_2}\right) = 0,$$

thus, we can take $u_{22} = v_{22} = 0$.

At order $\left(\epsilon^{3/2}, e^{i\theta_1}, e^{i\theta_2}\right)$ the equation is

$$\left[-2iC_0\omega\frac{\partial u_{11}}{\partial T_1} - \frac{2ik}{L}\frac{\partial u_{11}}{\partial X_1} - 3C_0b\omega\left(\omega + i\frac{R}{L}\right)\left(|u_{11}|^2 + 2\,|v_{11}|^2\right)u_{11}\right] e^{i\theta_1} +$$
$$+\left[-2iC_0\omega\frac{\partial v_{11}}{\partial T_1} + \frac{2ik}{L}\frac{\partial v_{11}}{\partial Y_1} - 3C_0b\omega\left(\omega + i\frac{R}{L}\right)\left(|v_{11}|^2 + 2\,|u_{11}|^2\right)v_{11}\right] e^{i\theta_2} = 0.$$

From this equation we obtain the system

$$\begin{aligned}
&\frac{\partial u_{11}}{\partial T_1} + \frac{\partial\omega}{\partial k}\frac{\partial u_{11}}{\partial X_1} + A_3\left(|u_{11}|^2 + 2|v_{11}|^2\right)u_{11} = 0,\\
&\frac{\partial v_{11}}{\partial T_1} - \frac{\partial\omega}{\partial k}\frac{\partial v_{11}}{\partial X_1} + A_3\left(|v_{11}|^2 + 2|u_{11}|^2\right)v_{11} = 0,
\end{aligned} \tag{3.3}$$

where

$$A_3 = \frac{3b}{2}\left(\frac{R}{L} - i\omega\right), \qquad \frac{\partial\omega}{\partial k} = \frac{1}{C_0L}\,\frac{k}{\omega}. \tag{3.4}$$

At order $\left(\epsilon^{3/2}, e^{i\theta_1}, e^{i\theta_2}\right)$ we have

$$\left[\left(-9C_0\omega^2 + 9\frac{k^2}{L} + \frac{RG}{L}\right)u_{33} - 3C_0b\omega\left(3\omega + \frac{R}{L}\right)u_{11}^3\right] e^{3i\theta_1} +$$
$$+\left[\left(-9C_0\omega^2 + 9\frac{k^2}{L} + \frac{RG}{L}\right)v_{33} - 3C_0b\omega\left(3\omega + \frac{R}{L}\right)v_{11}^3\right] e^{3i\theta_2} = 0.$$

By the dispersion relation (3.2), the last equation becomes

$$u_{33} = A_4u_{11}^3, \quad v_{33} = A_4v_{11}^3, \tag{3.5}$$

with

$$A_4 = \frac{3C_0b\omega}{8\left(k^2/L - C_0\omega^2\right)}\left(3\omega + \frac{R}{L}\right). \tag{3.6}$$

The equation of order $\left(\epsilon^2, e^{2i\theta_1}, e^{2i\theta_2}\right)$ is

$$\left[\left(-4C_0\omega^2 + 4\frac{k^2}{L} + \frac{RG}{L}\right)u_{42}\right] e^{2i\theta_1} + \left[\left(-4C_0\omega^2 + 4\frac{k^2}{L} + \frac{RG}{L}\right)v_{42}\right] e^{2i\theta_2} = 0$$

from which we take

$$u_{42} = v_{42} = 0. \tag{3.7}$$

At order $\left(\epsilon^{5/2}, e^{i\theta_1}, e^{i\theta_2}\right)$ we have

$$\Bigg\{C_0\left(\frac{\partial^2 u_{11}}{\partial T_1^2}-2i\omega\frac{\partial u_{11}}{\partial T_2}\right)-\frac{1}{L}\frac{\partial^2 u_{11}}{\partial X_1^2}+4C_0b\omega\left(\omega+i\,\frac{R}{L}\right)u_{11}^{*2}u_{33}-$$

$$-i\mu_1\omega u_{11}+3C_0b\left(\frac{R}{L}-2i\omega\right)\frac{\partial}{\partial T_1}\left[\left(|u_{11}|^2+2\,|v_{11}|^2\right)u_{11}\right]\Bigg\}\,e^{i\theta_1}+$$

$$+\Bigg\{C_0\left(\frac{\partial^2 v_{11}}{\partial T_1^2}-2i\omega\frac{\partial v_{11}}{\partial T_2}\right)-\frac{1}{L}\frac{\partial^2 v_{11}}{\partial X_1^2}+4C_0b\omega\left(\omega+i\,\frac{R}{L}\right)v_{11}^{*2}v_{33}-$$

$$-i\mu_1\omega v_{11}+3C_0b\left(\frac{R}{L}-2i\omega\right)\frac{\partial}{\partial T_1}\left[\left(|v_{11}|^2+2\,|u_{11}|^2\right)v_{11}\right]\Bigg\}\,e^{i\theta_2}\,=0.$$

From this equation, we obtain, after using (3.5), the following system:

$$\begin{aligned}&-\frac{C_0}{\omega_1}\frac{\partial^2 u_{11}}{\partial T_1^2}+2iC_0\frac{\partial u_{11}}{\partial T_2}+\frac{1}{L\omega}\frac{\partial^2 u_{11}}{\partial X_1^2}+4C_0A_4b\left(\omega+i\,\frac{R}{L}\right)|u_{11}|^4\,u_{11}+\\&\quad+i\mu_1 u_{11}-\frac{3C_0b}{\omega}\left(\frac{R}{L}-2i\omega\right)\frac{\partial}{\partial T_1}\left[\left(|u_{11}|^2+2\,|v_{11}|^2\right)u_{11}\right]\,=0,\\&-\frac{C_0}{\omega}\frac{\partial^2 v_{11}}{\partial T_1^2}+2iC_0\frac{\partial v_{11}}{\partial T_2}+\frac{1}{L\omega_2}\frac{\partial^2 v_{11}}{\partial X_1^2}+4C_0A_4b\left(\omega+i\,\frac{R}{L}\right)|v_{11}|^4\,v_{11}+\\&\quad+i\mu_1 v_{11}-\frac{3C_0b}{\omega}\left(\frac{R}{L}-2i\omega\right)\frac{\partial}{\partial T_1}\left[\left(|v_{11}|^2+2\,|u_{11}|^2\right)v_{11}\right]\,=0.\end{aligned}\tag{3.8}$$

If we write the systems (3.3) and (3.8) in terms of the original coordinates x, y, t, multiply each equation of system (3.3) by $\epsilon^{1/2}\,(1+2iC_0)$ and each equation of system (3.8) by $\epsilon^{1/2}$ and then sum each side of these systems, we obtain, after using the transformations $u=\epsilon^{1/2}u_{11}$ and $v=\epsilon^{1/2}v_{11}$,

$$\begin{aligned}&(1+2iC_0)\left[\frac{\partial u}{\partial t'}+\frac{\partial\omega}{\partial k}\frac{\partial u}{\partial x}\right]-\frac{C_0}{\omega}\frac{\partial^2 u}{\partial t^2}+\frac{1}{L\omega}\frac{\partial^2 u}{\partial x^2}+4C_0A_4b\left(\omega+i\,\frac{R}{L}\right)|u|^4\,u+\\&\quad+A_3\,(1+2iC_0)\left(|u|^2+2\,|v|^2\right)u+i\left(G+\frac{RC_0}{L}\right)u-\\&\quad-\frac{3C_0b}{\omega}\left(\frac{R}{L}-2i\omega\right)\frac{\partial}{\partial t}\left[\left(|u|^2+2\,|v|^2\right)u\right]\,=0,\\&(1+2iC_0)\left[\frac{\partial v}{\partial t'}-\frac{\partial\omega}{\partial k}\frac{\partial v}{\partial x}\right]-\frac{C_0}{\omega}\frac{\partial^2 v}{\partial t^2}+\frac{1}{L\omega}\frac{\partial^2 v}{\partial x^2}+4C_0A_4b\left(\omega+i\,\frac{R}{L}\right)|v|^4\,v+\\&\quad+A_3\,(1+2iC_0)\left(|v|^2+2\,|u|^2\right)v+i\left(G+\frac{RC_0}{L}\right)v-\\&\quad-\frac{3C_0b}{\omega}\left(\frac{R}{L}-2i\omega\right)\frac{\partial}{\partial t}\left[\left(|v|^2+2\,|u|^2\right)v\right]\,=0,\end{aligned}\tag{3.9}$$

where $t' = t/2$. Now we can use (3.3) to eliminate the terms

$$\frac{\partial u}{\partial t^2}, \quad \frac{\partial v}{\partial t^2}, \quad \frac{\partial}{\partial t}\left[\left(|v|^2 + 2|u|^2\right) v\right], \quad \frac{\partial}{\partial t}\left[\left(|u|^2 + 2|v|^2\right) u\right]$$

from (3.9) and obtain

$$\frac{\partial u}{\partial t} + S_0 \frac{\partial u}{\partial x} + P\frac{\partial^2 u}{\partial x^2} + \gamma u + Q\left(|u|^2 + 2|v|^2\right)u + \\ + \left(D|u|^4 + F|u|^2|v|^2 + K|v|^4\right)u + E\frac{\partial}{\partial x}\left[\left(|u|^2 + 2|v|^2\right)u\right] + Hu\frac{\partial}{\partial x}|v|^2 = 0, \tag{3.10}$$

$$\frac{\partial v}{\partial t} - S_0 \frac{\partial v}{\partial x} + P\frac{\partial^2 v}{\partial x^2} + \gamma v + Q\left(|v|^2 + 2|u|^2\right)v + \\ + \left(D|v|^4 + F|v|^2|u|^2 + K|u|^4\right)v - E\frac{\partial}{\partial x}\left[\left(|v|^2 + 2|u|^2\right)v\right] - Hv\frac{\partial}{\partial x}|u|^2 = 0, \tag{3.11}$$

where

$$\begin{aligned}
& t = t', \quad P = \frac{1-2iC_0}{1+4C_0^2}\left[\frac{1}{L\omega} - \frac{C_0}{\omega}\left(\frac{\partial\omega}{\partial k}\right)^2\right], \quad \gamma = \frac{i+2C_0}{1+4C_0^2}\left(G + \frac{RC_0}{L}\right), \\
& Q = A_3 = \frac{3b}{2}\left(\frac{R}{L} - i\omega\right), \quad E = \frac{1-2iC_0}{1+4C_0^2}\left[\frac{3C_0 b}{\omega}\left(\frac{R}{L} - 2i\omega\right) - \frac{2A_3C_0}{\omega}\right]\frac{\partial\omega}{\partial k}, \\
& D = \frac{1-2iC_0}{1+4C_0^2}\left[\frac{3C_0 b}{\omega}\left(\frac{R}{L} - 2i\omega\right)(A_3^* + 2A_3) + 4C_0A_4b\left(\omega + i\frac{R}{L}\right)\right], \\
& H = 4\frac{1-2iC_0}{1+4C_0^2}\left[A_3\frac{C_0}{\omega} - \frac{3C_0 b}{\omega}\left(\frac{R}{L} - 2i\omega\right)\right]\frac{\partial\omega}{\partial k}, \\
& F = \frac{1-2iC_0}{1+4C_0^2}\frac{3C_0 b}{\omega}\left(\frac{R}{L} - 2i\omega\right)(6A_3^* + 10A_3), \\
& K = \frac{1-2iC_0}{1+4C_0^2}\frac{3C_0 b}{\omega}\left(\frac{R}{L} - 2i\omega\right)(6A_3 + 2A_3^*), \quad S_0 = \frac{\partial\omega}{\partial k} = \frac{1}{C_0 L}\frac{k}{\omega}.
\end{aligned} \tag{3.12}$$

Equations (3.10), (3.11) are the required GCGL equations: they form a cubic-quintic GL system with derivatives in the cubic terms. These equations are amplitude equations describing the slow modulations of a right-traveling mode with (x, t) dependence $\exp\left[i\left(kx - \omega t\right)\right]$ and amplitude u, coupled with a left-traveling mode $\exp\left[i\left(-kx - \omega t\right)\right]$ with amplitude v. In the special case where $D = F = K = E = H = 0$, system (3.10), (3.11) was studied in [4–9]. The existence and stability of the modulated amplitude waves in the complex plane were studied in [4] and the existence of soliton-like solutions has been shown.

For wave propagation occurring in one direction ($v = 0$ or $u = 0$), system (3.10), (3.11) give the following GGL equation:

$$\frac{\partial U}{\partial t} \pm S_0\frac{\partial U}{\partial x} + P\frac{\partial^2 U}{\partial x^2} + \gamma U + Q|U|^2U + D|U|^4U + E\frac{\partial}{\partial x}|U|^2U = 0. \tag{3.13}$$

The term $\frac{\partial}{\partial x}|U|^2U = U^2\frac{\partial U^*}{\partial x} + 2|U|^2\frac{\partial U}{\partial x}$ appears in the asymptotic derivation. Deissler et al. [10] showed numerically that this term can significantly slow down the propagating speed of pulses and also cause nonsymmetric pulses. For equation (3.13), Saarloos and Hohenberg [11] present a framework for the discussion of front, pulse and domain wall dynamics. Doelman and Eckhaus [12] (Theorem 3.4) find some homoclenic domain walls with wave speed $c = 0$, via Poincaré maps and Melnikov integrals, extending and correcting the work of Holmes [13]. Kengne [14] studied the Benjamin – Feir instability of the monochromatic wave solutions of Eq. (3.13). In the case where $S_0 = 0$, Kengne and Liu [15] found some exact solutions of Eq. (3.13).

In the special case where $E = 0$ (for example when $k = 0$), Eq. (3.13) becomes the so-called quintic GL equation

$$\frac{\partial U}{\partial t} \pm S_0\frac{\partial U}{\partial x} + P\frac{\partial^2 U}{\partial x^2} + \gamma U + Q|U|^2U + D|U|^4U = 0. \tag{3.14}$$

Thual and Fauve [16] and Fauve and Thual [17] discuss pulses for this equation. Kolyshkin et al. [18, 19] investigated Eq. (3.14) in the special case where $D = 0$ for a suddenly blocked unsteady channel.

It is seen from (3.12) that all the coefficients of system (3.10), (3.11), except for the linear group velocity S_0, are complex-valued functions of the wavenumber k. In what follows, we shall denote by f^r and f^i the real and imaginary parts of the complex f, respectively.

In the CGL Eqs. (3.10), (3.11), S_0 is the linear group velocity, i.e., the group velocity of the fast modes. It is important to realize that the group velocity s is different from S_0. To see this, note that the CGL equations admit single mode traveling waves of the form

$$u(x,t) = a\,e^{i(qx-\omega_u t)}, \quad v(x,t) = 0, \quad \text{or} \quad v(x,t) = ae^{i(qx-\omega_v t)}, \quad u(x,t) = 0. \tag{3.15}$$

Substituting these wave solutions in the amplitude equations (3.10), (3.11) we obtain the nonlinear dispersion relation

$$\omega_{u,v} = \pm\left(S_0 + E^i a^2\right)q - P^i q^2 + \gamma^i + Q^i a^2 + D^i a^4, \tag{3.16}$$

where the real amplitude a is a solution of the equation

$$D^r a^4 + \left(Q^r - E^i q\right)a^2 - P^r q^2 + \gamma^r = 0. \tag{3.17}$$

Solving (3.17) for a we get

$$a^2 = \begin{cases} \dfrac{1}{2D^r}\left[E^i q - Q^r \pm \sqrt{\left((E^i)^2 + 4D^r P^r\right)q^2 - 2E^i Q^r q + (Q^r)^2 - 4D^r\gamma^r}\right], & \text{if } D^r \neq 0, \\ (P^r q^2 - \gamma^r)/(Q^r - Eq^i), & \text{if } D^r = 0. \end{cases} \tag{3.18}$$

Therefore, if $D^r \neq 0$, the group velocity $s = \partial\omega/\partial k$ of these traveling waves, as functions of the wavenumber q, becomes

$$s_u = S_0 + E^i a^2 - 2P^i q, \qquad s_v = -S_0 - E^i a^2 - 2P^i q. \tag{3.19}$$

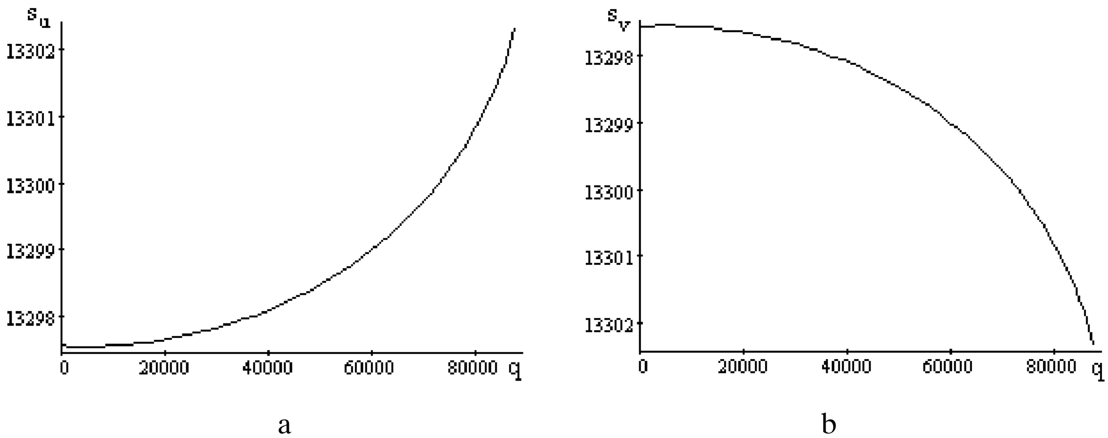


Fig. 2. Behavior of the group velocities s_u (*a*) and s_v (*b*) as function of the wavenumber q, corresponding to the line parameters $(\mathrm{L}_{\mathrm{p1}})$ for $q \in [0, q_- = 89580]$, respectively.

We see that, if either $D^r = 0$ or $\left(E^i\right)^2 + 4D^rP^r \geq 0$, the amplitude a goes to infinity as q increases. Therefore we frequently assume the global existence condition $\left(E^i\right)^2 + 4D^rP^r < 0$. For the sign of the square root in (3.18) to make sense, we assume that $D^r < \left(Q^r\right)^2/(4\gamma^r)$ (it is seen from Appendix A that $\gamma^r > 0$). In this case, plane waves exist for q in a bounded interval $[q_+, q_-]$, where

$$q_\pm = \frac{E^iQ^r \pm \sqrt{4D^r\gamma^r\left(E^i\right)^2 + 16\left(D^r\right)^2P^r\gamma^r - 4\left(Q^r\right)^2D^rP^r}}{\left(E^i\right)^2 + 4D^rP^r}.$$

For the line parameters [4]

$$C_0 = 540\,pF, \quad L = 28\,\mu H, \quad R = 10^5\,\Omega, \qquad G = 10^{-4}\,\Omega^{-1}, \qquad b = 0.16\,V^{-1}, \qquad (\mathrm{L}_{\mathrm{p1}})$$

$$C_0 = 540\,pF, \quad L = 28\,\mu H, \quad R = 0.0525\,\Omega, \quad G = 190.48\,\Omega^{-1}, \quad b = 0.16\,V^{-1}, \qquad (\mathrm{L}_{\mathrm{p2}})$$

and wavenumber $k=0.2$, the analytical expressions for the coefficients of the GCGL Eqs. (3.10), (3.11) are given in Appendix B. For the line parameters $(\mathrm{L}_{\mathrm{p1}})$ the above conditions for the existence of wave solutions (3.15) are not satisfied. These conditions are satisfied for $(\mathrm{L}_{\mathrm{p2}})$, and the curves of the group velocities s_u and s_v are shown in Fig. 2. As one can see from Figs. 2(*a*) and 2(*b*), s_u is monotone increasing and s_v is monotone decreasing for wavenumbers $q \in [0, q_- = 89580]$.

4. Amplitude dynamics of phase winding solutions. Having computed the analytical expressions for the coefficients of the GCGL equations (3.10), (3.11) (see Appendix A), we are now able to study their stability. We shall analyze the so-called phase winding solutions which possess a spatial periodic structure. A phase winding solution to (3.10), (3.11) is a pair of functions (u, v) of the form

$$u(x,t) = a(t)\,e^{i(k_ux+\Omega_u(t))}, \qquad v(x,t) = b(t)\,e^{i(k_vx+\Omega_v(t))}, \tag{4.1}$$

for $(x,t) \in \mathbf{R}\times\mathbf{R}^+$, where a, b, and Ω_u, Ω_v are real amplitudes and phases, respectively, depending only on time $t \in \mathbf{R}^+$, and $k_u, k_v \in \mathbf{R}$ are phase winding numbers. Note that under assumption (4.1) only amplitude instabilities can be analyzed. The phase functions do not affect the stability properties in this case. If we insert Ansatz (4.1) into system (3.10), (3.11), we obtain the following planar system of ordinary differential equations (ODEs) for the real amplitudes a and b:

$$a' = \left(P^r k_u^2 - \gamma^r\right) a - \left(Q^r + k_u E^i\right)\left(a^2 + 2b^2\right) a - \left(D^r a^4 + F^r a^2 b^2 + K^r b^4\right) a, \tag{4.2}$$

$$b' = \left(P^r k_v^2 - \gamma^r\right) b - \left(Q^r - k_v E^i\right)\left(b^2 + 2a^2\right) b - \left(D^r b^4 + F^r a^2 b^2 + K^r a^4\right) b. \tag{4.3}$$

The phase functions Ω_u and Ω_v are given by

$$\Omega_u(t) = \Omega_u^0 + \left(P^i k_u^2 - S_0 k_u - \gamma^i\right) t -$$
$$- \int_0^t \left[\left(Q^i + k_u E^r\right)\left(a^2(\tau) + 2b^2(\tau)\right) + \left(D^i a^4(\tau) + F^i a^2(\tau) b^2(\tau) + K^i b^4(\tau)\right)\right] d\tau,$$

$$\Omega_v(t) = \Omega_v^0 + \left(P^i k_v^2 + S_0 k_v - \gamma^i\right) t -$$
$$- \int_0^t \left[\left(Q^i - k_v E^r\right)\left(b^2(\tau) + 2a^2(\tau)\right) + \left(D^i b^4(\tau) + F^i a^2(\tau) b^2(\tau) + K^i a^4(\tau)\right)\right] d\tau,$$

where Ω_u^0 and Ω_v^0 are the initial phases.

Let (u, v) be a solution of system (4.2), (4.3) corresponding to the phase winding numbers $k_u, k_v \in \mathbf{R}$. A straightforward phase-plane analysis enables us to conclude that the first quadrant $\mathbf{R}_0^+ \times \mathbf{R}_0^+$ is invariant with respect to solutions of (4.2), (4.3). Furthermore, by introducing the logarithmic transformation of the variables $A = \log a$, $B = \log b$ and taking into account the Poincaré – Bendixon criterion applied to the transformed planar system of ODEs, we conclude that there are no periodic orbits and no heteroclinic cycles in system (4.2), (4.3).

The system of ODEs (4.2), (4.3) can have a number of fixed points. If (a_0, b_0) is a fixed point of this system, then the linear flow near the stationary solution (a_0, b_0) is

$$A' = \left[P^r k_u^2 - \gamma^r - 2a_0^2\left(Q^r + k_u E^i\right) - 4D^r a_0^4 - 2F^r a_0^2 b_0^2\right] A -$$
$$- \left[4a_0 b_0\left(Q^r + k_u E^i\right) + 2F^r b_0 a_0^3 + 4K^r a_0 b_0^3\right] B, \tag{4.4}$$

$$B' = \left[4a_0 b_0\left(k_v E^i - Q^r\right) - 2F^r a_0 b_0^3 - 4K^r b_0 a_0^3\right] A +$$
$$+ \left[P^r k_v^2 - \gamma^r - 2b_0^2\left(Q^r - k_v E^i\right) - 4D^r b_0^4 - 2F^r a_0^2 b_0^2\right] B. \tag{4.5}$$

As in Jones, Kapitula and Powell [20], we can show that system (4.4), (4.5) characterizes the behavior of the solutions of system (4.2), (4.3) as $a \to a_0$ and $b \to b_0$.

Since there are two flow equations, there are two eigenvalues of the linear flow near each fixed point. When performing the counting analysis for these fixed points we will only need the signs of the real parts of the two eigenvalues, since these determine whether the flow along the corresponding eigendirection is inwards $(-)$ or outwards $(+)$.

Let

$$M = \begin{pmatrix} P^r k_u^2 - \gamma^r - 2a_0^2\left(Q^r + k_u E^i\right) - & -4a_0 b_0\left(Q^r + k_u E^i\right) - \\ -4D^r a_0^4 - 2F^r a_0^2 b_0^2 & -2F^r b_0 a_0^3 - 4K^r a_0 b_0^3 \\ 4a_0 b_0\left(k_v E^i - Q^r\right) - & P^r k_v^2 - \gamma^r - 2b_0^2\left(Q^r - k_v E^i\right) - \\ -2F^r a_0 b_0^3 - 4K^r b_0 a_0^3 & -4D^r b_0^4 - 2F^r a_0^2 b_0^2 \end{pmatrix}$$

be the matrix of the flow (4.4), (4.5) near the stationary solution (a_0, b_0) of system (4.2), (4.3) with eigenvalues λ_1 and λ_2. To have a stable stationary solution (a_0, b_0) of system (3.2), (3.3), it is necessary and sufficient that $\operatorname{Re}\lambda_j > 0$, $j \in \{1, 2\}$. For example, for the stability of the zero amplitude wave critical point $(0, 0)$ of system (4.2), (4,3), it is necessary and sufficient that $P^r k_u^2 - \gamma^r < 0$ and $P^r k_v^2 - \gamma^r < 0$. In what follows, we give some numerical solutions of system (4.4), (4.5) for (L_{p1}) and (L_{p2}).

For given coefficients of the GCGL Eqs. (3.10), (3.11), the matrix M contains two parameters, the phase winding numbers k_u and k_v. For some arbitrary values of k_u and k_v we find some stationary solutions to system (4.2), (4.3) and in Figs. 3 and 4 (near the obtained stationary solutions) we show the phase graph for (a, b), $|u(x,t)| = |a(t)|$ and $|v(x,t)| = |b(t)|$, according to (4.1), and for $a(t)$ and $b(t)$. Figures 5(*a*) and 5(*b*) show the behavior of the real parts of $u(x,t)$ and $v(x,t)$ corresponding to Figs. 3 and 4, respectively.

***4.1. The line parameters* (L_{p1}).** In this subsection, for notational simplicity, we set $\alpha = 1.4653 \times 10^{-7}$. For the line parameters (L_{p1}) and the phase winding numbers $k_u = 0.2$ and $k_v = -0.2$, we find that

$$(-\alpha, -\alpha), \quad (\alpha, \alpha), \quad (-\alpha, \alpha), \quad (\alpha, -\alpha),$$

are stationary solutions of system (4.2), (4.3) with $a = b$ or $a = -b$. For the stationary solutions (α, α) the matrix M and eigenvalues are

$$M = \begin{pmatrix} 0.000018407 & -0.000073615 \\ 0.000073615 & 0.000018407 \end{pmatrix}, \qquad \begin{matrix} \lambda_1 = 1.8407 \times 10^{-5} + 7.3615 \times 10^{-5} i, \\ \lambda_2 = 1.8407 \times 10^{-5} - 7.3615 \times 10^{-5} i. \end{matrix}$$

For the stationary solutions $(\alpha, -\alpha)$ the matrix M and eigenvalues are

$$M = \begin{pmatrix} 1.8407 \times 10^{-5} & 7.3615 \times 10^{-5} \\ 7.3615 \times 10^{-5} & 1.8407 \times 10^{-5} \end{pmatrix}, \qquad \begin{matrix} \lambda_1' = -5.5208 \times 10^{-5}, \\ \lambda_2' = 9.2022 \times 10^{-5}. \end{matrix}$$

Because $\operatorname{Re}\lambda_1 > 0$ and $\operatorname{Re}\lambda_2 > 0$, the stationary solutions (α, α) and $(-\alpha, -\alpha)$ are unstable foci. For the stationary solutions $(-\alpha, \alpha)$ and $(\alpha, -\alpha)$, the eigenvalues λ_1' and λ_2' of the matrix M are real and $\lambda_1' < 0$ and $\lambda_2' > 0$. Therefore $(-\alpha, \alpha)$ and $(\alpha, -\alpha)$ are two saddle points.

Figure 3 corresponds to the stationary solution (α, α) of system (4.2), (4.3). Figure 3(*a*) shows the solution in the (a, b) phase plane. Figures 3(*b*) and 3(*c*) show the plots of $|u(x,t)| = |a(t)|$ and $|v(x,t)| = |b(t)|$, respectively Figures 3(*d*) and 3(*e*) show that the waves travel from left to right and are unbounded as $t \to +\infty$.

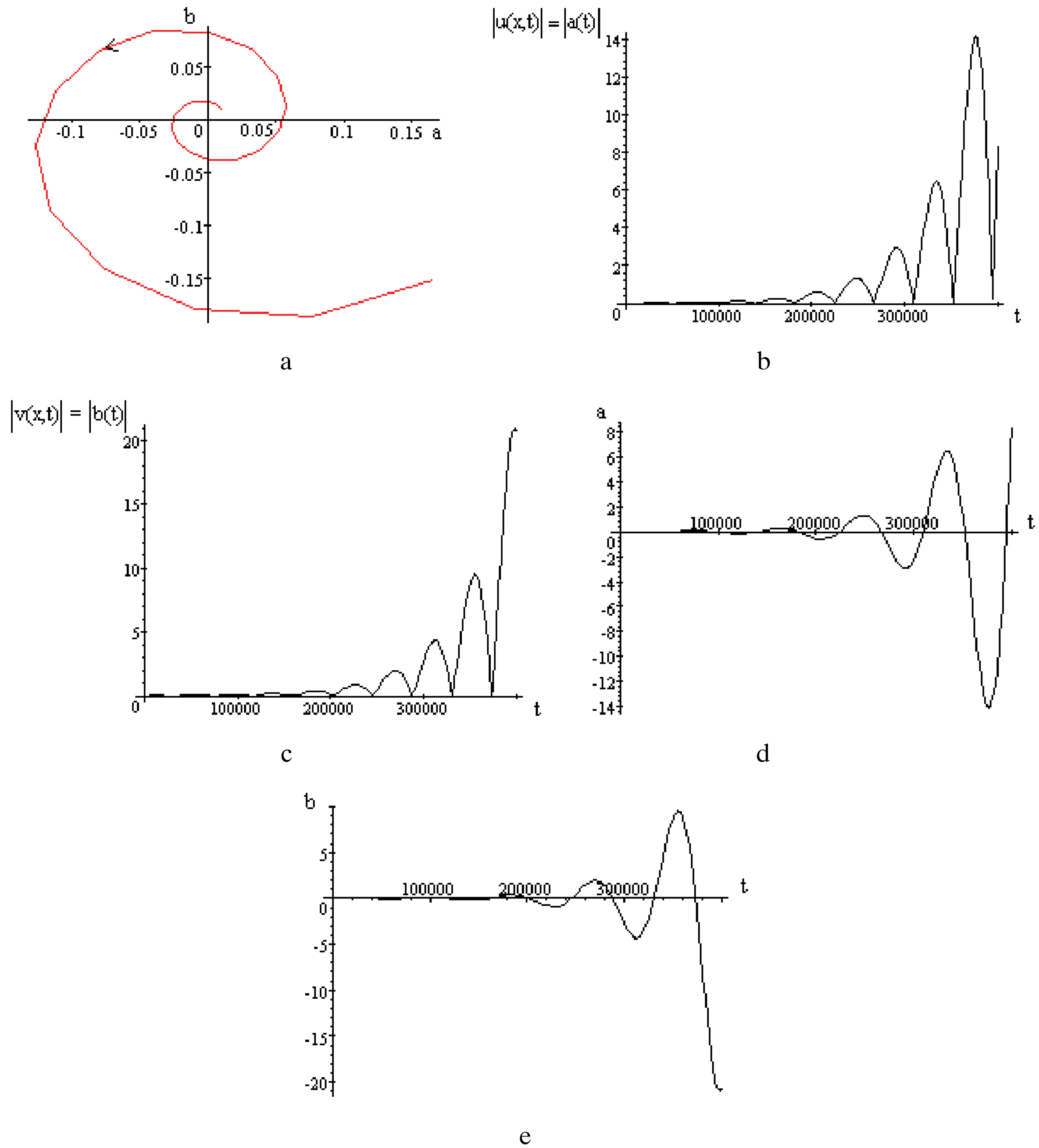


Fig. 3. (a): Plot of the stationary solution (α, α) of system (4.2), (4.3) in the (a, b) phase plane; (b) and (c): plots of $|u(x,t)| = |a(t)|$ and $|v(x,t)| = |b(t)|$, respectively; (d) and (e): graphs of $a(t)$ and $b(t)$, respectively.

***4.2. The line parameters* ($\mathbf{L_{p2}}$).** For the line parameters (L_{p2}) and the phase winding numbers $k_u = 0.57664$ and $k_v = 0.80947$ we find that $\left(10^{-3}, 10^{-4}\right)$ is a stationary solution to system (4.2), (4.3), and the matrix M and eigenvalues corresponding to this solution are

$$M = \begin{pmatrix} -6.21 \times 10^{-4} & -1.8 \times 10^{-4} \\ -1.8 \times 10^{-4} & -1.755 \times 10^{-4} \end{pmatrix}, \qquad \begin{aligned} \lambda_1 &= -6.8464 \times 10^{-4}, \\ \lambda_2 &= -1.1186 \times 10^{-4}. \end{aligned}$$

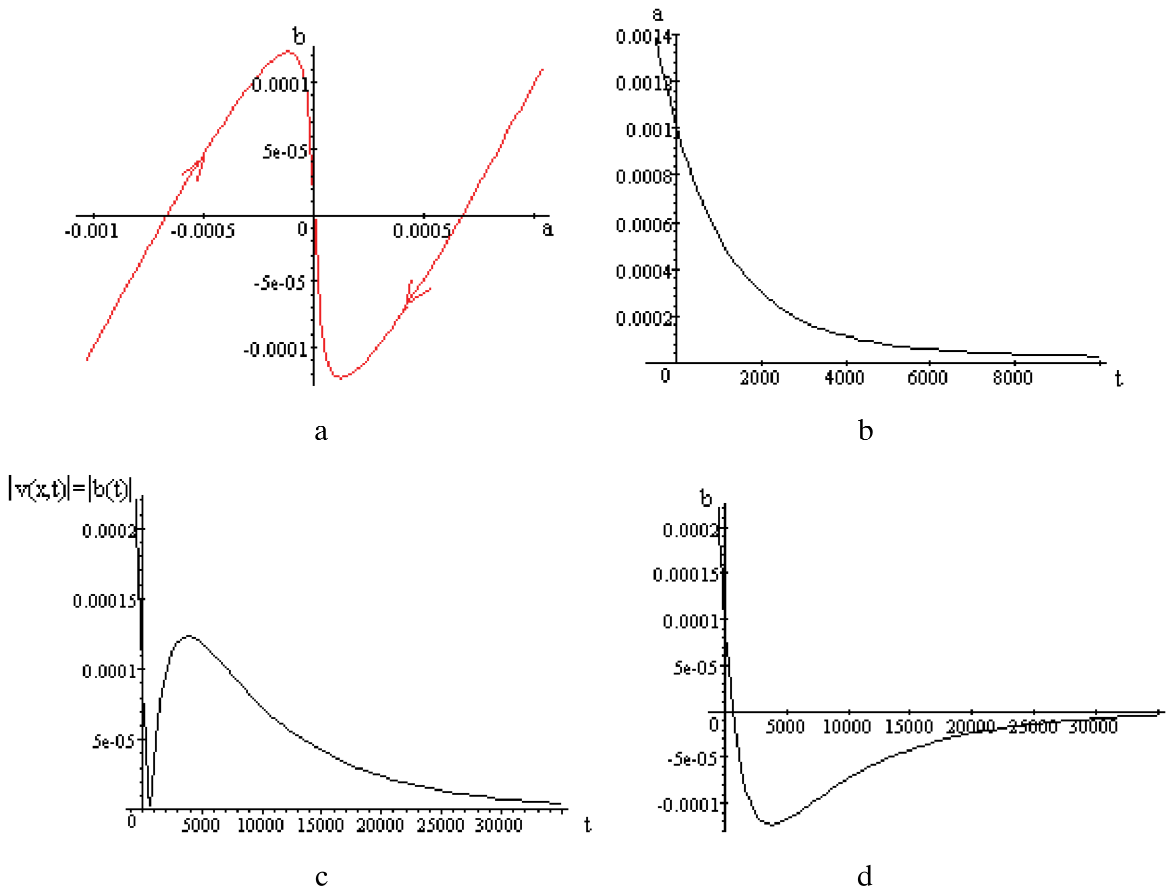


Fig. 4. Graphs corresponding to the stationary solution $(10^{-3}, 10^{-4})$ to system (4.2), (4.3). (*a*): Plot of the solution in the (a, b) phase plane; (*b*) and (*c*): plots of $|u(x,t)| = |a(t)|$ and $|v(x,t)| = |b(t)|$, respectively; (*b*) and (*d*): plots of $a(t)$ and $b(t)$, respectively.

Because $\lambda_1 < 0$ and $\lambda_2 < 0$, $\left(10^{-3}, 10^{-4}\right)$ is a stable node. Figure 4(*a*) shows the plot of the solution in the (a, b) phase plane. Figure 4(*b*) shows the curves $a(t)$ and $|u(x,t)| = |a(t)|$, because $a(t) > 0$ for all t. As one can see from Figs. 4(*b*), (*c*), and (*d*), the curves $a(t)$, $|u(x,t)|$, $|v(x,t)|$, and $b(t)$ are bounded as $t \to +\infty$.

5. Coherent structures. Many patterns that occur in experiments on traveling wave systems or numerical simulations of the single and CGL equations exhibit local structures that have an essentially time-independent shape and propagate with a constant velocity v. For these so-called coherent structures, the spatial and temporal degrees of freedom are not independent: apart from a phase factor, they are stationary in the co-moving frame $z = x - v_{\text{coh}}t$. Since the appropriate functions that describe the profiles of these coherent structures depend only on the single variable z, these functions can be determined by ODEs. These are obtained by substituting the appropriate Ansatz in the original CGL equations, which of course are partial differential equations. Since the ODEs can themselves be written as a set of first-order flow equations in a simple phase space, the coherent structures of the amplitude equations correspond to certain orbits of these ODEs. Note that plane waves, since they have constant profiles, are trivial examples of coherent structures (in the flow equations they correspond to fixed points stationary solutions). Sources and sinks connect, asymptotically, plane waves, and so the corresponding orbits in the ODEs connect fixed points. Many different coherent structures have been identified within this framework [21–26].

The counting arguments that give the multiplicity of such solutions are essentially based on determining the dimensions of the stable and unstable manifolds near the fixed points. These dimensions, together with the parameters of the Ansatz such as v_{coh}, determine, for some orbit, the number of constraints and the number of free parameters that can be varied to fulfill these constraints. We may illustrate the theoretical importance of counting arguments by recalling that for the single CGL equation a continuous family of hole solutions has been known to exist for some time [24]. Later, however, counting arguments showed that these source type solutions were on general grounds expected to come as discrete sets, not as a continuous one-parameter family [22, 23]. This suggested that there is some accidental degeneracy or hidden symmetry in the single CGL equation, so that by adding a seemingly innocuous perturbation to the CGL equation, the family of hole solutions should collapse to a discrete set. This was indeed found to be the case [27, 28]. For further details of the results and implications of these counting arguments for coherent structures in the single CGL equation, we refer to [22, 23, 29].

It should be stressed that counting arguments cannot prove the existence of certain coherent structures nor can they establish the dynamical relevance of the solutions. They can only establish the multiplicity of the solutions, assuming that the equations have no hidden symmetries. Imagine that we know either by an explicit construction or from numerical experiments that a certain type of coherent structure solution does exist. The counting arguments then establish whether this should be an isolated or discrete solution (at most a member of a discrete set of them), or a member of a one-parameter family of solutions, etc. In the case of an isolated solution, there are no nearby solutions if we slightly change one of the parameters (like the velocity v_{coh}). For a one-parameter family, the counting argument implies that when we start from a known solution and change the velocity, we have enough other free parameters available to make sure that there is a perturbed trajectory that flows into the proper fixed point as $z \to \infty$.

For the two CGL Eqs. (3.10), (3.11) the counting can be performed by a straightforward extension of the counting for the single CGL equation [22, 23, 30]. The Ansatz for coherent structures of the CGL equations (3.10), (3.11) is the following generalization of the Ansatz for the single CGL equation

$$\begin{aligned} u(x,t) &= a(z = x - v_{\mathrm{coh}}t)\exp\left[i\int \Phi(z)dz - i\omega_u t\right], \\ v(x,t) &= b(z = x - v_{\mathrm{coh}}t)\exp\left[i\int \Psi(z)dz - i\omega_v t\right]. \end{aligned} \tag{5.1}$$

Note that we take the velocities of the structures in the left and right modes equal, while the frequencies ω are allowed to be different. This is due to the form of the coupling of the left- and right-traveling modes, which is through the moduli of the amplitudes. It obviously does not make sense to choose the velocities of u and v differently: for large times the cores of the structures in u and v would then get arbitrarily far apart, and at the technical level, this would be reflected by the fact that with different velocities we would not obtain simple ODEs for a and b. Since the phases of u and v are not directly coupled, there is no a priori reason to take the frequencies ω_u and ω_v equal. In (5.1) $a(z)$ and $b(z)$ are real amplitudes while $\Phi(z)$ and $\Psi(z)$

are the local wavenumbers and are functions of $z = x - v_{\text{coh}}t$. In other words, $\int \Phi(z)dz$ and $\int \Psi(z)dz$ are real phases and are functions of $z = x - v_{\text{coh}}t$.

Substituting Ansatz (5.1) into the CGL Eqs. (3.10), (3.11), we obtain the following set of ODEs:

$$a' = X, \tag{5.2a}$$

$$b' = Y, \tag{5.3a}$$

$$\begin{aligned} X' = {} & \Phi^2 a + \frac{P^r(v_{\text{coh}} - S_0)}{|P|^2} X - \frac{E^r P^r + E^i P^i}{|P|^2}\left(3a^2 + 2b^2\right) X - \\ & - 2\left(\frac{H^r P^r + H^i P^i}{|P|^2} + 2\,\frac{E^r P^r + E^i P^i}{|P|^2}\right) abY - \frac{\gamma^r P^r + \gamma^i P^i - \omega_u P^i}{|P|^2}\, a - \\ & - \frac{Q^r P^r + Q^i P^i}{|P|^2}\left(a^2 + 2b^2\right) a - \left(\frac{D^r P^r + D^i P^i}{|P|^2} a^4 + \frac{F^r P^r + F^i P^i}{|P|^2} a^2 b^2 + \right. \\ & \left. + \frac{K^r P^r + K^i P^i}{|P|^2} b^4\right) a - \frac{E^r P^i - E^i P^r}{|P|^2}\left(a^3 + 2ab^2\right)\Phi + \left(\frac{v_{\text{coh}} P^i - S_0 P^i}{|P|^2}\right) a\Phi, \end{aligned} \tag{5.4a}$$

$$\begin{aligned} Y' = {} & \Psi^2 b + \frac{P^r(v_{\text{coh}} + S_0)}{|P|^2} Y + \frac{E^r P^r + E^i P^i}{|P|^2}\left(3b^2 + 2a^2\right) Y + \\ & + 2\left(\frac{H^r P^r + H^i P^i}{|P|^2} + 2\,\frac{E^r P^r + E^i P^i}{|P|^2}\right) abX - \frac{\gamma^r P^r + \gamma^i P^i - \omega_v P^i}{|P|^2}\, b - \\ & - \frac{Q^r P^r + Q^i P^i}{|P|^2}\left(b^2 + 2a^2\right) b - \left(\frac{D^r P^r + D^i P^i}{|P|^2} b^4 + \frac{F^r P^r + F^i P^i}{|P|^2} a^2 b^2 + \right. \\ & \left. + \frac{K^r P^r + K^i P^i}{|P|^2} a^4\right) b + \frac{E^r P^i - E^i P^r}{|P|^2}\left(b^3 + 2ba^2\right)\Psi + \left(\frac{v_{\text{coh}} P^i + S_0 P^i}{|P|^2}\right) b\Psi, \end{aligned} \tag{5.5a}$$

$$\begin{aligned} \Phi' = {} & -2\Phi\,\frac{X}{a} + \frac{P^i(S_0 - v_{\text{coh}})}{|P|^2}\frac{X}{a} - \frac{E^i P^r - E^r P^i}{|P|^2}\left(3aX + 2b^2\,\frac{X}{a}\right) - \\ & - 2\left(\frac{H^i P^r - H^r P^i}{|P|^2} + 2\,\frac{E^i P^r - E^r P^i}{|P|^2}\right) bY + \frac{\gamma^r P^i - \gamma^i P^r - \omega_u P^r}{|P|^2} - \\ & - \frac{Q^i P^r - Q^r P^i}{|P|^2}\left(a^2 + 2b^2\right) - \left(\frac{D^i P^r - D^r P^i}{|P|^2} a^4 + \frac{F^i P^r - F^r P^i}{|P|^2} a^2 b^2 + \right. \\ & \left. + \frac{K^i P^r - K^r P^i}{|P|^2} b^4\right) - \frac{E^r P^r + E^i P^i}{|P|^2}\Phi\left(a^2 + 2b^2\right) + \left(\frac{v_{\text{coh}} P^i - S_0 P^r}{|P|^2}\right)\Phi, \end{aligned} \tag{5.6a}$$

$$\Psi' = -2\Psi\,\frac{Y}{b} - \frac{P^i(S_0 + v_{\text{coh}})}{|P|^2}\frac{Y}{b} + \frac{E^i P^r - E^r P^i}{|P|^2}\left(3bY + 2a^2\,\frac{Y}{b}\right) +$$

$$+2\left(\frac{H^iP^r-H^rP^i}{|P|^2}+2\,\frac{E^iP^r-E^rP^i}{|P|^2}\right)aX+\frac{\gamma^rP^i-\gamma^iP^r-\omega_vP^r}{|P|^2}-$$

$$-\frac{Q^iP^r-Q^rP^i}{|P|^2}\left(b^2+2a^2\right)-\left(\frac{D^iP^r-D^rP^i}{|P|^2}\,b^4+\frac{F^iP^r-F^rP^i}{|P|^2}\,a^2b^2+\right.$$

$$\left.+\frac{K^iP^r-K^rP^i}{|P|^2}\,a^4\right)+\frac{E^rP^r+E^iP^i}{|P|^2}\,\Psi\left(b^2+2a^2\right)+\left(\frac{v_{\rm coh}P^i+S_0P^r}{|P|^2}\right)\Psi. \tag{5.7a}$$

The solutions of these ODEs correspond to coherent structures of the GCGL equations (3.10), (3.11) and vice-versa. From Appendix A, we have $E^rP^r+E^iP^i=0$.

System (5.2a) – (5.7a) for (a, X, Φ, b, Y, Ψ) has singularities at $a=0$ and $b=0$. To overcome this difficulty we introduce the "blow up" transform or σ-process [31]. Letting

$$\frac{X}{a}=x, \qquad \frac{Y}{b}=y, \tag{5.8}$$

we compute $\dfrac{X'}{a}=x^2+x'$ and $\dfrac{Y'}{b}=y^2+y'$ so that (5.2a) – (5.7a) become the following regularized system for (a, x, Φ, b, y, Ψ):

$$a'=ax, \tag{5.2}$$

$$b'=by, \tag{5.3}$$

$$x'=-x^2+\Phi^2+\frac{P^r\left(v_{\rm coh}-S_0\right)}{|P|^2}\,x-\frac{2\left(H^rP^r+H^iP^i\right)}{|P|^2}\,b^2y-\frac{\gamma^rP^r+\gamma^iP^i-\omega_uP^i}{|P|^2}-$$

$$-\frac{Q^rP^r+Q^iP^i}{|P|^2}\left(a^2+2b^2\right)-\left(\frac{D^rP^r+D^iP^i}{|P|^2}\,a^4+\frac{F^rP^r+F^iP^i}{|P|^2}\,a^2b^2+\right.$$

$$\left.+\frac{K^rP^r+K^iP^i}{|P|^2}\,b^4\right)-\frac{E^rP^i-E^iP^r}{|P|^2}\left(a^2+2b^2\right)\Phi+\left(\frac{v_{\rm coh}P^i-S_0P^i}{|P|^2}\right)\Phi, \tag{5.4}$$

$$y'=-y^2+\Psi^2+\frac{P^r\left(v_{\rm coh}+S_0\right)}{|P|^2}\,y+\frac{2\left(H^rP^r+H^iP^i\right)}{|P|^2}a^2x-\frac{\gamma^rP^r+\gamma^iP^i-\omega_vP^i}{|P|^2}-$$

$$-\frac{Q^rP^r+Q^iP^i}{|P|^2}\left(b^2+2a^2\right)-\left(\frac{D^rP^r+D^iP^i}{|P|^2}\,b^4+\frac{F^rP^r+F^iP^i}{|P|^2}\,a^2b^2+\right.$$

$$\left.+\frac{K^rP^r+K^iP^i}{|P|^2}\,a^4\right)+\frac{E^rP^i-E^iP^r}{|P|^2}\left(b^2+2a^2\right)\Psi+\left(\frac{v_{\rm coh}P^i+S_0P^i}{|P|^2}\right)\Psi, \tag{5.5}$$

$$\Phi'=-2\Phi x+\frac{P^i\left(S_0-v_{\rm coh}\right)}{|P|^2}\,x-\frac{E^iP^r-E^rP^i}{|P|^2}\left(3a^2+2b^2\right)x-$$

$$-2\left(\frac{H^iP^r-H^rP^i+2\left(E^iP^r-E^rP^i\right)}{|P|^2}\right)b^2y+\frac{\gamma^rP^i-\gamma^iP^r-\omega_uP^r}{|P|^2}-$$

$$- \frac{Q^i P^r - Q^r P^i}{|P|^2} \left(a^2 + 2b^2\right) - \left(\frac{D^i P^r - D^r P^i}{|P|^2} a^4 + \frac{F^i P^r - F^r P^i}{|P|^2} a^2 b^2 + \right.$$

$$\left. + \frac{K^i P^r - K^r P^i}{|P|^2} b^4 \right) + \left(\frac{v_{\text{coh}} P^i - S_0 P^r}{|P|^2} \right) \Phi, \tag{5.6}$$

$$\Psi' = -2\Psi y - \frac{P^i \left(S_0 + v_{\text{coh}}\right)}{|P|^2} y + \frac{E^i P^r - E^r P^i}{|P|^2} \left(3b^2 + 2a^2\right) y +$$

$$+ 2 \left(\frac{H^i P^r - H^r P^i + 2\left(E^i P^r - E^r P^i\right)}{|P|^2} \right) a^2 x + \left(\frac{\gamma^r P^i - \gamma^i P^r - \omega_v P^i}{|P|^2} \right) -$$

$$- \frac{Q^i P^r - Q^r P^i}{|P|^2} \left(b^2 + 2a^2\right) - \left(\frac{D^i P^r - D^r P^i}{|P|^2} b^4 + \frac{F^i P^r - F^r P^i}{|P|^2} a^2 b^2 + \right.$$

$$\left. + \frac{K^i P^r - K^r P^i}{|P|^2} a^4 \right) + \left(\frac{v_{\text{coh}} P^i + S_0 P^r}{|P|^2} \right) \Psi. \tag{5.7}$$

Compared to the flow equations for the single CGL equation [22, 23], there are two important differences that should be noted: (i) Instead of the velocity v_{coh} we now have velocities $v_{\text{coh}} \pm S_0$. This is simply due to the fact that the linear group velocity terms cannot be transformed away. (ii) The nonlinear coupling term in the CGL equations shows up only in the flow equations for x and y.

The fixed points of these flow equations, the points in phase space at which the right-hand sides of Eqs. (5.2)–(5.7) vanish, describe the asymptotic states for $z \to \pm\infty$ of the coherent structures. What are these fixed points? From Eq. (5.2) we find that either x or a is equal to zero at a fixed point, and similarly, from Eq. (5.3) it follows that either y or b vanishes. For the sources and sinks of (3.10) and (3.11) that we wish to study, the asymptotic states are left- and right-traveling waves. Therefore the fixed points of interest to us have either both x and b or both y and a equal to zero, and we search for heteroclinic orbits connecting these two fixed points.

5.1. Coherent structures in systems described by the GGL equation (3.13). As it is noted above the GGL Eqs. (3.13) is obtained from the GCGL Eqs. (3.10), (3.11) by setting either $u = 0$ or $v = 0$. Suppose that the GGL Eq. (3.13) is obtained from (3.10), (3.11) by setting $v(x,t) = 0$. Then by setting $b = 0, Y = 0, F = K = H = 0, \Psi(z) = 0$ in system (5.2a)–(5.7a) and using (5.8) for Y and a, we obtain Eqs. (5.1), (5.3), and (5.5) in which we set $b = 0$:

$$a' = ax, \tag{5.9}$$

$$x' = -x^2 + \Phi^2 + \frac{P^r \left(v_{\text{coh}} - S_0\right)}{|P|^2} x - \frac{\gamma^r P^r + \gamma^i P^i - \omega_u P^i}{|P|^2} - \frac{Q^r P^r + Q^i P^i}{|P|^2} a^2 -$$

$$- \frac{D^r P^r + D^i P^i}{|P|^2} a^4 - \frac{E^r P^i - E^i P^r}{|P|^2} a^2 \Phi + \frac{P^i \left(v_{\text{coh}} - S_0\right)}{|P|^2} \Phi, \tag{5.10}$$

$$\Phi' = -2\Phi x + \frac{P^i\left(S_0 - v_{\rm coh}\right)}{|P|^2}\,x - 3\,\frac{E^iP^r - E^rP^i}{|P|^2}\,a^2x + \frac{\gamma^rP^i - \gamma^iP^r - \omega_uP^r}{|P|^2} -$$

$$- \frac{Q^iP^r - Q^rP^i}{|P|^2}\,a^2 - \frac{D^iP^r - D^rP^i}{|P|^2}\,a^4 + \frac{v_{\rm coh}P^i - S_0P^r}{|P|^2}\,\Phi. \qquad (5.11)$$

The solutions of these ODEs correspond to coherent structures of the GGL equation (3.13) and vice-versa.

The fixed points of the ODEs have, according to (5.9), either $a = 0$ or $x = 0$. The values of x and Φ for the fixed points with $a = 0$ are related through the dispersion relation of the linearized equation, or, what amounts to the same, by the equation obtained by setting the right-hand side of (5.10) and (5.11) equal to zero and taking $a = 0$. We refer to these fixed points as *linear fixed points* [22, 23] and we denote them by $L_\pm$, where the subscript indicates the sign of x. This means that the behavior near some L_+ corresponds to a situation in which the amplitude is growing away from zero to the right, while the behavior near some L_- describes the situation in which the amplitude a decays to zero.

Since a fixed point with $a \neq 0$ and $x = 0$ corresponds to nonlinear traveling waves, the corresponding fixed points are referred to as nonlinear fixed points [22, 23] which we denote by $N_\pm$, where the subscript now indicates the sign of the nonlinear *group velocity* $s_u = s$ of the corresponding traveling wave. Thus, the amplitude near an N_+ can either grow ($x > 0$) or decay ($x < 0$) with increasing z.

Coherent structures correspond to orbits which go from one of the fixed point to another one or back to the original one, and the counting analysis amounts to establishing the dimensions of the in- and out-going manifolds of these fixed points. In combination with the number of free parameters (in this case $v_{\rm coh}$ and ω_u), this yields the multiplicity of orbits connecting these fixed points, and, therefore, of the multiplicity of the corresponding coherent structures.

Since there are three flow Eqs. (5.9)–(5.11), there are three eigenvalues of the linear flow near each fixed point. When we perform the counting analysis for these fixed points we will only need the signs of the real parts of the three eigenvalues, since they determine whether the flow along the corresponding eigendirection is inwards ($-$) or outwards ($+$). We will denote the signs by pluses and minuses, so that $L_-(+,+,-)$ denotes an L_- fixed point with two eigenvalues with positive real parts, and one with a negative real part.

From Eqs. (5.9)–(5.11), we obtain the fixed point equations:

$$ax = 0, \qquad (5.12)$$

$$-|P|^2x^2 + |P|^2\Phi^2 + P^r\left(v_{\rm coh} - S_0\right)x - \gamma^rP^r - \gamma^iP^i + \omega_uP^i - \left(Q^rP^r + Q^iP^i\right)a^2 -$$

$$- \left(D^rP^r + D^iP^i\right)a^4 + \left(E^iP^r - E^rP^i\right)a^2\Phi + P^i\left(v_{\rm coh} - S_0\right)\Phi = 0, \qquad (5.13)$$

$$-2|P|^2\Phi x + P^i\left(S_0 - v_{\rm coh}\right)x + \left(Q^rP^i - Q^iP^r\right)a^2 + \gamma^rP^i - \gamma^iP^r -$$

$$- \omega_uP^r + \left(D^rP^i - D^iP^r\right)a^4 - \left(E^rP^r + E^iP^i\right)\Phi a^2 + \left(v_{\rm coh}P^i - S_0P^r\right)\Phi = 0. \qquad (5.14)$$

From (5.12) we immediately obtain that fixed points either have $a = 0$ (linear fixed points denoted as L) or $a \neq 0$ and $x = 0$ (nonlinear fixed points denoted by N). Let (a_0, x_0, Φ_0) be a fixed point of the ODEs (5.9) – (5.11). Then the linear flow equations in its neighborhood is

$$a' = x_0 a + a_0 x, \tag{5.15}$$

$$x' = \frac{-2a_0\left(Q^r P^r + Q^i P^i\right) - 4a_0^3\left(D^r P^r + D^i P^i\right) - 2\Phi_0 a_0\left(E^i P^r - E^r P^i\right)}{|P|^2}\, a + \\ + \frac{P^r\left(v_{\text{coh}} - S_0\right) - 2|P|^2 x_0}{|P|^2} x + \frac{v_{\text{coh}} P^i - S_0 P^i + 2\,|P|^2\,\Phi_0 \Phi + a_0^2}{|P|^2}\,\Phi, \tag{5.16}$$

$$\Phi' = \frac{2a_0\left(Q^r P^i - Q^i P^r\right) + 6a_0 x_0\left(E^r P^i - E^i P^r\right) + a_0^3\left(D^r P^i - D^i P^r\right)}{|P|^2}\, a + \\ + \frac{P^i\left(S_0 - v_{\text{coh}}\right) - 2|P|^2\Phi_0 + 3a_0^2\left(E^r P^i - E^i P^r\right)}{|P|^2}\, x + \frac{v_{\text{coh}} P^i - S_0 P^r - 2|P|^2 x_0}{|P|^2}\,\Phi, \tag{5.17}$$

with matrix

$$M_L = \begin{pmatrix} x_0 & a_0 & 0 \\ \widetilde{a}_1 & \widetilde{x}_1 & \widetilde{\Phi}_1 \\ \widetilde{\hat{a}}_2 & \widetilde{x}_2 & \widetilde{\Phi}_2 \end{pmatrix}$$

where

$$\widetilde{a}_1 = \frac{-2a_0\left(Q^r P^r + Q^i P^i\right) - 4a_0^3\left(D^r P^r + D^i P^i\right) - 2\Phi_0 a_0\left(E^i P^r - E^r P^i\right)}{|P|^2},$$

$$\widetilde{a}_2 = \frac{2a_0\left(Q^r P^i - Q^i P^r\right) + 6a_0 x_0\left(E^r P^i - E^i P^r\right) + a_0^3\left(D^r P^i - D^i P^r\right)}{|P|^2},$$

$$\widetilde{x}_1 = \frac{P^r\left(v_{\text{coh}} - S_0\right) - 2\,|P|^2\,x_0}{|P|^2},$$

$$\widetilde{x}_2 = \frac{P^i\left(S_0 - v_{\text{coh}}\right) - 2\,|P|^2\,\Phi_0 + 3a_0^2\left(E^r P^i - E^i P^r\right)}{|P|^2},$$

$$\widetilde{\Phi}_1 = \frac{v_{\text{coh}} P^i - S_0 P^i + 2\,|P|^2\,\Phi_0 + a_0^2}{|P|^2},$$

$$\widetilde{\Phi}_2 = \frac{v_{\text{coh}} P^i - S_0 P^r - 2\,|P|^2\,x_0}{|P|^2}.$$

Solving the fixed point Eqs. (5.12) – (5.14) and calculating the eigenvalues of M_L yield the dimensions of the incoming and outgoing manifolds of these fixed points. Note that according to our convention, a fixed point with a two-dimensional outgoing and one-dimensional ingoing manifold is denoted as $(+, +, -)$. We can restrict calculations to the case of positive v_{coh}, since the case of negative $v_{\rm coh}$ can be found by the left-right symmetry operation $z \to -z$, $v_{\rm coh} \to -v_{\rm coh}$, $x \to -x$, and $\Phi \to -\Phi$ (for some examples we may take $v_{\rm coh}$ negative).

The eigenvalues of M_L are solutions of the cubic equation

$$\lambda^3 - \left(x_0 + \widetilde{x}_1 + \widetilde{\Phi}_2\right)\lambda^2 + \left(x_0\widetilde{x}_1 - \widetilde{a}_1 a_0 + x_0\widetilde{\Phi}_2 - \widetilde{x}_1\widetilde{\Phi}_2 - \widetilde{x}_2\widetilde{\Phi}_1\right)\lambda -$$
$$- \left(x_0\widetilde{x}_1\widetilde{\Phi}_2 - x_0\widetilde{x}_2\widetilde{\Phi}_1 - a_0\widetilde{a}_1\widetilde{\Phi}_2 + a_0\widetilde{a}_2\widetilde{\Phi}_1\right) = 0,$$

with coefficients

$$P_2 = -\left(x_0 + \widetilde{x}_1 + \widetilde{\Phi}_2\right), \quad P_1 = x_0\widetilde{x}_1 - \widetilde{a}_1 a_0 + x_0\widetilde{\Phi}_2 - \widetilde{x}_1\widetilde{\Phi}_2 - \widetilde{x}_2\widetilde{\Phi}_1,$$

$$P_0 = -x_0\widetilde{x}_1\widetilde{\Phi}_2 + x_0\widetilde{x}_2\widetilde{\Phi}_1 + a_0\widetilde{a}_1\widetilde{\Phi}_2 - a_0\widetilde{a}_2\widetilde{\Phi}_1.$$

We may read the signs of the real parts of the solution of these three equations from the following [22, 23]:

$$P_0 > 0 \quad \begin{cases} P_2 > 0, \quad P_1P_2 > P_0: & (-,-,-) \quad \text{case (i)}, \\ \text{else}: & (+,+,-) \quad \text{case (ii)}, \end{cases}$$

$$P_0 < 0 \quad \begin{cases} P_2 < 0, \quad P_1P_2 < P_0: & (+,+,+) \quad \text{case (iii)}, \\ \text{else}: & (+,-,-) \quad \text{case (iv)}. \end{cases}$$

According to these rules, we need to know the sign of the real parts of the eigenvalues of M_L for the three combinations of the coefficients, namely, P_0, P_2 and

$$P_1P_2 - P_0 = (a_0x_0 + a_0\widetilde{x}_1)\,\widetilde{a}_1 + \left(\widetilde{x} - x_0^2\right)\widetilde{\Phi}_2 + (\widetilde{x}_1\widetilde{x}_2 + a_0\widetilde{a}_2)\,\widetilde{\Phi}_1 -$$
$$- x_0\widetilde{x}_1^2 - x_0^2\widetilde{x}_1 + (\widetilde{x}_1 - x_0)\,\widetilde{\Phi}_2^2 + \widetilde{x}_2\widetilde{\Phi}_1\widetilde{\Phi}_2.$$

For $a_0 = 0$ (a linear fixed point) and $x_0 = 0$ and $a_0 \neq 0$ (a nonlinear fixed point), we have the respectively relations:

$$P_1P_2 - P_0 = \left(\widetilde{x} - x_0^2\right)\widetilde{\Phi}_2 + \widetilde{x}_1\widetilde{x}_2\widetilde{\Phi}_1 - x_0\widetilde{x}_1^2 - x_0^2\widetilde{x}_1 + (\widetilde{x}_1 - x_0)\,\widetilde{\Phi}_2^2 + \widetilde{x}_2\widetilde{\Phi}_1\widetilde{\Phi}_2,$$

$$P_1P_2 - P_0 = a_0\widetilde{x}_1\widetilde{a}_1 + \widetilde{x}\widetilde{\Phi}_2 + (\widetilde{x}_1\widetilde{x}_2 + a_0\widetilde{a}_2)\,\widetilde{\Phi}_1 + \widetilde{x}_1\widetilde{\Phi}_2^2 + \widetilde{x}_2\widetilde{\Phi}_1\widetilde{\Phi}_2.$$

5.1.1. Linear fixed points. In the case of a linear fixed point, we have $a = 0$, and from (5.12) – (5.14) we obtain the fixed-point equations

$$|P|^2\Phi^2 + P^i\,(v_{\rm coh} - S_0)\,\Phi + P^r\,(v_{\rm coh} - S_0)\,x - |P|^2x^2 - \gamma^r P^r - \gamma^i P^i + \omega_u P^i = 0, \tag{5.18}$$

$$\left(v_{\rm coh}P^i - S_0P^r\right)\Phi - 2|P|^2\Phi x + P^i\,(S_0 - v_{\rm coh})\,x + \gamma^r P^i - \gamma^i P^r - \omega_u P^r = 0. \tag{5.19}$$

Since we may choose ω_u and v_{coh} freely, we can take x such that $P^i\,(S_0 - v_{\text{coh}})\,x + \gamma^r P^i - \gamma^i P^r - \omega_u P^r = 0$. Equation (5.19) then gives

$$x_0 = \frac{v_{\text{coh}} P^i - S_0 P^r}{2|P|^2},$$

and from (5.18) we obtain

$$\Phi_0 = \frac{P^i S_0 - v_{\text{coh}} \pm \sqrt{[P^i\,(v_{\text{coh}} - S_0)]^2 + 4|P|^2 \left(P^r\,(S_0 - v_{\text{coh}})\,x_0 + |P|^2\,x_0^2 + \gamma^r P^r + \gamma^i P^i - \omega_u P^i\right)}}{2|P|^2}$$

with

$$\omega_u = \frac{2|P|^2\left(\gamma^r P^i - \gamma^i P^r\right) + P^i\,(S_0 - v_{\text{coh}})\left(v_{\text{coh}} P^i - S_0 P^r\right)}{2|P|^2 P^r}.$$

Under these conditions, we obtain three linear fixed points

$$\left(0, \qquad \frac{v_{\text{coh}} P^i - S_0 P^r}{2|P|^2}, \qquad \Phi_0\right). \tag{5.20}$$

To obtain all the solutions of system (5.18), (5.19), we can proceed as follows: we solve (5.19) for x and obtain

$$x = \frac{\left(v_{\text{coh}} P^i - S_0 P^r\right)\Phi + \gamma^r P^i - \gamma^i P^r - \omega_u P^r}{2|P|^2\Phi + P^i(v_{\text{coh}} - S_0)}. \tag{5.21a}$$

Replacing this expression for x in (5.18) we obtain the following quartic equation for Φ:

$$\begin{aligned}\left(|P|^2\Phi^2 + P^i(v_{\text{coh}} - S_0)\Phi\right)\left[2|P|^2\Phi + P^i(v_{\text{coh}} - S_0)\right]^2 + \\ + P^r(v_{\text{coh}} - S_0)\left[\left(v_{\text{coh}} P^i - S_0 P^r\right)\Phi + \gamma^r P^i - \gamma^i P^r - \omega_u P^r\right]\left[2|P|^2\Phi + P^i\,(v_{\text{coh}} - S_0)\right] - \\ - |P|^2\left[\left(v_{\text{coh}} P^i - S_0 P^r\right)\Phi + \gamma^r P^i - \gamma^i P^r - \omega_u P^r\right]^2 - \\ - \left[2|P|^2\Phi + P^i\,(v_{\text{coh}} - S_0)\right]^2\left(\gamma^r P^r + \gamma^i P^i - \omega_u P^i\right) = 0.\end{aligned} \tag{5.21}$$

At the fixed points, the eigenvalues are given by

$$x_0, \qquad \frac{\widetilde{x}_1 + \widetilde{\Phi}_1 \pm \sqrt{\left(\widetilde{x}_1 + \widetilde{\Phi}_1\right)^2 - 4\widetilde{x}_1\widetilde{\Phi}_2 + 4\widetilde{x}_2\widetilde{\Phi}_1}}{2} := \frac{\widetilde{x}_1 + \widetilde{\Phi}_1 \pm \sqrt{\Delta}}{2}.$$

To establish the signs of the real parts of the eigenvalues, if $\Delta < 0$ we need to determine the signs of x_0 and $\widetilde{x}_1 + \widetilde{\Phi}_1$. But if $\Delta > 0$ we need to determine the signs of x_0 and $\widetilde{x}_1 + \widetilde{\Phi}_1 \pm \sqrt{\Delta}$.

Let us first establish the signs of x_0. This is important in establishing whether the evanescent wave decays to the left (L_+) or to the right (L_-). For the fixed points (5.20) we have

$$x_0 = \frac{v_{\text{coh}} P^i - S_0 P^r}{2|P|^2}.$$

Because $P^r > 0$ and $P^i < 0$ and we are in the case where $v_{\text{coh}} > 0$, we conclude that $x_0 < 0$, and this means that the evanescent wave decays to the right (L_-).

For the line parameters ($\mathrm{L_{p1}}$) and $v_{\text{coh}} = 6 \times 10^5$ and $\omega_u = 10^6$, (5.20) gives us the two fixed points

$$p_1 = (0, -1.8617 \times 10^8, -1.5765 \times 10^{11}), \qquad p_2 = (0, -1.8617 \times 10^8, -1.5722 \times 10^{11}).$$

For p_1, the eigenvalues of M_L are

$$-1.8617 \times 10^8, \qquad 2.1757 \times 10^8 \pm 3.153 \times 10^{11} i$$

which means that p_1 is an $L_-(+,+,-)$ fixed point. Now, p_2 is an $L_-(+,+,-)$ fixed point because the corresponding matrix M_L has two eigenvalues with positive real parts and one eigenvalue with negative real part.

For the above values of v_{coh} and ω_u and line parameters ($\mathrm{L_{p1}}$), (5.21) gives

$$\Phi_0 \in \left\{-0.11601, \quad 3.1256 \times 10^9, \quad 3.155 \times 10^9, \quad 5.477 \times 10^9\right\}.$$

For these values of Φ we obtain the following values of x (see (5.21a)):

$$x \in \left\{10045, \quad -1.8617 \times 10^8, \quad -1.8617 \times 10^8, \quad -1.8617 \times 10^8\right\}.$$

We then have four linear fixed points:

$$(0, 10045, -0.11601), \quad (0, -1.8617 \times 10^8, 3.1256 \times 10^9),$$
$$(0, -1.8617 \times 10^8, 3.155 \times 10^9), \quad (0, -1.8617 \times 10^8, 5.477 \times 10^9),$$

that is, one L_+ fixed point and three L_- fixed points. The L_+ fixed point is an $L_+(+,+,-)$ fixed point while the three L_- fixed points are $L_+(+,+,-)$ fixed points. For the $L_+(+,+,-)$ fixed point $(0, 10045, -0.11601)$ and the $L_-(+,+,-)$ fixed point $\left(0, -1.8617 \times 10^8, 3.1256 \times 10^9\right)$, Figs. 5 and 6 plot the real and imaginary parts of u, respectively, as functions of x at fixed time t. In these figures we observe a left-right symmetry of the waves that is broken in domains where they are traveling to the left and domains where they are traveling to the right. In Fig. 5 the waves are emitted from a point and in Fig. 6, they are emitted from a wall. This point or wall is a source.

If we take $\Phi = 0$ in (5.21a) – (5.21), we have the fixed point

$$\left(0, \frac{\gamma^r P^i - \gamma^i P^r - \omega_u P^r}{P^i (v_{\text{coh}} - S_0)}, 0\right),$$

where v_{coh} and ω_u are such that

$$P^r P^i (v_{\text{coh}} - S_0)^2 \left(\gamma^r P^i - \gamma^i P^r - \omega_u P^r\right) - |P|^2 \left(\gamma^r P^i - \gamma^i P^r - \omega_u P^r\right)^2 -$$
$$- \left[P^i (v_{\text{coh}} - S_0)\right]^2 \left(\gamma^r P^r + \gamma^i P^i - \omega_u P^i\right) = 0.$$

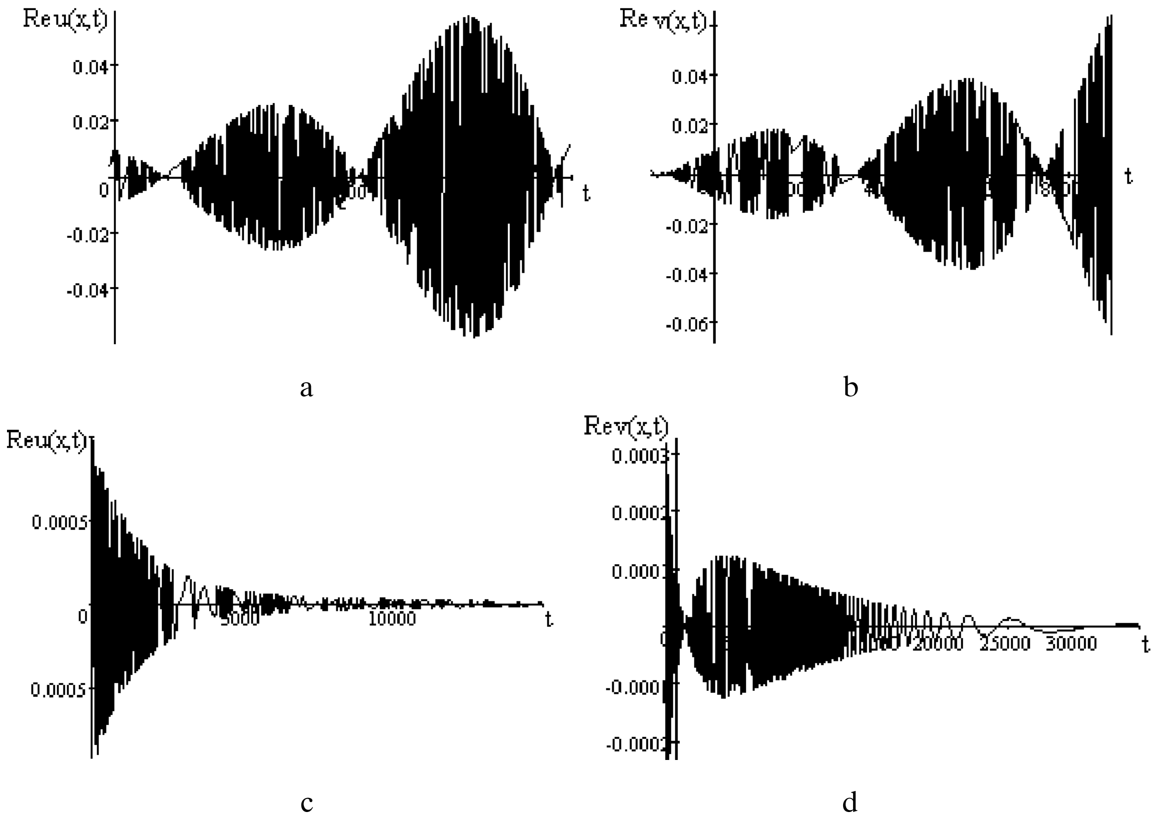


Fig. 5. Plots of (*a*) Re $u(x,t)$ and (*b*) Re $v(x,t)$ as functions of t at fixed spatial variable x. Both waves travel to the right changing form as x increases.

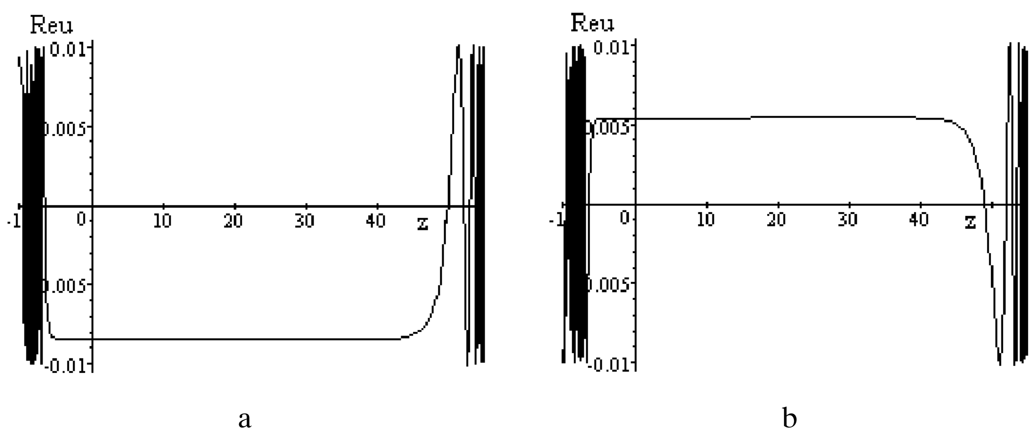


Fig. 6. Plots of Re $u(x,t)$ as a function of z, (*a*) at $t = 10^{-5}$ and (*b*) at $t = 10^{-6}$.

If these values of v_{coh} and ω_u verify the condition

$$\frac{\gamma^r P^i - \gamma^i P^r - \omega_u P^r}{(v_{\text{coh}} - S_0)} < 0,$$

then $x_0 > 0$. Reversing the inequality, we have $x_0 < 0$.

If we choose v_{coh} so that $v_{\text{coh}} = S_0$, then (5.21a) – (5.21) gives the linear fixed points

$$\left(0, \frac{S_0\left(P^i - P^r\right)\Phi_0 + \gamma^r P^i - \gamma^i P^r - \omega_u P^r}{2|P|^2\Phi_0}, \Phi_0\right),$$

where Φ_0 is any real solution of the quartic equation

$$4|P|^6\Phi^4 - \left(4|P|^4\left(\gamma^r P^r + \gamma^i P^i - \omega_u P^i\right) + |P|^2 S_0^2\left(P^i - P^r\right)^2\Phi^2\right)\Phi^2 -$$

$$-\,2|P|^2 S_0\left(P^i - P^r\right)\left(\gamma^r P^i - \gamma^i P^r - \omega_u P^r\right)\Phi - |P|^2\left(\gamma^r P^i - \gamma^i P^r - \omega_u P^r\right)^2 = 0.$$

If, for a given Φ_0, any real solution of this last equation satisfies one of the conditions

$$\frac{\gamma^r P^i - \gamma^i P^r - \omega_u P^r}{2|P|^2 + S_0\left(P^r - P^i\right)} > \Phi_0 > 0 \quad \text{or} \quad \frac{\gamma^r P^i - \gamma^i P^r - \omega_u P^r}{2|P|^2 + S_0\left(P^r - P^i\right)} < \Phi_0 < 0,$$

then $x_0 > 0$; else $x_0 < 0$.

5.1.2. Nonlinear fixed points. The analysis of the nonlinear fixed points goes along the same lines. Since the nonlinear fixed point has $x = 0$ and $a \neq 0$, the fixed point becomes

$$|P|^2\Phi^2 + \left(E^i P^r - E^r P^i\right)a^2\Phi + P^i\left(v_{\text{coh}} - S_0\right)\Phi - \left(D^r P^r + D^i P^i\right)a^4 -$$

$$-\left(Q^r P^r + Q^i P^i\right)a^2 - \gamma^r P^r - \gamma^i P^i + \omega_u P^i = 0, \tag{5.22}$$

$$\left(D^r P^i - D^i P^r\right)a^4 + \left(Q^r P^i - Q^i P^r\right)a^2 + \gamma^r P^i - \gamma^i P^r - \omega_u P^r +$$

$$+\left(v_{\text{coh}} P^i - S_0 P^r\right)\Phi = 0. \tag{5.23}$$

System (5.22), (5.23) gives

$$\Phi = \frac{\left(D^r P^i - D^i P^r\right)a^4 + \left(Q^r P^i - Q^i P^r\right)a^2 + \gamma^r P^i - \gamma^i P^r - \omega_u P^r}{S_0 P^r - v_{\text{coh}} P^i},$$

where a is any real solution of the equation of degree eight:

$$0 = |P|^2\left(\left(D^r P^i - D^i P^r\right)a^4 + \left(Q^r P^i - Q^i P^r\right)a^2 + \gamma^r P^i - \gamma^i P^r - \omega_u P^r\right)^2 +$$

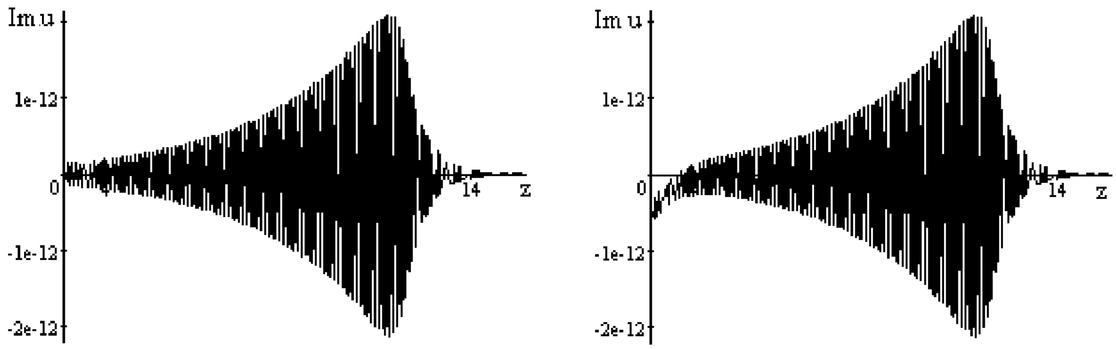
$$+\left(S_0 P^r - v_{\text{coh}} P^i\right)\left(\left(E^i P^r - E^r P^i\right)a^2 + P^i\left(v_{\text{coh}} - S_0\right)\right)\left(\left(D^r P^i - D^i P^r\right)a^4 +\right.$$

$$\left.+\left(Q^r P^i - Q^i P^r\right)a^2 + \gamma^r P^i - \gamma^i P^r - \omega_u P^r\right) - \left(S_0 P^r - v_{\text{coh}} P^i\right)^2\left(D^r P^r + D^i P^i\right)a^4 -$$

$$-\left(\left(Q^r P^r + Q^i P^i\right)a^2 + \gamma^r P^r + \gamma^i P^i - \omega_u P^i\right)\left(S_0 P^r - v_{\text{coh}} P^i\right)^2.$$

For the line parameters ($\mathrm{L_{p1}}$), we find the following $N_+(+,+,-)$:

$$(3301.8, 0, 2.7937\times 10^9) \quad \text{and} \quad (-3301.6, 0, 2.7937\times 10^9),$$



a b

Fig. 7. Plot of $\operatorname{Im} u(z,t)$ as function of z, (a) at $t = 0$ and (b) at $t = 10^{-14}$.

and corresponding to $\omega_u = 10^6$, $v_{\text{coh}} = 6 \times 10^5$, and the following $N_+(+,+,-)$:

$$(3301.6, 0, 2.7935 \times 10^9) \quad \text{and} \quad (-3301.8, 0, 2.7937 \times 10^9),$$

corresponding to $v_{\text{coh}} = S_0 = 5.1332 \times 10^5$ and $\omega_u = 10^6$. For the nonlinear fixed point $(3301.8, 0, 2.7937 \times 10^9)$ we show the evolution of (the real part of) $u(x,t) = u(z,t)$ in Fig. 7. This figure shows the dependence of u on a fixed time t. From Fig. 7, we observe the existence of a sink, a point that absorbs waves.

5.2. Coherent structures in systems described by the GCGL equations (3.10), (3.11). While the counting for the CGL equations follows unambiguously from that for the single CGL, there are various nontrivial subtleties in the extension of those results to the CGL equations that require careful discussion.

Suppose we want to perform the fixed points with $x = b = 0$, which corresponds to the case in which only a right-traveling wave is present. The fixed point equations that follow from (5.5)–(5.7) are, up to a change of $v_{\text{coh}} \to v_{\text{coh}} + S_0$, equal to the fixed point equations for the nonlinear fixed points of the single CGL Eq. (3.13) with $U = u$ and $\pm S_0 = S_0$. To solve the fixed point equations that follow from (5.4)–(5.6), note that a is a constant at the fixed point and so term $-\gamma^r P^r - \gamma^i P^i - a^2\left(Q^r P^r + Q^i P^i\right) - a^4\left(D^r P^r + D^i P^i\right)$ can be absorbed in the $P^i \omega_u$ term. Since we may choose ω_u freely, for the counting analysis we can forget about the $-\gamma^r P^r - \gamma^i P^i - a^2\left(Q^r P^r + Q^i P^i\right) - a^4\left(D^r P^r + D^i P^i\right)$ as we may think of it as having been absorbed into the frequency. The fixed point equations that follow from (5.4) and (5.6) give

$$a = \pm\sqrt{\frac{(v_{\text{coh}} - S_0)\, P^i}{E^r P^i - E^i P^r}}, \tag{5.24}$$

and

$$\Phi = \left(E^r P^i - E^i P^r\right)^{-1} \Big[\left(\gamma^r P^i - \gamma^i P^r - \omega_u P^r\right)\left(E^r P^i - E^i P^r\right)^2 + \\ + \left(v_{\text{coh}} P^i - S_0 P^i\right)\left(Q^r P^i - Q^i P^r\right)\left(E^r P^i - E^i P^r\right) + \left(v_{\text{coh}} P^i - S_0 P^i\right)^2\left(D^r P^i - D^i P^r\right)\Big] \times \\ \times \left[\left(E^r P^i - E^i P^r\right)\left(S_0 P^r - v_{\text{coh}} P^i\right)\right]^{-1} \tag{5.25}$$

and we obtain the expression for ω_u:

$$\omega_u = \frac{\gamma^r P^r + \gamma^i P^i}{P^i} + \frac{\left(v_{\text{coh}} P^i - S_0 P^i\right)\left(Q^r P^r + Q^i P^i\right)}{\left(E^r P^i - E^i P^r\right) P^i} + \frac{\left(v_{\text{coh}} P^i - S_0 P^i\right)^2 \left(D^r P^r + D^i P^i\right)}{P^i \left(E^r P^i - E^i P^r\right)^2}. \tag{5.26}$$

Here v_{coh} is a parameter and will be chosen so that $v_{\text{coh}} - S_0 < 0$, because $P^i\left(E^r P^i - E^i P^r\right) < 0$.

The fixed point equations that follow from (5.5) – (5.7) are two equations with two unknowns, y and Ψ. These two equations contain two parameters, v_{coh} and ω_v. Because a is a constant at the fixed point term $\gamma^r P^i - \gamma^i P^r + 2a^2\left(Q^r P^i - Q^i P^r\right) + \left(K^r P^i - K^i P^r\right) a^4$ can be absorbed in the $-\omega_v P^i$ term. We then choose ω_v from condition

$$\gamma^r P^i - \gamma^i P^r - \omega_v P^i + 2a^2\left(Q^r P^i - Q^i P^r\right) + \left(K^r P^i - K^i P^r\right) a^4 = 0$$

and obtain

$$\omega_v = \frac{\gamma^r P^i - \gamma^i P^r}{P^i} + \frac{2\left(v_{\text{coh}} P^i - S_0 P^i\right)\left(Q^r P^i - Q^i P^r\right)}{\left(E^r P^i - E^i P^r\right) P^i} + \frac{\left(K^r P^i - K^i P^r\right)\left(v_{\text{coh}} P^i - S_0 P^i\right)^2}{\left(E^r P^i - E^i P^r\right)^2 P^i}. \tag{5.27}$$

One of the fixed point equations that follow from (5.5) – (5.7) then becomes

$$-\left(P^i\left(S_0 + v_{\text{coh}}\right) + 2a^2\left(E^r P^i - E^i P^r\right)\right) y + \left(v_{\text{coh}} P^i + S_0 P^r - 2|P|^2 y\right)\Psi = 0.$$

Solving this last equation in Ψ, we obtain

$$\Psi = \frac{\left[P^i\left(S_0 + v_{\text{coh}}\right) + 2a^2\left(E^r P^i - E^i P^r\right)\right] y}{v_{\text{coh}} P^i + S_0 P^r - 2|P|^2 y}. \tag{5.28}$$

If we replace this expression for Ψ in the second equation of the fixed points equations, we obtain for determination of y the following fourth degree equation:

$$\begin{aligned}
&- |P|^2\left(v_{\text{coh}} P^i + S_0 P^r - 2|P|^2 y\right)^2 y^2 + |P|^2\left[P^i\left(S_0 + v_{\text{coh}}\right) + 2a^2\left(E^r P^i - E^i P^r\right)\right]^2 y^2 + \\
&+ \left(v_{\text{coh}} P^i + S_0 P^r - 2|P|^2 y\right)^2\left[P^r\left(v_{\text{coh}} + S_0\right)\right] y + \left(v_{\text{coh}} P^i + S_0 P^r - 2|P|^2 y\right) \times \\
&\quad \times \left(2a^2\left(E^r P^i - E^i P^r\right) + v_{\text{coh}} P^i + S_0 P^i\right)\left[P^i\left(S_0 + v_{\text{coh}}\right) + 2a^2\left(E^r P^i - E^i P^r\right)\right] y + \\
&+ \left\{a^4\left(K^r P^r + K^i P^i\right) - \left[\gamma^r P^r + \gamma^i P^i - \omega_v P^i - 2a^2\left(Q^r P^r + Q^i P^i\right)\right]\right\} \times \\
&\quad \times \left(v_{\text{coh}} P^i + S_0 P^r - 2|P|^2 y\right)^2 = 0.
\end{aligned} \tag{5.29}$$

Since the fixed points of interest in for sources and sinks always have either $b = 0$ or $a = 0$, the linearization around them largely parallels the analysis of the single GCGL equation. For, when we linearize about the fixed point $b = 0$, we do not have to take the variation of a into account in the coupling term and this allows us, for the counting argument, to absorb these terms into a frequency and redefined ω as discussed above. Once this is done, the linear

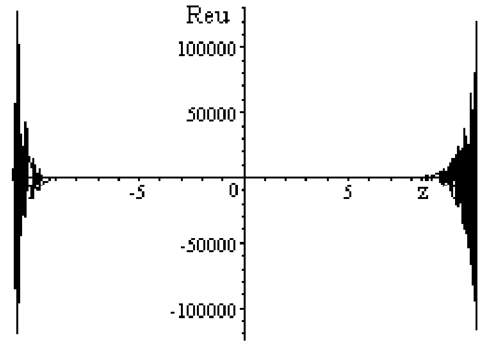


Fig. 8. Plot of $\mathrm{Re}\, u(z,t)$ as a function of z at fixed time t. One wave travels to the right and the other to the left.

equations for the mode whose amplitude vanishes at the fixed point do not involve the other mode variables at all.

For the line parameters (L_{p1}) and for $v_{coh} = 6 \times 10^5$, equations (5.24)–(5.29) give

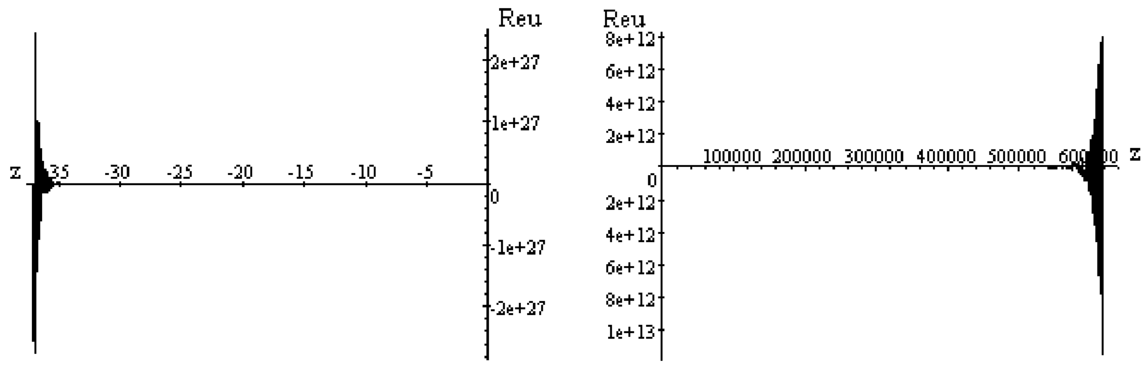
$$\omega_u = \left(-8.681 \times 10^{16}\right), \qquad \omega_v = \left(-2.5841 \times 10^{14}\right),$$
$$a = \pm 328.83, \qquad \Phi = 19.842,$$
$$y = -1.5382 \times 10^8, \qquad \Psi = 1.7486 \times 10^5,$$
$$\text{or} \quad y = 8.879 \times 10^8, \qquad \Psi = 4.881 \times 10^5,$$

whence the following fixed points of system (5.2)–(5.7) for (a, x, Φ, b, y, Ψ):

$$p_{\pm}(\pm 328.83, 0, 19.842, 0, -1.5382 \times 10^8, 1.7486 \times 10^5),$$
$$q_{\pm}(\pm 328.83, 0, 19.842, 0, 8.879 \times 10^8, 4.881 \times 10^5).$$

After some investigation we find that p_+ is an $(N_+(-,-,-), L_-(+,+,-))$ fixed point, that is, the matrix of the flow equations corresponding to this fixed point possesses four eigenvalues with negative real parts and two eigenvalues with positive real parts. For this fixed point we show the evolution of (the real parts of) $u(z,t)$ and $v(z,t)$ in Figs. 8 and 9, respectively. Figure 8 show the dependence of (the real part of) u on z (or, what is the same, on x) for a given time t. In this figure, we notice the existence of a source (see Figs. 9(a) and 9(b)). Figure 10 shows the dependence of v on z (that is, on x) for a given time t. In this figure, one can see a wave traveling form $-\infty$ to the right, and vanishing somewhere before $z = -26.5$ (see Fig. 10(a)). From $z = -25.5$ to -3.6, there is not any movement (see Fig. 10(b)). There exists a source at some point after $z = -2.2$ and before $z = 2000$ (see Fig. 10(c, d)). The wave that propagates from this source to the left vanish before $z = -3.6$, while the wave that travels from the source to the right vanishes before $z = 6200$ (see Fig. 10(e)).

System (5.2)–(5.7) has the invariant planes $a = 0$ and $b = 0$. Thus, by the σ-process, a singular point is transformed to a (singular) invariant plane. On the invariant planes $a = 0$ and



a b

Fig. 9. Plot of $\operatorname{Re} u(z,t)$ as a function of z at fixed time t; (a) wave traveling to the right and (b) wave traveling to the left.

$b = 0$, the system for (a, x, Φ, b, y, Ψ) reduces to

$$x' = -x^2 + \Phi^2 + \frac{P^r(v_{\text{coh}} - S_0)}{|P|^2}\,x - \left(v_{\text{coh}}\omega_u \frac{P^r}{|P|^2} + \frac{\gamma^r P^r + \gamma^i P^i}{|P|^2}\right) + \left(\frac{v_{\text{coh}}P^i - S_0 P^i}{|P|^2}\right)\Phi, \tag{5.4b}$$

$$y' = -y^2 + \Psi^2 + \frac{P^r(v_{\text{coh}} + S_0)}{|P|^2}\,y + \left(v_{\text{coh}}\omega_v \frac{P^r}{|P|^2} + \frac{\gamma^r P^r + \gamma^i P^i}{|P|^2}\right) + \left(\frac{v_{\text{coh}}P^i + S_0 P^i}{|P|^2}\right)\Psi, \tag{5.5b}$$

$$\Phi' = -2\Phi x + \frac{P^i(S_0 - v_{\text{coh}})}{|P|^2}\,x + \left(v_{\text{coh}}\omega_u \frac{P^i}{|P|^2} - \frac{\gamma^i P^r - \gamma^r P^i}{|P|^2}\right) + \left(\frac{v_{\text{coh}}P^i - S_0 P^r}{|P|^2}\right)\Phi, \tag{5.6b}$$

$$\Psi' = -2\Psi y - \frac{P^i(S_0 + v_{\text{coh}})}{|P|^2}\,y + \left(v_{\text{coh}}\omega_v \frac{P^i}{|P|^2} - \frac{\gamma^i P^r - \gamma^r P^i}{|P|^2}\right) + \left(\frac{v_{\text{coh}}P^i + S_0 P^r}{|P|^2}\right)\Psi. \tag{5.7b}$$

Because (5.4b) and (5.6b) depend only on x and Φ, and (5.5b) and (5.7b) depend only on y and Ψ, we obtain the following two systems of ODEs:

$$x' = -x^2 + \Phi^2 + \frac{P^r(v_{\text{coh}} - S_0)}{|P|^2}\,x - \left(v_{\text{coh}}\omega_u \frac{P^r}{|P|^2} + \frac{\gamma^r P^r + \gamma^i P^i}{|P|^2}\right) + \left(\frac{v_{\text{coh}}P^i - S_0 P^i}{|P|^2}\right)\Phi, \tag{5.8a}$$

$$\Phi' = -2\Phi x + \frac{P^i(S_0 - v_{\text{coh}})}{|P|^2}\,x + \left(v_{\text{coh}}\omega_u \frac{P^i}{|P|^2} - \frac{\gamma^i P^r - \gamma^r P^i}{|P|^2}\right) + \left(\frac{v_{\text{coh}}P^i - S_0 P^r}{|P|^2}\right)\Phi \tag{5.9a}$$

and

$$y' = -y^2 + \Psi^2 + \frac{P^r(v_{\text{coh}} + S_0)}{|P|^2}\,y + \left(v_{\text{coh}}\omega_v \frac{P^r}{|P|^2} + \frac{\gamma^r P^r + \gamma^i P^i}{|P|^2}\right) + \left(\frac{v_{\text{coh}}P^i + S_0 P^i}{|P|^2}\right)\Psi, \tag{5.10a}$$

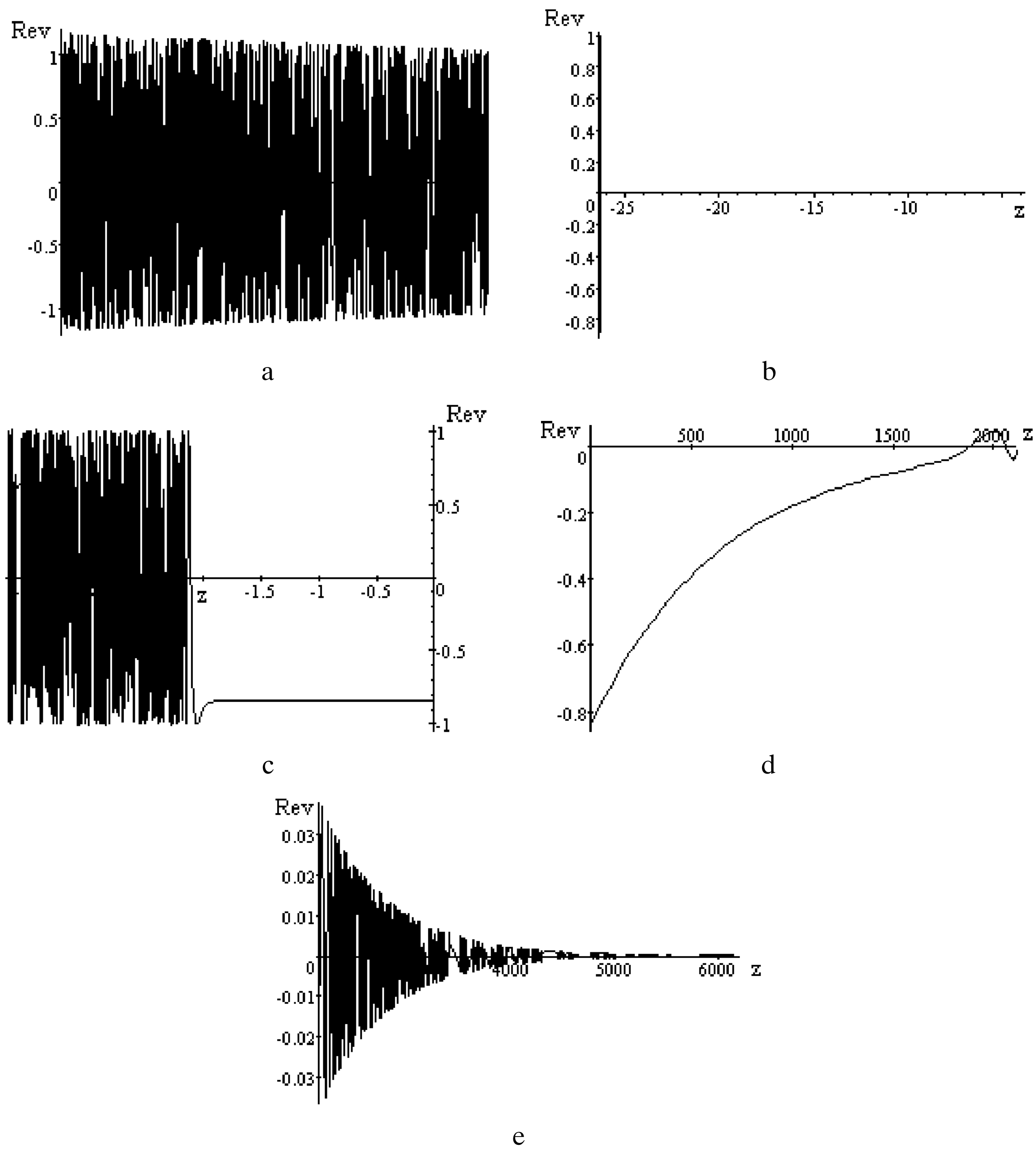


Fig. 10. Plot of $\mathrm{Re}\, v(z,t)$ as a function of z at fixed time t. (a) $z \in [-\infty, -26.33]$, (b) $z \in [-26.33, -3.71]$, (c) $z \in [-3.71, 0]$, (d) $z \in [0,\ 2120]$, (e) $z \in [2120, 6200]$.

$$\Psi' = -2\Psi y - \frac{P^i\,(S_0 + v_{\text{coh}})}{|P|^2}\, y + \left(v_{\text{coh}}\omega_v \,\frac{P^i}{|P|^2} - \frac{\gamma^i P^r - \gamma^r P^i}{|P|^2} \right) + \left(\frac{v_{\text{coh}} P^i + S_0 P^r}{|P|^2} \right) \Psi. \tag{5.11a}$$

Note that, if (x_0, Φ_0) and (y_0, Ψ_0) are fixed points of systems (5.8a), (5.9a) and (5.10a), (5.11a), respectively, then $(0, x_0, \Phi_0, b_0, y_0, \Psi_0)$ is a fixed point of system (5.2)–(5.7). Because we wish to study the sources and sinks of (3.10) and (3.11), the fixed points of interest to us have either x or y equal to zero.

If we consider the six variables $\widetilde{a}$, $\widetilde{x}$, $\widetilde{\Phi}$, $\widetilde{b}$, $\widetilde{y}$, and $\widetilde{\Psi}$ as the elements of a vector w and linearize the flow Eqs. (5.2)–(5.7) about the fixed point $(0, x_0, \Phi_0, 0, y_0, \Psi_0)$, we can write the linearized equations in the form $w_i' = \sum_j M_{ij} w_j$, where the 6×6 matrix M has the following structure:

$$M = \begin{pmatrix} x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{matrix} -2x_0+ \\ +\dfrac{P^r(v_{\mathrm{coh}} - S_0)}{|P|^2} \end{matrix} & \begin{matrix} 2\Phi_0+ \\ +\dfrac{v_{\mathrm{coh}}P^i - S_0P^i}{|P|^2} \end{matrix} & 0 & 0 & 0 \\ 0 & \begin{matrix} -2\Phi_0+ \\ +\dfrac{P^i(S_0 - v_{\mathrm{coh}})}{|P|^2} \end{matrix} & \begin{matrix} -2x_0+ \\ +\dfrac{v_{\mathrm{coh}}P^i - S_0P^r}{|P|^2} \end{matrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & y_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{matrix} -2y_0+ \\ +\dfrac{P^r(v_{\mathrm{coh}} + S_0)}{|P|^2} \end{matrix} & \begin{matrix} 2\Psi_0+ \\ +\dfrac{v_{\mathrm{coh}}P^i + S_0P^i}{|P|^2} \end{matrix} \\ 0 & 0 & 0 & 0 & \begin{matrix} -2\Psi_0- \\ -\dfrac{P^i(S_0 + v_{\mathrm{coh}})}{|P|^2} \end{matrix} & \begin{matrix} -2y_0+ \\ +\dfrac{v_{\mathrm{coh}}P^i + S_0P^r}{|P|^2} \end{matrix} \end{pmatrix}.$$

The eigenvalue of M are simply given by the eigenvalues of the upper-left and lower-right block matrices. The secular equation for the eigenvalues of each of the two blocks has the form

$$P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0 = 0.$$

We need only know the number of solutions of the secular equation that have positive real part, and instead of solving the equation explicitly, we can proceed as follows.

For the cubic equation of the above form where $P_3 > 0$, we may read off the signs of the real parts of the solution to this equation from the following [22, 23]:

$$P_0 > 0 \quad \begin{cases} P_2 > 0, \quad P_1P_2 > P_0P_3: & (-,-,-) \quad \text{case (i)}, \\ \text{else}: & (+,+,-) \quad \text{case (ii)}, \end{cases}$$

$$P_0 < 0 \quad \begin{cases} P_2 < 0, \quad P_1P_2 < P_0P_3: & (+,+,+) \quad \text{case (iii)}, \\ \text{else}: & (+,-,-) \quad \text{case (iv)}. \end{cases}$$

According to these rules, there are six combinations of the coefficients whose sign we need to know, namely, P_0, P_2, $P_1P_2 - P_0P_3$ for each of the two blocks (see (5.30)–(5.35)).

For the upper-left block matrices, we have

$$\lambda^3 + \left(3x_0 - \frac{(P^i + P^r)\,v_{\mathrm{coh}} - 2S_0P^r}{|P|^2}\right)\lambda^2 + \left(4\Phi_0^2 - \right.$$

$$-\frac{\left(P^i\right)^2\left(v_{\text{coh}}-S_0\right)^2-2x_0P^r\left(v_{\text{coh}}-S_0\right)\left(v_{\text{coh}}P^i-S_0P^r\right)+x_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r\right]}{|P|^4}\Bigg)\lambda+$$

$$+x_0\left(\frac{\left(P^i\right)^2\left(v_{\text{coh}}-S_0\right)^2-2x_0P^r\left(v_{\text{coh}}-S_0\right)\left(v_{\text{coh}}P^i-S_0P^r\right)+2x_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r\right]}{|P|^4}-\right.$$

$$\left.-4\Phi_0^2-4x_0^2\right)=0,$$

and

$$P_0=x_0\times$$

$$\times\left(\frac{\left(P^i\right)^2\left(v_{\text{coh}}-S_0\right)^2-2x_0P^r\left(v_{\text{coh}}-S_0\right)\left(v_{\text{coh}}P^i-S_0P^r\right)+2x_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r\right]}{|P|^4}-\right.$$

$$\left.-4\Phi_0^2-4x_0^2\right), \tag{5.30}$$

$$P_2=3x_0-\frac{\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r}{|P|^2}, \tag{5.31}$$

$$P_1P_2-P_0P_3=\Big(\left(P^i\right)^2\left(v_{\text{coh}}-S_0\right)^2-2x_0P^r\left(v_{\text{coh}}-S_0\right)\left(v_{\text{coh}}P^i-S_0P^r\right)+$$

$$+x_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r\right]\Big)\frac{\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r}{|P|^6}-$$

$$-\frac{4\Phi_0^2\left[\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r\right]}{|P|^2}+4x_0^3+16x_0\Phi_0^2-x_0\times$$

$$\times\frac{3\left(P^i\right)^2\left(v_{\text{coh}}-S_0\right)^2-8x_0P^r\left(v_{\text{coh}}-S_0\right)\left(v_{\text{coh}}P^i-S_0P^r\right)+5x_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}-2S_0P^r\right]}{|P|^4}. \tag{5.32}$$

For the lower-right block, we have

$$\lambda^3+\left(3y_0-\frac{\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r}{|P|^2}\right)\lambda^2-$$

$$-\left(\frac{\left(P^i\right)^2\left(v_{\text{coh}}+S_0\right)^2-2x_0P^r\left(v_{\text{coh}}+S_0\right)\left(v_{\text{coh}}P^i+S_0P^r\right)+y_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r\right]}{|P|^4}-\right.$$

$$\left.-4\Psi_0^2\right)\lambda+y_0\times$$

$$\times\left(\frac{\left(P^i\right)^2\left(v_{\text{coh}}+S_0\right)^2-2y_0P^r\left(v_{\text{coh}}+S_0\right)\left(v_{\text{coh}}P^i+S_0P^r\right)+2y_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r\right]}{|P|^4}-\right.$$

$$\left.-4\Psi_0^2-4y_0^2\right)=0$$

and

$$P_0 = y_0\times$$
$$\times\left(\frac{\left(P^i\right)^2\left(v_{\text{coh}}+S_0\right)^2-2y_0P^r\left(v_{\text{coh}}+S_0\right)\left(v_{\text{coh}}P^i+S_0P^r\right)+2y_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r\right]}{|P|^4}-\right.$$
$$\left.-4\Psi_0^2-4y_0^2\right), \tag{5.33}$$

$$P_2 = 3y_0-\frac{\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r}{|P|^2}, \tag{5.34}$$

$$P_1P_2-P_0P_3 = \left(\left(P^i\right)^2\left(v_{\text{coh}}+S_0\right)^2-2y_0P^r\left(v_{\text{coh}}+S_0\right)\left(v_{\text{coh}}P^i+S_0P^r\right)+\right.$$
$$\left.+x_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r\right]\right)\frac{\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r}{|P|^6}-\frac{4\Psi_0^2\left[\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r\right]}{|P|^2}+$$
$$+16y_0\Psi_0^2+4y_0^3-y_0\times$$
$$\times\frac{3\left(P^i\right)^2\left(v_{\text{coh}}+S_0\right)^2-8y_0P^r\left(v_{\text{coh}}+S_0\right)\left(v_{\text{coh}}P^i+S_0P^r\right)+5y_0|P|^2\left[\left(P^i+P^r\right)v_{\text{coh}}+2S_0P^r\right]}{|P|^4}. \tag{5.35}$$

It is seen from (5.30) and (5.33) that for P_0 to be different from zero it is necessary that $x_0\neq 0$ and $y_0\neq 0$. Let

$$F(\omega,\beta) = |P|^2\left(\gamma^iP^r-\gamma^rP^i-v_{\text{coh}}\omega P^i\right)^2+$$
$$+v_{\text{coh}}\left(v_{\text{coh}}P^i+(-1)^\beta S_0P^r\right)\left(P^i+(-1)^\beta S_0P^i\right)\left(\gamma^iP^r-\gamma^rP^i-v_{\text{coh}}\omega P^i\right)-$$
$$-\left(v_{\text{coh}}P^i+(-1)^\beta S_0P^r\right)^2\left(v_{\text{coh}}\omega P^r+\gamma^rP^r+\gamma^iP^i\right) = 0.$$

For $x = 0$, system (5.8a), (5.9a) admits the fixed point

$$\left(0,\frac{\gamma^iP^r-\gamma^rP^i-v_{\text{coh}}\omega_uP^i}{v_{\text{coh}}P^i-S_0P^r}\right),$$

if v_{coh} and ω_u verify the condition

$$F(\omega_u,1) = 0. \tag{5.12a}$$

For $y = 0$, system (5.10a), (5.11a) admits the fixed point

$$\left(0,\frac{\gamma^iP^r-\gamma^rP^i-v_{\text{coh}}\omega_vP^i}{\left(v_{\text{coh}}P^i+S_0P^r\right)}\right),$$

if $v_{\rm coh}$ and ω_v verify the condition

$$F(\omega_v, 2) = 0. \tag{5.13a}$$

If conditions (5.12a) and (5.13a) are satisfied, then

$$\left(0,0,0,0,\frac{\gamma^i P^r - (\gamma^r + v_{\rm coh}\omega_u)\,P^i}{v_{\rm coh}P^i - S_0P^r}, \frac{\gamma^i P^r - (\gamma^r + v_{\rm coh}\omega_v)\,P^i}{v_{\rm coh}P^i + S_0P^r}\right) \tag{5.36}$$

is a fixed point of system (5.2)–(5.7).

Now let

$$G(\omega,\beta) = -|P|^2\left(v_{\rm coh}\omega_u P^i - \gamma^i P^r + \gamma^r P^i\right)^2 + \\ + P^iP^r\left(v_{\rm coh} + (-1)^\beta S_0\right)^2\left(v_{\rm coh}\omega_u P^i - \gamma^i P^r + \gamma^r P^i\right) - \\ - P^{i2}\left(v_{\rm coh} + (-1)^\beta S_0\right)^2\left(v_{\rm coh}\omega_u P^r + \gamma^r P^r + \gamma^i P^i\right).$$

If $v_{\rm coh}$ and ω_u are such that

$$G(\omega_u, 1) = 0, \tag{5.14a}$$

then

$$\left(\frac{(v_{\rm coh}\omega_u + \gamma^r)\,P^i - \gamma^i P^r}{P^i\,(v_{\rm coh} - S_0)}, 0\right)$$

is a fixed point of system (5.8a), (5.9a), and if $v_{\rm coh}$ and ω_v are such that

$$G(\omega_v, 2) = 0, \tag{5.15a}$$

then

$$\left(\frac{(v_{\rm coh}\omega_v + \gamma^r)\,P^i - \gamma^i P^r}{P^i\,(S_0 + v_{\rm coh})}, 0\right)$$

is a fixed point of system (5.10a), (5.11a). Thus if conditions (5.14a) and (5.13a) or (5.15a) and (5.12a) are satisfied, then

$$\left(0,0,\frac{v_{\rm coh}\omega_u P^i - \gamma^i P^r + \gamma^r P^i}{P^i\,(v_{\rm coh} - S_0)}, 0, 0, \frac{\gamma^i P^r - \gamma^r P^i - v_{\rm coh}\omega_v P^i}{v_{\rm coh}P^i + S_0P^r}\right) \tag{5.37}$$

or

$$\left(0,0,0,\frac{v_{\rm coh}\omega_v P^i - \gamma^i P^r + \gamma^r P^i}{P^i\,(S_0 + v_{\rm coh})}, \frac{\gamma^i P^r - \gamma^r P^i - v_{\rm coh}\omega_u P^i}{v_{\rm coh}P^i - S_0P^r}, 0\right) \tag{5.38}$$

is a fixed point of system (5.2)–(5.7).

If

$$v_{\rm coh} = -S_0 \quad \text{and} \quad \omega_v = \frac{\gamma^r P^i - \gamma^i P^r}{S_0 P^i},$$

then

$$\left(\frac{S_0 \left(P^r - P^i\right)}{2|P|^2}, \frac{\pm 1}{2|P|^2} \sqrt{S_0^2 \left(P^r - P^i\right)^2 - \frac{4|P|^4 \gamma^i}{P^i}} \right)$$

are fixed points of system (5.10a), (5.11a). Moreover, if v_{coh} and ω_u verify (5.12a) (with $v_{\text{coh}} = -S_0$), then

$$\left(0, 0, 0, \frac{S_0 \left(P^r - P^i\right)}{2|P|^2}, \frac{\left(\gamma^r - S_0 \omega_u\right) P^i - \gamma^i P^r}{S_0 \left(P^i + P^r\right)}, \frac{\pm 1}{2|P|^2} \sqrt{S_0^2 \left(P^r - P^i\right)^2 - \frac{4|P|^4 \gamma^i}{P^i}} \right) \qquad (5.39)$$

are fixed points of system (5.2) – (5.7). We should note that the quantity $S_0^2 \left(P^r - P^i\right)^2 - \dfrac{4|P|^4 \gamma^i}{P^i}$ is always positive, because $P^i \gamma^i < 0$ (see Appendix A).

If

$$v_{\text{coh}} = S_0 \quad \text{and} \quad \omega_u = \frac{\gamma^i P^r - \gamma^r P^i}{S_0 P^i},$$

then

$$\left(\frac{S_0 \left(P^i - P^r\right)}{2|P|^2}, \frac{\pm 1}{2|P|^2} \sqrt{-\frac{P^i S_0^2 \left(P^i - P^r\right)^2 + 4|P|^4 \gamma^i}{P^i}} \right)$$

are fixed points of system (5.8a), (5.9a), if the line parameters and the wavenumber k in coefficients of the GCGL Eqs. (3.10), (3.11) satisfy the inequality

$$k^4 \left(k^2 + RG\right) < 4 \frac{C_0^4 \left(R^2 G^2 + 4 C_0^2 R^2 G^2\right)^4}{\left(1 + 4 C_0^2\right)^4 L^4} \frac{\left(GL + RC_0\right)^2 C_0 L}{C_0^4 R^2 G^2 \left(RG + 2 C_0 RG\right)^4}.$$

Moreover, if condition (5.13a) is satisfied with $v_{\text{coh}} = S_0$, then

$$\left(0, 0, \frac{S_0 \left(P^i - P^r\right)}{2|P|^2}, 0, \frac{\pm 1}{2|P|^2} \sqrt{-\frac{P^i S_0^2 \left(P^i - P^r\right)^2 + 4|P|^4 \gamma^i}{P^i}}, \frac{\gamma^i P^r - \left(\gamma^r + S_0 \omega_v\right) P^i}{S_0 \left(P^i + P^r\right)} \right) \qquad (5.40)$$

are fixed points of system (5.2) – (5.7).

Thus, we have obtained the five particular fixed points, (5.36) – (5.40). We now study the stability of these fixed points. Linearizing (5.2) – (5.7) at the obtained fixed points we obtain the following five sets of eigenvalues. To save space and make reading easier, we let $\lambda_j^{[+]} \equiv \lambda_j$ and $\lambda_{j+1}^{[-]} \equiv \lambda_{j+1}$ be the eigenvale with the "+" sign and "−" sign, respecively for $j = 3$ and 5, in the following formulae.

1. For (5.36), $x_0 = y_0 = 0$ and we have

$$\lambda_1 = \lambda_2 = 0,$$

$$\lambda_3^{[+]}, \lambda_4^{[-]} = \frac{1}{2|P|^2} \Big\{ -2 S_0 P^r \left(P^r + P^i\right) v_{\text{coh}} \pm \big[16 |P|^4 \Phi_0^2 + v_{\text{coh}}^2 P^{r2} +$$

$$+\left(8S_0v_{\rm coh}-3v_{\rm coh}^2-6S_0^2\right)P^{i2}-2v_{\rm coh}^2P^rP^i\Big]^{1/2}\Big\},$$

$$\lambda_5^{[+]},\lambda_6^{[-]}=\frac{1}{2|P|^2}\Big\{2S_0P^r+\left(P^r+P^i\right)v_{\rm coh}\pm\Big[v_{\rm coh}^2P^{r2}-2v_{\rm coh}^2P^rP^i-$$

$$-\left(4S_0^2+3v_{\rm coh}^2+8S_0v_{\rm coh}\right)P^{i2}-16|P|^4\Psi_0^2-16P^i|P|^2\left(S_0+v_{\rm coh}\right)\Psi_0\Big]^{1/2}\Big\}.$$

2. For (5.37), $y_0=\Phi_0=0$ and we find

$$\lambda_1=\frac{v_{\rm coh}\omega_uP^i-\gamma^iP^r+\gamma^rP^i}{P^i\left(v_{\rm coh}-S_0\right)},\qquad \lambda_2=0,$$

$$\lambda_3^{[+]},\lambda_4^{[-]}=\frac{1}{2|P|^2}\Big\{-4x_0|P|^2-2S_0P^r+\left(P^r+P^i\right)v_{\rm coh}\pm$$

$$\pm\Big[v_{\rm coh}^2P^{r2}-\left(3v_{\rm coh}^2+4S_0^2-8S_0v_{\rm coh}\right)P^{i2}-2v_{\rm coh}^2P^iP^r\Big]^{1/2}\Big\},$$

$\lambda_5^{[+]},\lambda_6^{[-]}$ same as $\lambda_5^{[+]},\lambda_6^{[-]}$ in case 1.

3. For (5.38), $x_0=\Psi_0=0$ and we obtain

$$\lambda_1=0,\qquad \lambda_2=\frac{v_{\rm coh}\omega_vP^i-\gamma^iP^r+\gamma^rP^i}{P^i\left(S_0+v_{\rm coh}\right)},$$

$\lambda_3^{[+]},\lambda_4^{[-]}$ same as $\lambda_3^{[+]},\lambda_4^{[-]}$ in case 1,

$$\lambda_5^{[+]},\lambda_6^{[-]}=\frac{1}{2|P|^2}\Big\{-4x_0|P|^2-2S_0P^r+\left(P^r+P^i\right)v_{\rm coh}\pm$$

$$\pm\Big[v_{\rm coh}^2P^{r2}-\left(3v_{\rm coh}^2+4S_0^2+8S_0v_{\rm coh}\right)P^{i2}-2v_{\rm coh}^2P^iP^r\Big]^{1/2}\Big\}.$$

4. For (5.39), $x_0=0$, $v_{\rm coh}=-S_0$, and $\omega_v=(\gamma^rP^i-\gamma^iP^r)/(S_0P^i)$ and we obtain

$$\lambda_1=0,\qquad \lambda_2=\frac{S_0\left(P^r-P^i\right)}{2|P|^2},$$

$$\lambda_3^{[+]},\lambda_4^{[-]}=\frac{1}{2|P|^2S_0}\left\{-S_0^2\left(3P^r+P^i\right)\pm\right.$$

$$\left.\pm\left[16\,|P|^4\left(\frac{\gamma^iP^r-\gamma^rP^i+S_0\omega_uP^i}{P^i+P^r}\right)^2+S_0^4\left(P^{r2}-2P^rP^i-15P^{i2}\right)\right]^{1/2}\right\},$$

$$\lambda_5^{[+]},\lambda_6^{[-]}=\frac{1}{2|P|^2\sqrt{-P^i}}\left\{\left[S_0\left(P^r-P^i\right)-4|P|^2y_0\right]\sqrt{-P^i}\pm\left[3P^iS_0^2\left(P^r-P^i\right)^2-64|P|^8\gamma^i\right]^{1/2}\right\}.$$

5. For (5.40), $y_0 = 0, v_{\rm coh} = S_0,$ and $\omega_u = (\gamma^i P^r - \gamma^r P^i)/(S_0 P^i)$ we have

$$\lambda_1 = \frac{S_0\left(P^i - P^r\right)}{2|P|^2}, \qquad \lambda_2 = 0,$$

$$\lambda_3^{[+]}, \lambda_4^{[-]} = \frac{1}{2|P|^2 P^i}\Big\{-P^i\left[S_0\left(P^i - P^r\right) - 4x_0|P|^2\right] \pm$$

$$\pm\left[4P^i S_0^2\left(P^i - P^r\right)^2 + 16|P|^4\gamma^i - P^i S_0^2\left(P^i - P^r\right)^2\right]^{1/2}\Big\},$$

$$\lambda_5^{[+]}, \lambda_6^{[-]} = \frac{\left(3P^r + P^i\right)S_0}{2|P|^2} \pm \frac{1}{2S_0|P|^2|P^i + P^r|}\Big[S_0^4\left(P^{r2} - 2P^rP^i - 15P^{i2}\right)\left(P^i + P^r\right)^2 - 16|P|^4 \times$$

$$\times\left(\gamma^i P^r - \gamma^r P^i - S_0\omega_v P^i\right)^2 - 32S_0^2 P^i|P|^2\left(\gamma^i P^r - \gamma^r P^i - S_0\omega_v P^i\right)\left(P^i + P^r\right)\Big]^{1/2}.$$

6. Conclusion. The dynamics of modulated wave trains in a distributed nonlinear mono-inductance electrical line was analyzed in this paper. The GGL system of complex modulation equations was derived. From this system we have derived the cubic-quintic and the GGL equations. The coefficients of these equations are analytically given in terms of the line parameters. The stability properties of the GCGL equations were studied for a class of phase winding solutions. The coherent structures have been studied for the GGL equation and system. While, in general, it is not clear how to go from the tools developed for low-dimensional dynamic systems to an effective description of systems with many degrees of freedom, the coherent structure framework sketched in this paper may be such a bridge in the case of CGL. Possibly the greatest advantage of studying 1D systems is that their time evolution can be captured in two-dimensional space-time plots, which may help one understand these systems. Without such plots, the discovery of many dynamical properties would have been much more difficult.

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7. Appendix. ***7.1. Appendix A.*** We use the dispersion relation (3.2)

$$\omega = \sqrt{\frac{k^2 + RG}{C_0 L}}, \qquad \frac{\partial\omega}{\partial k} = \frac{k}{\sqrt{C_0 L\left(k^2 + RG\right)}}$$

and (3.12) to compute the coefficients of system (3.10), (3.11):

$$P = P^r + iP^i = \frac{RG}{\left(1 + 4C_0^2\right)L}\sqrt{\frac{C_0 L}{\left(k^2 + RG\right)^3}} - i\frac{2C_0 RG}{\left(1 + 4C_0^2\right)L}\sqrt{\frac{C_0 L}{\left(k^2 + RG\right)^3}},$$

$$\gamma = \gamma^r + i\gamma^i = \frac{2C_0\,(GL + RC_0)}{L\left(1 + 4C_0^2\right)} + i\frac{GL + RC_0}{L(1 + 4C_0^2)},$$

$$Q = Q^r + iQ^i = \frac{3bR}{2L} - i\frac{3b}{2}\sqrt{\frac{k^2 + RG}{C_0L}},$$

$$E = E^r + iE^i = -\frac{6C_0^2bk}{\left(1 + 4C_0^2\right)\sqrt{C_0L\,(k^2 + RG)}} - i\frac{3C_0bk}{\left(1 + 4C_0^2\right)\sqrt{C_0L\,(k^2 + RG)}},$$

$$D = D^r + iD^i = \frac{3C_0b}{1+4C_0^2}\left[\frac{9bR^2}{2L^2}\sqrt{\frac{C_0L}{k^2+RG}} - 3b\sqrt{\frac{k^2+RG}{C_0L}} - \frac{LC_0b}{2RG}\left(3\sqrt{\left(\frac{(k^2+RG)}{C_0L}\right)^3} +\right.\right.$$

$$\left.\left. + \frac{R}{L}\frac{k^2 + RG}{C_0L}\right) - \frac{21bRC_0}{L} - \frac{bC_0}{GL}\left(3\left(k^2 + RG\right) + \frac{R}{L}\sqrt{C_0L\,(k^2 + RG)}\right)\right] +$$

$$+ i\frac{3C_0b}{1 + 4C_0^2}\left[-\frac{9bC_0R^2}{L^2}\sqrt{\frac{C_0L}{k^2 + RG}} + 6C_0b\sqrt{\frac{k^2 + RG}{C_0L}} + \frac{LC_0^2b}{RG}\left(3\sqrt{\left(\frac{(k^2 + RG)}{C_0L}\right)^3} +\right.\right.$$

$$\left.\left. + \frac{R}{L}\frac{k^2 + RG}{C_0L}\right) - \frac{21bR}{2L} - \frac{b}{2GL}\left(3\left(k^2 + RG\right) + \frac{R}{L}\sqrt{C_0L\,(k^2 + RG)}\right)\right],$$

$$H = H^r + iH^i = \frac{\left(36C_0^2b - \frac{6C_0bR}{L}\sqrt{\frac{C_0L}{k^2+RG}}\right)k}{\left(1 + 4C_0^2\right)\sqrt{C_0L\,(k^2 + RG)}} + i\frac{\left(\frac{12C_0bR}{L}\sqrt{\frac{C_0L}{k^2+RG}} + 18C_0b\right)k}{\left(1 + 4C_0^2\right)\sqrt{C_0L\,(k^2 + RG)}},$$

$$F = F^r + F^i = \frac{3C_0b^2}{1 + 4C_0^2}\left[\frac{24R^2}{L^2}\sqrt{\frac{C_0L}{k^2 + RG}} - 12\sqrt{\frac{k^2 + RG}{C_0L}} - \frac{108bC_0R}{L}\right] -$$

$$- i\frac{3C_0b^2}{1 + 4C_0^2}\left[\frac{48C_0R^2}{L^2}\sqrt{\frac{C_0L}{k^2 + RG}} - 24C_0\sqrt{\frac{k^2 + RG}{C_0L}} + \frac{54R}{L}\right],$$

$$K = K^r + iK^i = \frac{3C_0b^2}{1 + 4C_0^2}\left[\frac{12R^2}{L^2}\sqrt{\frac{C_0L}{k^2 + RG}} - 12\sqrt{\frac{k^2 + RG}{C_0L}} - \frac{60C_0R}{L}\right] -$$

$$- i\frac{3C_0b^2}{1 + 4C_0^2}\left[\frac{12C_0R^2}{L^2}\sqrt{\frac{C_0L}{k^2 + RG}} - 24C_0\sqrt{\frac{k^2 + RG}{C_0L}} + \frac{30bR}{L}\right],$$

$$S_0 = \frac{\partial\omega}{\partial k} = \frac{k}{\sqrt{C_0L\,(k^2 + RG)}}.$$

7.2. Appendix B (Analytical expressions for the coefficients of the GCGL equations (3.10), (3.11) corresponding to the line parameters **(L_p)).** Using Appendix A, we compute the coefficients of the GCGL equations (3.10), (3.11) corresponding to the line parameters (L_p).

(L_p)	(L_{p1})	(L_{p2})
P	$1.3804 \times 10^{-3} - 1.4909 \times 10^{-6}i$	$1.3804 \times 10^{-3} - 7.827 \times 10^{-14}i$
γ	$2.083 \times 10^{-9} + 1.9287i$	$2.0681 \times 10^{-13} + 1.9149 \times 10^{-4}i$
Q	$8.5714 \times 10^{8} - 1.9905 \times 10^{6}i$	$450.0 - 6.1845 \times 10^{6}i$
E	$-1.437 \times 10^{-13} - 1.3305 \times 10^{-4}i$	$-1.437 \times 10^{-13} - 1.3305 \times 10^{-4}i$
H	$-3.6881 \times 10^{-2} + 0.07456i$	$-1.9361 \times 10^{-8} + 7.9834 \times 10^{-4}i$
D	$92.297 - 12.083i$	$-4.8155 \times 10^{-3} - 9.3358 \times 10^{-7}i$
F	$492.66 - 7.9982i$	$-1.2824 \times 10^{-2} - 4.199 \times 10^{-6}i$
K	$246.32 - 4.4434i$	$-1.2824 \times 10^{-2} - 2.3328 \times 10^{-6}i$
S_0	5.1332×10^{5}	5.1331×10^{5}

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