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EIGENVALUE CHARACTERIZATION OF A SYSTEM OF DIFFERENCE EQUATIONS

ХАРАКТЕРИЗАЦІЯ СИСТЕМИ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ЗА ВЛАСНИМИ ЗНАЧЕННЯМИ

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We consider the following system of difference equations:

$$u_i(k) = \lambda \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \;\; k \in \{0,1,\dots,T\},\; 1 \le i \le n,$$

where $\lambda > 0$ and $T \ge N \ge 0$. Our aim is to determine those values of λ such that the above system has a constant-sign solution. In addition, explicit intervals for λ will be presented. The generality of the results obtained is illustrated through applications to several well known boundary-value problems. We also extend the above problem to that on $\{0,1,\dots\}$,

$$u_i(k) = \lambda \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \;\; k \in \{0,1,\dots\},\; 1 \le i \le n.$$

Finally, both systems above are extended to the general case when λ is replaced by λ_i.

Розглянуто систему диференціальних рівнянь

$$u_i(k) = \lambda \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \;\; k \in \{0,1,\dots,T\},\; 1 \le i \le n,$$

де $\lambda > 0$ і $T \ge N \ge 0$. Метою статті є знаходження тих значень λ, для яких наведена система має розв'язок постійного знаку. Також знайдено в явному вигляді інтервали для таких λ. Загальність отриманих результатів проілюстровано застосуваннями до низки добре відомих граничних задач. Наведена вище задача також узагальнюється до такої ж задачі на $\{0,1,\dots\}$,

$$u_i(k) = \lambda \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \dots, u_n(\ell)), \;\; k \in \{0,1,\dots,T\},\; 1 \le i \le n,$$

На завершення ці дві системи поширюються на загальний випадок, коли λ замінюється на λ_i.

1. Introduction. We shall use the notation $Z[a,b] = \{a, a+1, \ldots, b\}$ where $a, b\ (> a)$ are integers. In this paper two systems of difference equations are discussed. The first system is on a finite set of integers,

$$u_i(k) = \lambda \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \ldots, u_n(\ell)), \ \ k \in I \equiv Z[0,T], \ 1 \le i \le n, \tag{1.1}$$

where $T \ge N > 0$. The second system is on the infinite set of $\mathbb{N} = \{0, 1, \ldots\}$,

$$u_i(k) = \lambda \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \ldots, u_n(\ell)), \ \ k \in \mathbb{N}, \ 1 \le i \le n. \tag{1.2}$$

A solution $u = (u_1, u_2, \ldots, u_n)$ of (1.1) will be sought in $(C(I))^n = C(I) \times \ldots \times C(I)$ (n times), where $C(I)$ denotes the class of functions continuous on I (discrete topology). We say that u is a solution of *constant sign* of (1.1) if for each $1 \le i \le n$, we have $\theta_i u_i(k) \ge 0$ for $k \in I$ where $\theta_i \in \{1, -1\}$ is fixed. On the other hand, a solution $u = (u_1, u_2, \ldots, u_n)$ of (1.2) will be sought in a subset of $(C(\mathbb{N}))^n = C(\mathbb{N}) \times \ldots \times C(\mathbb{N})$ (n times) where $\lim_{k\to\infty} u_i(k)$ exists for each $1 \le i \le n$. Moreover, u is a solution of *constant sign* of (1.2) if for each $1 \le i \le n$, we have $\theta_i u_i(k) \ge 0$ for $k \in \mathbb{N}$ where $\theta_i \in \{1, -1\}$ is fixed.

For each of (1.1) and (1.2), we shall characterize those values of λ for which the system has a constant-sign solution. If, for a particular λ the system has a constant-sign solution $u = (u_1, u_2, \ldots, u_n)$, then λ is called an *eigenvalue* and u a corresponding *eigenfunction* of the system. Let E be the set of eigenvalues, i.e.,

$$E = \{\lambda \mid \lambda > 0 \text{ such that the system under consideration has a constant-sign solution}\}.$$

We shall establish criteria for E to be an interval (which may either be bounded or unbounded). In addition explicit subintervals of E are derived.

Finally, both (1.1) and (1.2) are extended to the following systems:

$$u_i(k) = \lambda_i \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \ldots, u_n(\ell)), \ \ k \in I, \ 1 \le i \le n, \tag{1.3}$$

$$u_i(k) = \lambda_i \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \ldots, u_n(\ell)), \ \ k \in \mathbb{N}, \ 1 \le i \le n. \tag{1.4}$$

For each of (1.3) and (1.4), we shall characterize those values of $\lambda_i,\ 1 \le i \le n$, for which the system has a constant-sign solution. If, for a particular $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ the system has a constant-sign solution $u = (u_1, u_2, \ldots, u_n)$, then λ is called an *eigenvalue* and u a corresponding *eigenfunction* of the system. The set of eigenvalues is denoted by

$$E = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda_i > 0, \ 1 \le i \le n \text{ such that the system under consideration has a constant-sign solution}\}.$$

Results analogous to those for (1.1) and (1.2) will be developed for systems (1.3) and (1.4).

Recently, Agarwal and O'Regan [1] have investigated the existence of positive solutions of the discrete equation

$$y(k) = \sum_{\ell=0}^{N} g(k,\ell) f(y(\ell)) + h(k),\ k \in Z[0,T]. \tag{1.5}$$

The continuous version of (1.5) is well known in the literature, see [2–4]. We remark that a generalization of (1.5) to a system with existence criteria for single and multiple constant-sign solutions has recently been presented in [5]. In the present paper, besides extending (1.5) to a system, we have added in the parameter λ (or λ_i) and we consider constant-sign solutions. As a result, it is the *eigenvalue problem* that is of interest in this paper. Note that the term $h(k)$ in (1.5) has been excluded as we intend to apply the results to homogeneous boundary-value problems (in which case $h(k) \equiv 0$), which have received almost all the attention in the recent literature. However, it is not difficult to develop parallel results with the inclusion of $h(k)$ or even $h_i(k),\ 1 \le i \le n$. Many papers have discussed eigenvalues of boundary-value problems (see the monographs [6, 7] and the references cited therein). Our eigenvalue problems (1.1)–(1.4) generalize almost all the work done in the literature to date as we are considering *systems* as well as more general nonlinear terms. Moreover, our present approach is not only generic, but also improves, corrects and completes the arguments in many papers in the literature. It is also noted that this paper provides a discrete extension to the recent work [8].

The outline of the paper is as follows. In Section 2, we shall state Krasnosel'skii's fixed-point theorem which is crucial in establishing subintervals of E. The system (1.1) is discussed in Sections 3 and 4. In Section 3, we develop criteria for E to contain an interval, and for E to be an interval, which may either be bounded or unbounded. Moreover, upper and lower bounds are established for an eigenvalue λ. Explicit subintervals of E are derived in Section 4. To illustrate the importance and generality of the results obtained, applications to six well known boundary-value problems are included in Section 5. The treatment of systems (1.2), (1.3) and (1.4) is respectively presented in Sections 6–9 and 10, 11.

2. Preliminaries. The following theorem will be needed. It is usually called *Krasnosel'skii's fixed point theorem in a cone*.

Theorem 2.1 [9]. *Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that, either

(a) $\|Su\| \le \|u\|,\ u \in C \cap \partial\Omega_1$, and $\|Su\| \ge \|u\|,\ u \in C \cap \partial\Omega_2$, or

(b) $\|Su\| \ge \|u\|,\ u \in C \cap \partial\Omega_1$, and $\|Su\| \le \|u\|,\ u \in C \cap \partial\Omega_2$.

Then S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Characterization of E for (1.1). Throughout we shall denote $u = (u_1, u_2, \ldots$ $\ldots, u_n)$. Let the Banach space

$$B = \left\{ u \;\middle|\; u \in (C(I))^n \right\} \tag{3.1}$$

be equipped with norm

$$\|u\| = \max_{1\le i\le n} \max_{k\in I} |u_i(k)| = \max_{1\le i\le n} |u_i|_0 \tag{3.2}$$

where we let $|u_i|_0 = \max_{k\in I} |u_i(k)|,\ 1 \le i \le n$. Moreover, for fixed $\theta_i \in \{1, -1\},\ 1 \le i \le n$, define

$$\tilde{K} = \left\{ u \in B \;\middle|\; \theta_i u_i \ge 0,\ 1 \le i \le n \right\}$$

and

$$K = \left\{ u \in \tilde{K} \;\middle|\; \theta_j u_j > 0 \text{ for some } j \in \{1, 2, \ldots, n\} \right\} = \tilde{K}\backslash\{0\}.$$

For the purpose of clarity, we shall list the conditions that are needed later. Note that in these conditions $\theta_i \in \{1, -1\},\ 1 \le i \le n$, are fixed.

(C_1) For each $1 \le i \le n$, assume that $P_i : Z[0, N] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and

$$g_i(k, \ell) \ge 0,\ (k, \ell) \in I \times Z[0, N].$$

(C_2) For each $1 \le i \le n$, there exists a constant $M_i \in (0, 1)$, a continuous function $H_i : Z[0, N] \to [0, \infty)$, and an interval $Z[a, b] \subseteq Z[0, N]$ such that

$$g_i(k, \ell) \ge M_i H_i(\ell) \ge 0,\ \ (k, \ell) \in Z[a, b] \times Z[0, N].$$

(C_3) For each $1 \le i \le n$,

$$g_i(k, \ell) \le H_i(\ell),\ \ (k, \ell) \in I \times Z[0, N].$$

(C_4) For each $1 \le i \le n$, assume that

$$\theta_i P_i(\ell, u) \ge 0,\ \ u \in \tilde{K},\ \ell \in Z[0, N] \qquad \text{and} \qquad \theta_i P_i(\ell, u) > 0,\ \ u \in K,\ \ell \in Z[0, N].$$

(C_5) For each $1 \le i \le n$, there exist continuous functions $f_i,\ a_i,\ b_i$ with $f_i : \mathbb{R}^n \to [0, \infty)$ and $a_i, b_i : Z[0, N] \to [0, \infty)$ such that

$$a_i(\ell) \le \frac{\theta_i P_i(\ell, u)}{f_i(u)} \le b_i(\ell),\ \ u \in \tilde{K},\ \ell \in Z[0, N].$$

(C_6) For each $1 \le i \le n$, the function a_i is not identically zero on any nondegenerate subinterval of $Z[0, N]$, and there exists a number $0 < \rho_i \le 1$ such that

$$a_i(\ell) \ge \rho_i b_i(\ell),\ \ell \in Z[0, N].$$

(C_7) For each $1 \le i, j \le n$, if $|u_j| \le |v_j|$, then

$$\theta_i P_i(\ell, u_1, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_n) \le \theta_i P_i(\ell, u_1, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_n),\ \ell \in Z[0, N].$$

(C_8) For each $1 \le i, j \le n$, if $|u_j| \le |v_j|$, then

$$f_i(u_1, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_n) \le f_i(u_1, \dots, u_{j-1}, v_j, u_{j+1}, \dots, u_n).$$

To begin the discussion, let the operator $S : B \to B$ be defined by

$$Su(k) = (Su_1(k), Su_2(k), \dots, Su_n(k)),\ k \in I, \tag{3.3}$$

where

$$Su_i(k) = \lambda \sum_{\ell=0}^{N} g_i(k, \ell) P_i(\ell, u(\ell)),\ \ k \in I,\ 1 \le i \le n. \tag{3.4}$$

Clearly, a fixed point of the operator S is a solution of the system (1.1).

Next, we define a cone in B as

$$C = \left\{ u \in B \;\middle|\; \text{for each } 1 \le i \le n,\ \theta_i u_i(k) \ge 0 \text{ for } k \in I, \right.$$
$$\left. \text{and } \min_{k \in Z[a,b]} \theta_i u_i(k) \ge M_i \rho_i |u_i|_0 \right\} \tag{3.5}$$

where M_i and ρ_i are defined in (C_2) and (C_6) respectively. Note that $C \subseteq \tilde{K}$. A fixed point of S obtained in C or $\tilde{K}$ will be a *constant-sign solution* of the system (1.1). For $R > 0$, let

$$C(R) = \{u \in C \mid \|u\| \le R\}.$$

If (C_1), (C_4) and (C_5) hold, then it is clear from (3.4) that for $u \in \tilde{K}$,

$$\lambda \sum_{\ell=0}^{N} g_i(k, \ell) a_i(\ell) f_i(u(\ell)) \le \theta_i Su_i(k) \le \lambda \sum_{\ell=0}^{N} g_i(k, \ell) b_i(\ell) f_i(u(\ell)),\ \ k \in I,\ 1 \le i \le n. \tag{3.6}$$

Lemma 3.1. *Let (C_1) hold. Then, the operator S is continuous and completely continuous.*

Proof. Using Ascoli – Arzela Theorem as in [10], (C_1) ensures that S is continuous and completely continuous.

Lemma 3.2. *Let (C_1) – (C_6) hold. Then, the operator S maps C into itself.*

Proof. Let $u \in C$. From (3.6) we have for $k \in I$ and $1 \le i \le n$,

$$\theta_i Su_i(k) \ge \lambda \sum_{\ell=0}^{N} g_i(k, \ell) a_i(\ell) f_i(u(\ell)) \ge 0. \tag{3.7}$$

Next, using (3.6) and (C_3) gives for $k \in I$ and $1 \le i \le n$,

$$|Su_i(k)| = \theta_i Su_i(k) \le \lambda \sum_{\ell=0}^{N} g_i(k, \ell) b_i(\ell) f_i(u(\ell)) \le \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i(u(\ell)).$$

Hence, we have

$$|Su_i|_0 \le \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i(u(\ell)), \ 1 \le i \le n. \tag{3.8}$$

Now, employing (3.6), (C_2), (C_6) and (3.8) we find for $k \in Z[a,b]$ and $1 \le i \le n$,

$$\theta_i Su_i(k) \ge \lambda \sum_{\ell=0}^{N} g_i(k,\ell) a_i(\ell) f_i(u(\ell)) \ge \lambda \sum_{\ell=0}^{N} M_i H_i(\ell) a_i(\ell) f_i(u(\ell)) \ge$$

$$\ge \lambda \sum_{\ell=0}^{N} M_i H_i(\ell) \rho_i b_i(\ell) f_i(u(\ell)) \ge M_i \rho_i |Su_i|_0.$$

This leads to

$$\min_{k \in Z[a,b]} \theta_i Su_i(k) \ge M_i \rho_i |Su_i|_0, \ 1 \le i \le n. \tag{3.9}$$

Inequalities (3.7) and (3.9) imply that $Su \in C$.

Theorem 3.1. *Let (C_1) – (C_6) hold. Then, there exists $c > 0$ such that the interval $(0, c] \subseteq E$.*

Proof. Let $R > 0$ be given. Define

$$c = R \left\{ \left[\max_{1 \le m \le n} \sup_{\substack{|u_j| \le R \\ 1 \le j \le n}} f_m(u_1, u_2, \ldots, u_n) \right] \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right\}^{-1}. \tag{3.10}$$

Let $\lambda \in (0, c]$. We shall prove that $S(C(R)) \subseteq C(R)$. To begin, let $u \in C(R)$. By Lemma 3.2, we have $Su \in C$. Thus, it remains to show that $\|Su\| \le R$. Using (3.6), (C_3) and (3.10), we get for $k \in I$ and $1 \le i \le n$,

$$|Su_i(k)| = \theta_i Su_i(k) \le \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i(u(\ell)) \le$$

$$\le \lambda \left[\sup_{\substack{|u_j| \le R \\ 1 \le j \le n}} f_i(u_1, u_2, \ldots, u_n) \right] \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \le$$

$$\le \lambda \left[\max_{1 \le m \le n} \sup_{\substack{|u_j| \le R \\ 1 \le j \le n}} f_m(u_1, u_2, \ldots, u_n) \right] \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \le$$

$$\le c \left[\max_{1 \le m \le n} \sup_{\substack{|u_j| \le R \\ 1 \le j \le n}} f_m(u_1, u_2, \ldots, u_n) \right] \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \ = \ R.$$

It follows immediately that

$$\|Su\| \le R.$$

Thus, we have shown that $S(C(R)) \subseteq C(R)$. Also, from Lemma 3.1 the operator S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in $C(R)$. Clearly, this fixed point is a constant-sign solution of (1.1) and therefore λ is an eigenvalue of (1.1). Since $\lambda \in (0, c]$ is arbitrary, we have proved that the interval $(0, c] \subseteq E$.

Theorem 3.2. *Let (C$_1$), (C$_4$) and (C$_7$) hold. Suppose that $\lambda^* \in E$. Then, for any $\lambda \in (0, \lambda^*)$, we have $\lambda \in E$, i.e., $(0, \lambda^*] \subseteq E$.*

Proof. Let $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ be the eigenfunction corresponding to the eigenvalue λ^*. Thus, we have

$$u_i^*(k) = \lambda^* \sum_{\ell=0}^{N} g_i(k, \ell) P_i(\ell, u^*(\ell)), \;\; k \in I, \; 1 \le i \le n. \tag{3.11}$$

Define

$$K^* = \left\{ u \in \tilde{K} \;\middle|\; \text{for each } 1 \le i \le n, \; \theta_i u_i(k) \le \theta_i u_i^*(k), \; k \in I \right\}.$$

For $u \in K^*$ and $\lambda \in (0, \lambda^*)$, applying (C$_1$), (C$_4$) and (C$_7$) yields

$$\begin{aligned} \theta_i S u_i(k) \; &= \theta_i \left[\lambda \sum_{\ell=0}^{N} g_i(k, \ell) P_i(\ell, u(\ell)) \right] \le \theta_i \left[\lambda^* \sum_{\ell=0}^{N} g_i(k, \ell) P_i(\ell, u^*(\ell)) \right] = \\ &= \theta_i u_i^*(k), \;\; k \in I, \; 1 \le i \le n, \end{aligned}$$

where the last equality follows from (3.11). This immediately implies that the operator S defined by (3.3) maps K^* into K^*. Moreover, from Lemma 3.1 the operator S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in K^*, which is a constant-sign solution of (1.1). Hence, λ is an eigenvalue, i.e., $\lambda \in E$.

Corollary 3.1. *Let (C$_1$), (C$_4$) and (C$_7$) hold. If $E \ne \emptyset$, then E is an interval.*

Proof. Suppose E is not an interval. Then, there exist $\lambda_0, \lambda_0' \in E$ $(\lambda_0 < \lambda_0')$ and $\tau \in (\lambda_0, \lambda_0')$ with $\tau \notin E$. However, this is not possible as Theorem 3.2 guarantees that $\tau \in E$. Hence, E is an interval.

We shall now establish conditions under which E is a bounded or an unbounded interval. For this, we need the following result.

Theorem 3.3. *Let (C$_1$) – (C$_6$) and (C$_8$) hold. Suppose that λ is an eigenvalue of (1.1) and $u \in C$ is a corresponding eigenfunction. Let $q_i = |u_i|_0, \; 1 \le i \le n$. Then, for each $1 \le i \le n$, we have*

$$\lambda \ge \frac{q_i}{f_i(q_1, q_2, \ldots, q_n)} \left[\sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1} \tag{3.12}$$

and

$$\lambda \le \frac{q_i}{f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n)} \left[\sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}. \tag{3.13}$$

Proof. First, we shall prove (3.12). For each $1 \le i \le n$, let $k_i^* \in I$ be such that

$$q_i = |u_i|_0 = \theta_i u_i(k_i^*),\ 1 \le i \le n.$$

Then, in view of (3.6), (C_3) and (C_8), we find

$$q_i = \theta_i u_i(k_i^*) = \theta_i S u_i(k_i^*) = \theta_i \lambda \sum_{\ell=0}^{N} g_i(k_i^*, \ell) P_i(\ell, u(\ell)) \le$$

$$\le \lambda \sum_{\ell=0}^{N} g_i(k_i^*, \ell) b_i(\ell) f_i(u(\ell)) \le \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i(q_1, q_2, \ldots, q_n)$$

from which (3.12) is immediate.

Next, to verify (3.13), we employ (3.6), (C_4), (C_8) and the fact that $\min_{k\in Z[a,b]} \theta_i u_i(k) \ge \ge M_i\rho_i |u_i|_0 = M_i \rho_i q_i$ to get

$$q_i = |u_i|_0 \ge \theta_i u_i(a) = \theta_i \lambda \sum_{\ell=0}^{N} g_i(a, \ell) P_i(\ell, u(\ell)) \ge$$

$$\ge \lambda \sum_{\ell=0}^{N} g_i(a, \ell) a_i(\ell) f_i(u(\ell)) \ge \lambda \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) f_i(u(\ell)) \ge$$

$$\ge \lambda \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n)$$

which reduces to (3.13).

Theorem 3.4. *Let (C_1) – (C_8) hold. For each* $1 \le i \le n$, *define*

$$F_i^B = \left\{ f : \mathbb{R}^n \to [0,\infty) \;\middle|\; \frac{|u_i|}{f(u_1, u_2, \ldots, u_n)} \text{ is bounded for } u \in \mathbb{R}^n \right\},$$

$$F_i^0 = \left\{ f : \mathbb{R}^n \to [0,\infty) \;\middle|\; \lim_{\min_{1\le j\le n} |u_j| \to \infty} \frac{|u_i|}{f(u_1, u_2, \ldots, u_n)} = 0 \right\},$$

$$F_i^\infty = \left\{ f : \mathbb{R}^n \to [0,\infty) \;\middle|\; \lim_{\min_{1\le j\le n} |u_j| \to \infty} \frac{|u_i|}{f(u_1, u_2, \ldots, u_n)} = 0 \right\}.$$

(a) If $f_i \in F_i^B$ for each $1 \le i \le n$, then $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$.
(b) If $f_i \in F_i^0$ for each $1 \le i \le n$, then $E = (0, c]$ for some $c \in (0, \infty)$.
(c) If $f_i \in F_i^\infty$ for each $1 \le i \le n$, then $E = (0, \infty)$.

Proof. (a) This is immediate from (3.13) and Corollary 3.1.

(b) Since $F_i^0 \subseteq F_i^B$, $1 \le i \le n$, it follows from Case (a) that $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$. In particular,

$$c = \sup E.$$

Let $\{\lambda_m\}_{m=1}^\infty$ be a monotonically increasing sequence in E which converges to c, and let

$$\{u^m = (u_1^m, u_2^m, \ldots, u_n^m)\}_{m=1}^\infty \in \tilde{K}$$

be a corresponding sequence of eigenfunctions. Further, let $q_i^m = |u_i^m|_0$, $1 \le i \le n$. Then, (3.13) together with $f_i \in F_i^0$ implies that no subsequence of $\{q_i^m\}_{m=1}^\infty$ can diverge to infinity. Thus, there exists $R_i > 0$, $1 \le i \le n$, such that $q_i^m \le R_i$, $1 \le i \le n$, for all m. So u_i^m is uniformly bounded for each $1 \le i \le n$. This together with $Su^m = u^m$ (note Lemma 3.1) implies that for each $1 \le i \le n$ there is a subsequence of $\{u_i^m\}_{m=1}^\infty$, relabeled as the original sequence, which converges uniformly to some $u_i \in \tilde{K}_i$, where

$$\tilde{K}_i = \left\{ y \in C(I) \;\middle|\; \theta_i y(k) \ge 0,\ k \in I \right\}.$$

Clearly, we have

$$u_i^m(k) = \lambda_m \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u_1^m(\ell), u_2^m(\ell), \ldots, u_n^m(\ell)),\ \ k \in I,\ 1 \le i \le n. \tag{3.14}$$

Since u_i^m converges to u_i and λ_m converges to c, letting $m \to \infty$ in (3.14) yields

$$u_i(k) = c \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u_1(\ell), u_2(\ell), \ldots, u_n(\ell)),\ \ k \in I,\ 1 \le i \le n.$$

Hence, c is an eigenvalue with corresponding eigenfunction $u = (u_1, u_2, \ldots, u_n)$, i.e., $c = \sup E \in E$. This completes the proof for Case (b).

(c) Let $\lambda > 0$ be fixed. Choose $\varepsilon > 0$ so that

$$\lambda \max_{1 \le i \le n} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \le \frac{1}{\varepsilon}. \tag{3.15}$$

By definition, if $f_i \in F_i^\infty$, $1 \le i \le n$, then there exists $R = R(\varepsilon) > 0$ such that the following holds for each $1 \le i \le n$:

$$f_i(u_1, u_2, \ldots, u_n) < \varepsilon |u_i|,\ \ |u_j| \ge R,\ 1 \le j \le n. \tag{3.16}$$

We shall prove that $S(C(R)) \subseteq C(R)$. To begin, let $u \in C(R)$. By Lemma 3.2, we have $Su \in C$. Thus, it remains to show that $\|Su\| \le R$. Using (3.6), (C_3), (C_8), (3.16) and (3.15), we find for $k \in I$ and $1 \le i \le n$,

$$|Su_i(k)| = \theta_i Su_i(k) \le \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i(u(\ell)) \le$$

$$\le \lambda f_i(R, R, \ldots, R) \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \le \lambda(\varepsilon R) \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \ \le \ R.$$

It follows that $\|Su\| \le R$ and hence $S(C(R)) \subseteq C(R)$. From Lemma 3.1 the operator S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in $C(R)$. Clearly, this fixed point is a constant-sign solution of (1.1) and therefore λ is an eigenvalue of (1.1). Since $\lambda > 0$ is arbitrary, we have proved that $E = (0, \infty)$.

4. Subintervals of *E* for (1.1). For each $f_i,\ 1 \le i \le n$, introduced in (C_5), we shall define

$$\overline{f}_{0,i} = \limsup_{\max_{1\le j\le n}|u_j|\to 0} \frac{f_i(u_1, u_2, \ldots, u_n)}{|u_i|}, \qquad \underline{f}_{0,i} = \liminf_{\max_{1\le j\le n}|u_j|\to 0} \frac{f_i(u_1, u_2, \ldots, u_n)}{|u_i|},$$

$$\overline{f}_{\infty,i} = \limsup_{\min_{1\le j\le n}|u_j|\to \infty} \frac{f_i(u_1, u_2, \ldots, u_n)}{|u_i|} \quad \text{and} \quad \underline{f}_{\infty,i} = \liminf_{\min_{1\le j\le n}|u_j|\to \infty} \frac{f_i(u_1, u_2, \ldots, u_n)}{|u_i|}.$$

Theorem 4.1. *Let (C_1) – (C_6) hold. If λ satisfies*

$$\gamma_{1,i} < \lambda < \gamma_{2,i},\ 1 \le i \le n, \tag{4.1}$$

where

$$\gamma_{1,i} = \left[\underline{f}_{\infty,i}\ M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\gamma_{2,i} = \left[\overline{f}_{0,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $\lambda \in E$.

Proof. Let λ satisfy (4.1) and let $\varepsilon_i > 0,\ 1 \le i \le n$, be such that

$$\left[(\underline{f}_{\infty,i} - \varepsilon_i) M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1} \le \lambda \le \left[(\overline{f}_{0,i} + \varepsilon_i) \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},\ 1 \le i \le n. \tag{4.2}$$

First, we choose $w > 0$ so that

$$f_i(u) \le (\overline{f}_{0,i} + \varepsilon_i)|u_i|, \;\; 0 < |u_i| \le w, \; 1 \le i \le n. \tag{4.3}$$

Let $u \in C$ be such that $\|u\| = w$. Then, applying (3.6), (C_3), (4.3) and (4.2) successively, we find for $k \in I$ and $1 \le i \le n$,

$$\begin{aligned} |Su_i(k)| \;=\; \theta_i Su_i(k) \;&\le\; \lambda \sum_{\ell=0}^{N} g_i(k,\ell) b_i(\ell) f_i(u(\ell)) \le \\ &\le\; \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i(u(\ell)) \le \\ &\le\; \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell)(\overline{f}_{0,i} + \varepsilon_i)|u_i(\ell)| \le \\ &\le\; \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell)(\overline{f}_{0,i} + \varepsilon_i)\|u\| \;\le\; \|u\|. \end{aligned}$$

Hence,

$$\|Su\| \le \|u\|. \tag{4.4}$$

If we set $\Omega_1 = \{u \in B \mid \|u\| < w\}$, then (4.4) holds for $u \in C \cap \partial\Omega_1$.

Next, pick $r > w > 0$ such that

$$f_i(u) \ge (\underline{f}_{\infty,i} - \varepsilon_i)|u_i|, \;\; |u_i| \ge r, \; 1 \le i \le n. \tag{4.5}$$

Let $u \in C$ be such that

$$\|u\| = r' \equiv \max_{1 \le j \le n} \frac{r}{M_j \rho_j} \qquad (> w).$$

Suppose $\|u\| = |u_z|_0$ for some $z \in \{1, 2, \dots, n\}$. Then, for $\ell \in Z[a,b]$ we have

$$|u_z(\ell)| \ge M_z \rho_z |u_z|_0 = M_z \rho_z \|u\| \ge M_z \rho_z \frac{r}{M_z \rho_z} = r,$$

which, in view of (4.5), yields

$$f_z(u(\ell)) \ge (\underline{f}_{\infty,z} - \varepsilon_z)|u_z(\ell)|, \; \ell \in Z[a,b]. \tag{4.6}$$

Using (3.6), (C_2), (4.6) and (4.2), we find

$$\begin{aligned} |Su_z(a)| = \theta_z Su_z(a) &\geq \lambda \sum_{\ell=0}^{N} g_z(a,\ell) a_z(\ell) f_z(u(\ell)) \geq \\ &\geq \lambda \sum_{\ell=0}^{N} M_z H_z(\ell) a_z(\ell) f_z(u(\ell)) \geq \\ &\geq \lambda \sum_{\ell=a}^{b} M_z H_z(\ell) a_z(\ell) f_z(u(\ell)) \geq \\ &\geq \lambda \sum_{\ell=a}^{b} M_z H_z(\ell) a_z(\ell) (\underline{f}_{\infty,z} - \varepsilon_z) |u_z(\ell)| \geq \\ &\geq \lambda \sum_{\ell=a}^{b} M_z H_z(\ell) a_z(\ell) (\underline{f}_{\infty,z} - \varepsilon_z) M_z \rho_z |u_z|_0 = \\ &= \lambda \sum_{\ell=a}^{b} M_z H_z(\ell) a_z(\ell) (\underline{f}_{\infty,z} - \varepsilon_z) M_z \rho_z \|u\| \geq \|u\|. \end{aligned}$$

Therefore, $|Su_z|_0 \geq \|u\|$ and this leads to

$$\|Su\| \geq \|u\|. \tag{4.7}$$

If we set $\Omega_2 = \{u \in B \mid \|u\| < r'\}$, then (4.7) holds for $u \in C \cap \partial\Omega_2$.

Now that we have obtained (4.4) and (4.7), it follows from Theorem 2.1 that S has a fixed point $u \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $w \leq \|u\| \leq r'$. Since this u is a constant-sign solution of (1.1), the conclusion of the theorem follows immediately.

The following corollary is immediate from Theorem 4.1.

Corollary 4.1. *Let (C_1) – (C_6) hold. Then,*

$$(\gamma_{1,i}, \gamma_{2,i}) \subseteq E,\ 1 \leq i \leq n,$$

where $\gamma_{1,i}$ and $\gamma_{2,i}$ are defined in Theorem 4.1.

Corollary 4.2. *Let (C_1) – (C_7) hold. Then,*

$$\left(\min_{1 \leq i \leq n} \gamma_{1,i},\ \max_{1 \leq i \leq n} \gamma_{2,i} \right) \subseteq E$$

where $\gamma_{1,i}$ and $\gamma_{2,i}$ are defined in Theorem 4.1.

Proof. This is immediate from Corollaries 4.1 and 3.1.

Theorem 4.2. *Let (C_1) – (C_6) hold. If λ satisfies*

$$\gamma_{3,i} < \lambda < \gamma_{4,i},\ 1 \leq i \leq n, \tag{4.8}$$

where

$$\gamma_{3,i} = \left[\underline{f}_{0,i}\, M_i\rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell)\right]^{-1}$$

and

$$\gamma_{4,i} = \left[\overline{f}_{\infty,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell)\right]^{-1},$$

then $\lambda \in E$.

Proof. Let λ satisfy (4.8) and let $\varepsilon_i > 0,\ 1 \le i \le n$, be such that

$$\left[(\underline{f}_{0,i} - \varepsilon_i) M_i\rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell)\right]^{-1} \le \lambda \le \left[(\overline{f}_{\infty,i} + \varepsilon_i) \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell)\right]^{-1},\ 1 \le i \le n. \tag{4.9}$$

First, pick $\bar{w} > 0$ such that

$$f_i(u) \ge (\underline{f}_{0,i} - \varepsilon_i)|u_i|,\ \ 0 < |u_i| \le \bar{w},\ 1 \le i \le n. \tag{4.10}$$

Let $u \in C$ be such that $\|u\| = \bar{w}$. Suppose $\|u\| = |u_z|_0$ for some $z \in \{1, 2, \ldots, n\}$. Employing (3.6), ($C_2$), (4.10) and (4.9) successively, we get

$$\begin{aligned} |Su_z(a)| = \theta_z Su_z(a) &\ge \lambda \sum_{\ell=0}^{N} g_z(a,\ell) a_z(\ell) f_z(u(\ell)) \ge \\ &\ge \lambda \sum_{\ell=0}^{N} M_z H_z(\ell) a_z(\ell) f_z(u(\ell)) \ge \\ &\ge \lambda \sum_{\ell=0}^{N} M_z H_z(\ell) a_z(\ell)(\underline{f}_{0,z} - \varepsilon_z)|u_z(\ell)| \ge \\ &\ge \lambda \sum_{\ell=a}^{b} M_z H_z(\ell) a_z(\ell)(\underline{f}_{0,z} - \varepsilon_z)|u_z(\ell)| \ge \\ &\ge \lambda \sum_{\ell=a}^{b} M_z H_z(\ell) a_z(\ell)(\underline{f}_{0,z} - \varepsilon_z) M_z\rho_z |u_z|_0 = \\ &= \lambda \sum_{\ell=a}^{b} M_z H_z(\ell) a_z(\ell)(\underline{f}_{0,z} - \varepsilon_z) M_z\rho_z \|u\| \ \ge\ \|u\|. \end{aligned}$$

Therefore, $|Su_z|_0 \ \ge\ \|u\|$ and inequality (4.7) follows immediately. By setting $\Omega_1 = \{u \in B \mid \|u\| < \bar{w}\}$, we see that (4.7) holds for $u \in C \cap \partial\Omega_1$.

Next, choose $\bar{r} > \bar{w} > 0$ such that

$$f_i(u) \le (\overline{f}_{\infty,i} + \varepsilon_i)|u_i|, \ \ |u_i| \ge \bar{r}, \ 1 \le i \le n. \tag{4.11}$$

For each $f_i, \ 1 \le i \le n$, we shall consider two cases, namely, f_i is bounded and f_i is unbounded. Let N_b and N_u be subsets of $\{1, 2, \ldots, n\}$ such that

$$N_b \cap N_u = \emptyset, \qquad N_b \cup N_u = \{1, 2, \ldots, n\},$$

$$f_i \text{ is bounded for } i \in N_b,$$

$$f_i \text{ is unbounded for } i \in N_u.$$

Case 1. Suppose that f_i, $i \in N_b$, is bounded. Then, there exists some $R_i > 0$ such that

$$f_i(u) \le R_i, \ \ u \in \mathbb{R}^n, \ i \in N_b. \tag{4.12}$$

We define

$$r' = \max_{i \in N_b} \gamma_{4,i} \, R_i \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell).$$

Let $u \in C$ be such that $\|u\| \ge r'$. Applying (3.6), (C_3), (4.12) and (4.8) gives for $i \in N_b$ and $k \in I$,

$$\begin{aligned} |Su_i(k)| \ = \ \theta_i Su_i(k) \ &\le \ \lambda \sum_{\ell=0}^{N} g_i(k,\ell) b_i(\ell) f_i(u(\ell)) \le \\ &\le \ \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) R_i < \\ &< \ \gamma_{4,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) R_i \ \le \ r' \ \le \ \|u\|. \end{aligned}$$

It follows that for $u \in C$ with $\|u\| \ge r'$,

$$\max_{i \in N_b} |Su_i|_0 \le \|u\|. \tag{4.13}$$

Case 2. Suppose that f_i, $i \in N_u$, is unbounded. Then, there exists

$$r'' > \max\{\bar{r}, r'\} \qquad (> \bar{w})$$

such that

$$f_i(u) \le \max_{\substack{\eta_j \in \{-1,1\} \\ 1 \le j \le n}} f_i(\eta_1 r'', \eta_2 r'', \ldots, \eta_n r''), \ \ |u_j| \le r'', \ 1 \le j \le n. \tag{4.14}$$

Let $u \in C$ be such that $\|u\| = r''$. Then, successive use of (3.6), (4.14), (4.11), (C_3) and (4.9) provides for $i \in N_u$ and $k \in I$,

$$\begin{aligned} |Su_i(k)| = \theta_i Su_i(k) &\le \lambda \sum_{\ell=0}^{N} g_i(k,\ell) b_i(\ell) f_i(u(\ell)) \le \\ &\le \lambda \sum_{\ell=0}^{N} g_i(k,\ell) b_i(\ell) \max_{\substack{\eta_j \in \{-1,1\} \\ 1 \le j \le n}} f_i(\eta_1 r'', \eta_2 r'', \ldots, \eta_n r'') \le \\ &\le \lambda \sum_{\ell=0}^{N} g_i(k,\ell) b_i(\ell) (\overline{f}_{\infty,i} + \varepsilon_i) r'' \le \\ &\le \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) (\overline{f}_{\infty,i} + \varepsilon_i) \|u\| \le \|u\|. \end{aligned}$$

Therefore, we have for $u \in C$ with $\|u\| = r''$,

$$\max_{i \in N_u} |Su_i|_0 \le \|u\|. \tag{4.15}$$

Combining (4.13) and (4.15), we obtain for $u \in C$ with $\|u\| = r''$,

$$\max_{i \in N_b \cup N_u} |Su_i|_0 \le \|u\|,$$

which is actually (4.4). Hence, by setting $\Omega_2 = \{u \in B \mid \|u\| < r''\}$, we see that (4.4) holds for $u \in C \cap \partial\Omega_2$.

Having obtained (4.7) and (4.4), an application of Theorem 2.1 leads to the existence of a fixed point u of S in $C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $\bar{w} \le \|y\| \le r''$. This u is a constant-sign solution of (1.1) and the conclusion of the theorem follows immediately.

Theorem 4.2 leads to the following corollary.

Corollary 4.3. *Let (C_1) – (C_6) hold. Then,*

$$(\gamma_{3,i}, \gamma_{4,i}) \subseteq E,\ 1 \le i \le n,$$

where $\gamma_{3,i}$ and $\gamma_{4,i}$ are defined in Theorem 4.2.

Corollary 4.4. *Let (C_1) – (C_7) hold. Then,*

$$\left(\min_{1 \le i \le n} \gamma_{3,i},\ \max_{1 \le i \le n} \gamma_{4,i} \right) \subseteq E$$

where $\gamma_{3,i}$ and $\gamma_{4,i}$ are defined in Theorem 4.2.

Proof. This is immediate from Corollaries 4.3 and 3.1.

Remark 4.1. For a fixed $i \in \{1, 2, \dots, n\}$, if f_i is *superlinear* (i.e., $\overline{f}_{0,i} = 0$ and $\underline{f}_{\infty,i} = \infty$) or *sublinear* (i.e., $\underline{f}_{0,i} = \infty$ and $\overline{f}_{\infty,i} = 0$), then we conclude from Corollaries 4.1 and 4.3 that $E = (0, \infty)$, i.e., (1.1) has a constant-sign solution for any $\lambda > 0$. We remark that superlinearity and sublinearity conditions have also been discussed for various boundary-value problems in the literature for the single equation case ($n = 1$), see for example [3, 6, 7, 11–14] and the references cited therein.

5. Applications to boundary-value problems. In this section we shall illustrate the generality of the results obtained in Sections 3 and 4 by considering various well known boundary-value problems in the literature. Indeed, we shall apply our results to systems of boundary-value problems of the following types: (m, p), Lidstone, focal, conjugate, Hermite and Sturm–Liouville.

Case 5.1. (m, p) ***Boundary-value problem.*** Consider the system of (m, p) boundary-value problems

$$\begin{gathered} \Delta^m u_i(k) + \lambda P_i(k, u(k)) = 0, \ k \in Z[0, N], \\ \Delta^j u_i(0) = 0, \ 0 \le j \le m-2, \qquad \Delta^{p_i} u_i(N + m - p_i) = 0 \end{gathered} \tag{5.1}$$

where $i = 1, 2, \dots, n$. It is assumed that $m \ge 2$, $N \ge m - 1$ and for each $1 \le i \le n$, $1 \le p_i \le m - 1$ is fixed and $P_i : Z[0, N] \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

Let $G_i(k, \ell)$ be the Green's function of the boundary-value problem

$$\begin{gathered} -\Delta^m y(k) = 0, \ k \in Z[0, N], \\ \Delta^j y(0) = 0, \ 0 \le j \le m-2; \qquad \Delta^{p_i} y(N + m - p_i) = 0. \end{gathered}$$

It is known that [6, p. 315]

(a) $$G_i(k, \ell) = \frac{1}{(m-1)!} \begin{cases} \dfrac{k^{(m-1)}(N + m - p_i - 1 - \ell)^{(m-p_i-1)}}{(N + m - p_i)^{(m-p_i-1)}} - (k - \ell - 1)^{(m-1)}, \\ \hfill \ell \in Z[0, k-m]; \\ \dfrac{k^{(m-1)}(N + m - p_i - 1 - \ell)^{(m-p_i-1)}}{(N + m - p_i)^{(m-p_i-1)}}, \ell \in Z[k - m + 1, N]; \end{cases}$$

(b) $\Delta^j G_i(k, \ell)$ $(w.r.t.\ k) \ge 0,\ \ 0 \le j \le p_i,\ (k, \ell) \in Z[0, N + m - j] \times Z[0, N]$;

(c) for $(k, \ell) \in Z[m - 1, N + m - p_i] \times Z[0, N]$, we have

$$G_i(k, \ell) \ge \frac{p_i}{(N + m - p_i)^{(m-p_i-1)}(N + 1)} (N + m - p_i - 1 - \ell)^{(m-p_i-1)};$$

(d) for $(k, \ell) \in Z[0, N + m] \times Z[0, N]$, we have

$$G_i(k, \ell) \le \frac{(N + m)^{(m-1)}}{(m-1)!(N + m - p_i)^{(m-p_i-1)}} (N + m - p_i - 1 - \ell)^{(m-p_i-1)}.$$

Now, with $I = Z[0, N+m]$, $u = (u_1, u_2, \ldots, u_n)$ is a solution of the system (5.1) if and only if u is a fixed point of the operator $S : B \to B$ defined by (3.3) where

$$Su_i(k) = \lambda \sum_{\ell=0}^{N} G_i(k,\ell) P_i(\ell, u(\ell)), \;\; k \in I, \; 1 \le i \le n. \tag{5.2}$$

In the context of Section 3, we have

$$\begin{gathered} g_i(k,\ell) = G_i(k,\ell), \;\; I = Z[0, N+m], \;\; Z[a,b] = Z[m-1, N], \;\; M_i = \frac{(m-1)!p_i}{(N+m)^{(m)}}, \\ H_i(\ell) = \frac{(N+m)^{(m-1)}}{(m-1)!(N+m-p_i)^{(m-p_i-1)}} \, (N+m-p_i-1-\ell)^{(m-p_i-1)}. \end{gathered} \tag{5.3}$$

Then, noting (a) – (d), we see that the conditions (C_1) – (C_3) are fulfilled.

The results in Sections 3 and 4 reduce to the following theorem, which improves and extends the earlier work of [11, 15] (for $n = 1$) — not only do we consider a more general P_i, our method is also generic in nature.

Theorem 5.1. *Let $E = \{\lambda \mid \lambda > 0$ such that (5.1) has a constant-sign solution$\}$. With g_i, a, b, M_i and H_i given in (5.3), we have the following:*

(i) (Theorem 3.1). Let (C_4) – (C_6) hold. Then, there exists $c > 0$ such that the interval $(0, c] \subseteq$ $\subseteq E$.

(ii) (Theorem 3.2 and Corollary 3.1). Let (C_4) and (C_7) hold. Suppose that $\lambda^* \in E$. Then, for any $\lambda \in (0, \lambda^*)$, we have $\lambda \in E$, i.e., $(0, \lambda^*] \subseteq E$. Indeed, if $E \ne \emptyset$, then E is an interval.

(iii) (Theorem 3.3). Let (C_4) – (C_6) and (C_8) hold. Suppose that $\lambda \in E$ and

$$u \in C = \left\{ u \in (C(I))^n \;\middle|\; \text{for each } 1 \le i \le n, \; \theta_i u_i(k) \ge 0 \text{ for } k \in I, \right.$$
$$\left. \text{and } \min_{k \in Z[a,b]} \theta_i u_i(k) \ge M_i \rho_i |u_i|_0 \right\}$$

is a corresponding eigenfunction. Let $q_i = |u_i|_0$, $1 \le i \le n$. Then, for each $1 \le i \le n$, we have

$$\lambda \ge \frac{q_i}{f_i(q_1, q_2, \ldots, q_n)} \left[\sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1}$$

and

$$\lambda \le \frac{q_i}{f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n)} \left[\sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}.$$

(iv) (Theorem 3.4). Let (C_4) – (C_8) hold. For each $1 \le i \le n$, let F_i^B, F_i^0 and F_i^∞ be defined as in Theorem 3.4.

(a) If $f_i \in F_i^B$ for each $1 \le i \le n$, then $E = (0,c)$ or $(0,c]$ for some $c \in (0,\infty)$.

(b) If $f_i \in F_i^0$ for each $1 \le i \le n$, then $E = (0,c]$ for some $c \in (0,\infty)$.

(c) If $f_i \in F_i^\infty$ for each $1 \le i \le n$, then $E = (0,\infty)$.

(v) (Theorem 4.1, Corollaries 4.1 and 4.2). Let (C_4) – (C_6) hold. For each $1 \le i \le n$, let $\overline{f}_{0,i}$ and, $\underline{f}_{\infty,i}$ be defined as in Section 4. If λ satisfies

$$\gamma_{1,i} < \lambda < \gamma_{2,i}, \ 1 \le i \le n,$$

where

$$\gamma_{1,i} = \left[\underline{f}_{\infty,i} \, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\gamma_{2,i} = \left[\overline{f}_{0,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $\lambda \in E$. Indeed,

$$(\gamma_{1,i}, \gamma_{2,i}) \subseteq E, \ 1 \le i \le n.$$

Moreover, if (C_7) holds, then

$$\left(\min_{1 \le i \le n} \gamma_{1,i}, \ \max_{1 \le i \le n} \gamma_{2,i} \right) \subseteq E.$$

(vi) (Theorem 4.2, Corollaries 4.3 and 4.4). Let (C_4) – (C_6) hold. For each $1 \le i \le n$, let $\underline{f}_{0,i}$ and $\overline{f}_{\infty,i}$ be defined as in Section 4. If λ satisfies

$$\gamma_{3,i} < \lambda < \gamma_{4,i}, \ 1 \le i \le n,$$

where

$$\gamma_{3,i} = \left[\underline{f}_{0,i} \, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\gamma_{4,i} = \left[\overline{f}_{\infty,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $\lambda \in E$. Indeed,

$$(\gamma_{3,i}, \gamma_{4,i}) \subseteq E, \ 1 \le i \le n.$$

Moreover, if (C_7) holds, then

$$\left(\min_{1 \le i \le n} \gamma_{3,i}, \ \max_{1 \le i \le n} \gamma_{4,i} \right) \subseteq E.$$

(vii) (Remark 4.1). Let (C_4) – (C_6) hold. If f_j is m superlinear (i.e., $\overline{f}_{0,j} = 0$ and $\underline{f}_{\infty,j} = \infty$) or sublinear (i.e., $\underline{f}_{0,j} = \infty$ and $\overline{f}_{\infty,j} = 0$) for some $j \in \{1, 2, \ldots, n\}$, then $E = (0,\infty)$.

Example 5.1. Consider the system of (m,p) boundary-value problems

$$\Delta^3 u_1(k) + \lambda\, \frac{[u_1(k)+1][u_2(k)+1]}{[k(k-1)(11-k)+1][k(k-1)(20-k)+1]} = 0,\ k\in Z[0,5],$$

$$\Delta^3 u_2(k) + \lambda\, \frac{[u_1(k)+20][u_2(k)+20]}{[k(k-1)(11-k)+20][k(k-1)(20-k)+20]} = 0,\ k\in Z[0,5], \tag{5.4}$$

$$u_1(0) = \Delta u_1(0) = 0,\ \Delta u_1(7) = 0; \qquad u_2(0) = \Delta u_2(0) = 0,\ \Delta^2 u_2(6) = 0.$$

In this example, $n=2,\ m=3,\ N=5,\ p_1=1,\ p_2=2,$

$$P_1(k,u(k)) = \frac{[u_1(k)+1][u_2(k)+1]}{[k(k-1)(11-k)+1][k(k-1)(20-k)+1]}$$

and

$$P_2(k,u(k)) = \frac{[u_1(k)+20][u_2(k)+20]}{[k(k-1)(11-k)+20][k(k-1)(20-k)+20]}.$$

Fix $\theta_1=\theta_2=1$. Clearly, (C_4) and (C_7) are satisfied. Now, choose

$$f_1(u) = [u_1(k)+1][u_2(k)+1], \qquad f_2(u) = [u_1(k)+20][u_2(k)+20],$$

$$a_1(k) = b_1(k) = \{[k(k-1)(11-k)+1][k(k-1)(20-k)+1]\}^{-1}$$

and

$$a_2(k) = b_2(k) = \{[k(k-1)(11-k)+20][k(k-1)(20-k)+20]\}^{-1}.$$

Then, (C_5), (C_6) (with $\rho_1=\rho_2=1$) and (C_8) are fulfilled. Moreover, we have $H_1(\ell)=4(6-\ell)$ and $H_2(\ell)=28$.

It is easy to see that

$$\overline{f}_{0,1} = \underline{f}_{0,1} = \infty, \qquad \overline{f}_{\infty,1} = \underline{f}_{\infty,1} = 1, \qquad \overline{f}_{0,2} = \underline{f}_{0,2} = \infty \qquad \text{and} \qquad \overline{f}_{\infty,2} = \underline{f}_{\infty,2} = 1.$$

Clearly, $f_i\in F_i^B,\ i=1,2$. Hence, Theorem 5.1(iv) guarantees that

$$E = \{\lambda \mid \lambda > 0 \text{ such that (5.4) has a constant-sign solution}\} = (0,c) \text{ or } (0,c] \tag{5.5}$$

for some $c\in(0,\infty)$.

By direct computation, we get

$$\gamma_{3,1} = \gamma_{3,2} = 0, \qquad \gamma_{4,1} = 0{,}02271 \qquad \text{and} \qquad \gamma_{4,2} = 6{,}3121.$$

It follows from Theorem 5.1(vi) that

$$\left(\min_{i=1,2}\gamma_{3,i},\ \max_{i=1,2}\gamma_{4,i}\right) = (0{,}63121) \subseteq E. \tag{5.6}$$

Coupling with (5.5), we further conclude that $E = (0, c)$ or $(0, c]$ where $c \geq 6,3121$. Indeed, when $\lambda = 6 \in E$, the system (5.4) has a positive solution given by

$$u(k) = (u_1(k), u_2(k)) = (k(k-1)(11-k),\ k(k-1)(20-k))\,,\ \ k \in Z[0,8].$$

Case 5.2. Lidstone boundary-value problem. Consider the system of Lidstone boundary-value problems

$$\begin{aligned} &(-1)^m \Delta^{2m} u_i(k) = \lambda P_i(k, u(k)),\ k \in Z[0,N], \\ &\Delta^{2j} u_i(0) = \Delta^{2j} u_i(N+2m-2j) = 0,\ 0 \leq j \leq m-1, \end{aligned} \tag{5.7}$$

where $i = 1, 2, \ldots, n$. It is assumed that $m \geq 1$ and $P_i : Z[0,N] \times \mathbb{R}^n \to \mathbb{R},\ 1 \leq i \leq n$, is continuous.

Let $G_m(k, \ell)$ be the Green's function of the boundary-value problem

$$\Delta^{2m} y(k) = 0,\ k \in Z[0,N],$$

$$\Delta^{2j} y(0) = \Delta^{2j} y(N+2m-2j) = 0,\ 0 \leq j \leq m-1.$$

It is given in [16] that

(a) $G_m(k,\ell) = \sum_{\tau=0}^{N+2m-2} G(k,\tau) G_{m-1}(\tau,\ell)$ where

$$G(k,\ell) = G_1(k,\ell) = -\frac{1}{N+2m} \begin{cases} (N+2m-k)(\ell+1), & \ell \in Z[0,k-2]; \\ k(N+2m-1-\ell), & \ell \in Z[k-1, N+2m-2]; \end{cases}$$

(b) $(-1)^m G_m(k,\ell) \geq 0,\ \ (k,\ell) \in Z[0, N+2m] \times Z[0,N]$;

(c) for $(k,\ell) \in Z[1, N+2m-1] \times Z[0,N]$, we have

$$(-1)^m G_m(k,\ell) \geq \beta_m \min\{\ell+1, N+1-\ell\} \geq \frac{\beta_m}{N+1}\,(\ell+1)(N+1-\ell)$$

where

$$\beta_m = \left[\prod_{j=1}^{m}(N+2j)\right]^{-1} \prod_{j=1}^{m-1} T_{2j-1}$$

and

$$T_j = \sum_{\tau=1}^{N+j} \min\{\tau+1, N+j+2-\tau\} = \frac{1}{4}\begin{cases} (N+j)^2 + 6(N+j) + 1, & (N+j) \text{ is odd}; \\ (N+j)(N+j+6), & (N+j) \text{ is even} \end{cases},\ j \geq 1;$$

(d) for $(k,\ell) \in Z[0, N+2m] \times Z[0,N]$, we have

$$(-1)^m G_m(k,\ell) \leq \alpha_m (\ell+1)(N+1-\ell)$$

where

$$\alpha_m = \left[\prod_{j=1}^{m}(N+2j)\right]^{-1}\prod_{j=1}^{m-1}s_{2j}$$

and

$$s_j = \sum_{\tau=0}^{N+j}(\tau+1)(N+j+1-\tau) = \frac{1}{6}(N+j+3)^{(3)},\ j \geq 2.$$

Clearly, with $I = Z[0, N+2m]$, $u = (u_1, u_2, \ldots, u_n)$ is a solution of the system (5.7) if and only if u is a fixed point of the operator $S : B \to B$ defined by (3.3) where

$$Su_i(k) = \lambda\sum_{\ell=0}^{N}(-1)^m G_m(k,\ell)P_i(\ell, u(\ell)),\ \ k \in I,\ 1 \leq i \leq n. \tag{5.8}$$

In the context of Section 3, let

$$\begin{gathered} g_i(k,\ell) = (-1)^m G_m(k,\ell), \qquad I = Z[0, N+2m], \qquad Z[a,b] = Z[1,N], \\ M_i = \frac{\beta_m}{\alpha_m(N+1)} \quad \text{and} \quad H_i(\ell) = \alpha_m(\ell+1)(N+1-\ell). \end{gathered} \tag{5.9}$$

Then, the conditions (C_1) – (C_3) are satisfied in view of (a) – (d).

Applying the results in Sections 3 and 4, we obtain the following theorem which improves and extends the earlier work of [16] (for $n = 1$). Note that the P_i considered in (5.7) as well as the methodology used are both more general.

Theorem 5.2. *Let $E = \{\lambda \mid \lambda > 0$ such that (5.7) has a constant-sign solution$\}$. With $g_i,\ a,$ b, M_i and H_i given in (5.9), the statements (i) – (vii) of Theorem 5.1 hold.*

Case 5.3. Focal boundary-value problem. Consider the system of focal boundary-value problems

$$\begin{gathered} (-1)^{m-p_i}\,\Delta^m u_i(k) = \lambda P_i(k, u(k)),\ k \in Z[0,N], \\ \Delta^j u_i(0) = 0,\ 0 \leq j \leq p_i - 1; \qquad \Delta^j u_i(N+1) = 0,\ p_i \leq j \leq m-1, \end{gathered} \tag{5.10}$$

where $i = 1, 2, \ldots, n$. It is assumed that $m \geq 2$, and for each $1 \leq i \leq n$, $1 \leq p_i \leq \min\{m - -1, N\}$ is fixed and $P_i : Z[0,N] \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

Let $G_i(k,\ell)$ be the Green's function of the boundary-value problem

$$\Delta^m y(k) = 0, \quad k \in Z[0,N],$$

$$\Delta^j y(0) = 0,\ 0 \leq j \leq p_i - 1; \qquad \Delta^j y(N+1) = 0,\ p_i \leq i \leq m-1.$$

In [17] it is given that

(a) $$G_i(k,\ell) = (-1)^{m-p_i} \begin{cases} \displaystyle\sum_{j=0}^{\ell} \frac{(k-j-1)^{(p_i-1)}(\ell+m-p_i-1-j)^{(m-p_i-1)}}{(p_i-1)!(m-p_i-1)!}, & \ell \in Z[0,k-1]; \\ \displaystyle\sum_{j=0}^{k-1} \frac{(k-j-1)^{(p_i-1)}(\ell+m-p_i-1-j)^{(m-p_i-1)}}{(p_i-1)!(m-p_i-1)!}, & \ell \in Z[k,N]; \end{cases}$$

(b) the signs of the differences of $G_i(k,\ell)$ w.r.t. k are as follows:

$$(-1)^{m-p_i}\Delta^j G_i(k,\ell) \ge 0, \quad (k,\ell) \in Z[0,N+m-j] \times Z[0,N],\ 0 \le j \le p_i - 1,$$

$$(-1)^{m-p_i+j}\Delta^{j+p_i} G_i(k,\ell) \ge 0,$$

$$(k,\ell) \in Z[0,N+m-j-p_i] \times Z[0,N],\ 0 \le j \le m-p_i-1;$$

(c) for a given $\delta_i \in Z[p_i,N]$, and $(k,\ell) \in Z[\delta_i,N+m] \times Z[0,N]$, we have

$$(-1)^{m-p_i}G_i(k,\ell) \ge L_i(-1)^{m-p_i}G_i(N+m,\ell)$$

where

$$L_i = \min_{\ell \in Z[0,N]} \frac{G_i(\delta_i,\ell)}{G_i(N+m,\ell)};$$

(d) $(-1)^{m-p_i}G_i(k,\ell) \le (-1)^{m-p_i}G_i(N+m,\ell),\ (k,\ell) \in Z[0,N+m] \times Z[0,N]$.

Obviously, with $I = Z[0,N+m]$, $u = (u_1,u_2,\ldots,u_n)$ is a solution of the system (5.10) if and only if u is a fixed point of the operator $S : B \to B$ defined by (3.3) where

$$Su_i(k) = \lambda \sum_{\ell=0}^{N} (-1)^{m-p_i}G_i(k,\ell)P_i(\ell,u(\ell)),\ \ k \in I,\ 1 \le i \le n. \tag{5.11}$$

Let $\delta_i \in Z[p_i,N],\ 1 \le i \le n$, be fixed and $\delta \equiv \max_{1\le i\le n}\delta_i$. In the context of Section 3, let

$$\begin{gathered} g_i(k,\ell) = (-1)^{m-p_i}G_i(k,\ell), \qquad I = Z[0,N+m], \qquad Z[a,b] = Z[\delta,N], \\ M_i = L_i \qquad \text{and} \qquad H_i(\ell) = (-1)^{m-p_i}G_i(N+m,\ell). \end{gathered} \tag{5.12}$$

Then, from (a) – (d) we see that the conditions (C_1) – (C_3) are satisfied.

The results in Sections 3 and 4 reduce to the following theorem which improves and extends the earlier work of [17] (for $n = 1$). We remark that the P_i considered in (5.10) as well as the methodology used are both more general.

Theorem 5.3. *Let* $E = \{\lambda \mid \lambda > 0$ *such that (5.10) has a constant-sign solution*$\}$. *With* $g_i,\ a,\ b,\ M_i$ *and* H_i *given in (5.12), the statements (i) – (vii) of Theorem 5.1 hold.*

Case 5.4. Conjugate boundary-value problem. Consider the system of conjugate boundary-value problems

$$(-1)^{m-p_i}\,\Delta^m u_i(k) = \lambda P_i(k,u(k)),\ k\in Z[0,N],$$

$$\Delta^j u_i(0) = 0,\ 0\le j\le p_i-1;\qquad \Delta^j u_i(N+p_i+1) = 0,\ 0\le j\le m-p_i-1, \tag{5.13}$$

where $i = 1,2,\ldots,n$. It is assumed that $m\ge 2$, and for each $1\le i\le n$, $1\le p_i\le m-1$, $N\ge \min_{1\le i\le n} p_i$ and $P_i : Z[0,N]\times\mathbb{R}^n\to\mathbb{R}$ is continuous.

Let $G_i(k,\ell)$ be the Green's function of the boundary-value problem

$$\Delta^m y(k) = 0,\ k\in Z[0,N],$$

$$\Delta^j y(0) = 0,\ 0\le j\le p_i-1;\qquad \Delta^j y(N+p_i+1) = 0,\ 0\le j\le m-p_i-1.$$

It is known that [18, 19]

(a) $$G_i(k,\ell) = \begin{cases} \displaystyle\sum_{j=0}^{p_i-1}\left[\sum_{\tau=0}^{p_i-j-1}\binom{m-p_i+\tau-1}{\tau}\frac{k^{(j+\tau)}}{(N+m-j)^{(m-p_i+\tau)}}\right]\frac{(-\ell-1)^{(m-j-1)}}{j!(m-j-1)!}\times \\ \qquad\qquad \times(N+m-k)^{(m-p_i)},\quad \ell\in Z[0,k-1], \\ \displaystyle -\sum_{j=0}^{m-p_i-1}\left[\sum_{\tau=0}^{m-p_i-j-1}\binom{p_i+\tau-1}{\tau}\frac{(N+p_i+j+\tau-k)^{(j+\tau)}}{(N+p_i+1+j+\tau)^{(p_i+\tau)}}\right](-1)^j\times \\ \qquad\qquad \displaystyle\times\frac{(N+p_i-\ell)^{(m-j-1)}}{j!(m-j-1)!}\,k^{(p_i)},\quad \ell\in Z[k,N]; \end{cases}$$

(b) $(-1)^{m-p_i}G_i(k,\ell)\ge 0,\ \ (k,\ell)\in Z[0,N+m]\times Z[0,N]$;

(c) for a given $\delta_i\in Z[p_i,N+p_i]$, and $(k,\ell)\in Z[\delta_i,N+p_i]\times Z[0,N]$, we have

$$(-1)^{m-p_i}G_i(k,\ell)\ge K_i\|G_i(\cdot,\ell)\|$$

where

$$\|G_i(\cdot,\ell)\| = \max_{k\in Z[0,N+m]}|G_i(k,\ell)| = \max_{k\in Z[0,N+m]}(-1)^{m-p_i}G_i(k,\ell),$$

$$K_i = \min\left\{\frac{\min_{k\in Z[\delta_i,N+p_i]} v(p_i+1,k)}{\max_{k\in Z[\delta_i,N+p_i]} v(p_i+1,k)},\ \frac{\min_{k\in Z[\delta_i,N+p_i]} v(p_i,k)}{\max_{k\in Z[\delta_i,N+p_i]} v(p_i,k)}\right\},$$

and the function v is defined as

$$v(x,k) = k^{(x-1)}(N+m-k)^{(m-x)};$$

(d) $(-1)^{m-p_i}G_i(k,\ell)\le\|G_i(\cdot,\ell)\|,\ \ (k,\ell)\in Z[0,N+m]\times Z[0,N]$.

Now, with $I = Z[0, N+m]$, $u = (u_1, u_2, \ldots, u_n)$ is a solution of the system (5.13) if and only if u is a fixed point of the operator $S : B \to B$ defined by (3.3) where

$$Su_i(k) = \lambda \sum_{\ell=0}^{N} (-1)^{m-p_i} G_i(k,\ell) P_i(\ell, u(\ell)), \;\; k \in I, \; 1 \le i \le n. \tag{5.14}$$

Let $\delta_i \in Z[p_i, N+p_i]$, $1 \le i \le n$, be fixed and $\delta \equiv \max_{1 \le i \le n} \delta_i$. In the context of Section 3, let

$$\begin{gathered} g_i(k,\ell) = (-1)^{m-p_i} G_i(k,\ell), \qquad I = Z[0, N+m], \qquad Z[a,b] = Z[\delta, N], \\ M_i = K_i \qquad \text{and} \qquad H_i(\ell) = \|G_i(\cdot, \ell)\|. \end{gathered} \tag{5.15}$$

Then, (a) – (d) ensures that the conditions (C_1) – (C_3) are fulfilled.

Applying the results in Sections 3 and 4, we obtain the following theorem which improves and extends the earlier work of [18] (for $n = 1$). Note that the P_i considered in (5.13) as well as the methodology used are both more general.

Theorem 5.4. *Let $E = \{\lambda \mid \lambda > 0$ such that (5.13) has a constant-sign solution$\}$. With g_i, a, b, M_i and H_i given in (5.15), the statements (i) – (vii) of Theorem 5.1 hold.*

Case 5.5. Hermite boundary-value problem. Consider the system of Hermite boundary-value problems

$$\begin{gathered} \Delta^m u_i(k) = \lambda F_i(k, u(k)), \; k \in Z[0, N], \\ \Delta^j u_i(k_\nu) = 0, \;\; j = 0, \ldots, m_\nu - 1, \; \nu = 1, \ldots, J, \end{gathered} \tag{5.16}$$

where $i = 1, 2, \ldots, n$. It is assumed that $J \ge 2$, $m_\nu \ge 1$ for $\nu = 1, \ldots, J$, $\sum_{\nu=1}^{J} m_\nu = m$, and k_ν's are integers such that $k_J \ge N$ and

$$0 = k_1 < k_1 + m_1 < k_2 < k_2 + m_2 < \ldots < k_J \le k_J + m_J - 1 = N + m.$$

Moreover, for each $1 \le i \le n$ and $k \in Z[0, N]$, we assume

$$F_i(k, u(k)) = \begin{cases} (-1)^{\gamma_\nu} P_i(k, u(k)), & k \in Z[k_\nu, k_{\nu+1} - 1], \; \nu = 1, \ldots, J-2; \\ (-1)^{\gamma_{J-1}} P_i(k, u(k)), & k \in Z[k_{J-1}, k_J], \end{cases} \tag{5.17}$$

where $P_i : Z[0, N] \times \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le n$, is continuous and

$$\gamma_\nu = \sum_{j=\nu+1}^{J} m_j, \; 1 \le \nu \le J - 1.$$

We shall also use the notation

$$I_\nu = Z[k_\nu + m_\nu, k_{\nu+1} - 1], \; 1 \le \nu \le J - 1.$$

Let $G(k,\ell)$ be the Green's function of the boundary-value problem

$$\Delta^m y(k) = 0, \quad k \in Z[0,N],$$

$$\Delta^j y(k_\nu) = 0, \qquad j = 0,\dots,m_\nu - 1, \quad \nu = 1,\dots,J.$$

It is known that [20, 21]

(a) the signs of $G(k,\ell)$ are as follows:

$$(-1)^{\gamma_\nu} G(k,\ell) \ge 0, (k,\ell) \in Z[k_\nu, k_{\nu+1}] \times Z[0,N],\ \nu = 1,\dots,J-1,$$

$$G(k,\ell) = 0, (k,\ell) \in Z[k_J, N+m] \times Z[0,N];$$

(b) for $(k,\ell) \in I_\nu \times Z[0,N],\ \nu = 1,\dots,J-1$, we have

$$(-1)^{\gamma_\nu} G(k,\ell) \ge L_\nu \|G(\cdot,\ell)\|$$

where

$$\|G(\cdot,\ell)\| = \max_{k\in Z[0,N+m]} |G(k,\ell)| = \max_{1\le\nu\le J-1} \max_{k\in Z[k_\nu,k_{\nu+1}]} (-1)^{\gamma_\nu} G(k,\ell),$$

$$L_\nu = \min\left\{ \frac{\min\{p(k_\nu+m_\nu), p(k_{\nu+1}-1)\}}{\max_{k\in Z[0,N+m]} p(k)},\ \frac{\min\{q(k_\nu+m_\nu), q(k_{\nu+1}-1)\}}{\max_{k\in Z[0,N+m]} q(k)} \right\}$$

and the functions p and q are defined as

$$p(k) = \left| \prod_{j=1}^{J-1} (k-k_j)^{(m_j)} \right| (N+m-k)^{(m_J-1)}, \qquad q(k) = k^{(m_1-1)} \left| \prod_{j=2}^{J} (k-k_j)^{(m_j)} \right|;$$

(c) $(-1)^{\gamma_\nu} G(k,\ell) \le \|G(\cdot,\ell)\|,\ \ (k,\ell) \in Z[0,N+m] \times Z[0,N],\ \nu = 1,\dots,J-1.$

Clearly, with $I = Z[0,N+m]$, $u = (u_1,u_2,\dots,u_n)$ is a solution of the system (5.16) if and only if u is a fixed point of the operator $S : B \to B$ defined by (3.3) where

$$Su_i(k) = \lambda \sum_{\ell=0}^{N} G(k,\ell) F_i(\ell, u(\ell)),\ \ k \in I,\ 1 \le i \le n. \tag{5.18}$$

In the context of Section 3, let

$$g_i(k,\ell) = (-1)^{\gamma_\nu} G(k,\ell), \qquad I = Z[0,N+m], \qquad Z[a,b] = I_\nu \cap Z[0,N],$$
$$M_i = L_\nu \qquad \text{and} \qquad H_i(\ell) = \|G(\cdot,\ell)\|. \tag{5.19}$$

Then, noting (a) – (c) the conditions (C_1), (C_3) and (C_2) (for $\nu = 1,2,\dots,J-1$) are fulfilled.

The results in Sections 3 and 4 reduce to the following theorem, which improves and extends the earlier work of [21] (for $n = 1$) — note that a more general F_i is considered by using a more general method.

Theorem 5.5. *Let $E = \{\lambda \mid \lambda > 0$ such that (5.16) has a constant-sign solution$\}$. With $g_i,\ a,\ b,\ M_i$ and H_i given in (5.19), the statements (i), (ii), (iv) and (vii) of Theorem 5.1 hold. Moreover, we have the following:*

(iii) (Theorem 3.3). Let (C_4) – (C_6) and (C_8) hold. Suppose that $\lambda \in E$ and

$$u \in C = \Big\{ u \in (C(I))^n \ \Big|\ \text{for each } 1 \le i \le n,\ \theta_i u_i(k) \ge 0 \text{ for } k \in I,$$

$$\text{and } \min_{k \in I_\nu \cap Z[0,N]} \theta_i u_i(k) \ge L_\nu \rho_i |u_i|_0,\ \nu = 1, 2, \ldots, J-1 \Big\}$$

is a corresponding eigenfunction. Let $q_i = |u_i|_0,\ 1 \le i \le n$. Then, we have

$$\lambda \ge \frac{q_i}{f_i(q_1, q_2, \ldots, q_n)} \left[\sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},\ 1 \le i \le n, \tag{5.20}$$

and

$$\lambda \le \frac{q_i}{f_i(L_\nu \rho_1 q_1, L_\nu \rho_2 q_2, \ldots, L_\nu \rho_n q_n)} \left[\sum_{\ell \in I_\nu \cap Z[0,N]} L_\nu H_i(\ell) a_i(\ell) \right]^{-1}, \qquad 1 \le i \le n,\ 1 \le \nu \le J-1. \tag{5.21}$$

(v) (Theorem 4.1, Corollaries 4.1 and 4.2). Let (C_4) – (C_6) hold. For each $1 \le i \le n$, let $\overline{f}_{0,i}$ and $\underline{f}_{\infty,i}$ be defined as in Section 4. If λ satisfies

$$\gamma_{1,i,\nu} < \lambda < \gamma_{2,i},\ \ 1 \le i \le n,\ 1 \le \nu \le J-1, \tag{5.22}$$

where

$$\gamma_{1,i,\nu} = \left[\underline{f}_{\infty,i}\, L_\nu \rho_i \sum_{\ell \in I_\nu \cap Z[0,N]} L_\nu H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\gamma_{2,i} = \left[\overline{f}_{0,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $\lambda \in E$. Indeed,

$$(\gamma_{1,i,\nu}, \gamma_{2,i}) \subseteq E,\ \ 1 \le i \le n,\ 1 \le \nu \le J-1.$$

Moreover, if (C$_7$) holds, then

$$\left(\min_{\substack{1\le i\le n\\ 1\le \nu\le J-1}}\gamma_{1,i,\nu},\ \max_{1\le i\le n}\gamma_{2,i}\right)\subseteq E.$$

(vi) (Theorem 4.2, Corollaries 4.3 and 4.4). Let (C$_4$) – (C$_6$) hold. For each $1\le i\le n$, let $\underline{f}_{0,i}$ and $\overline{f}_{\infty,i}$ be defined as in Section 4. If λ satisfies

$$\gamma_{3,i,\nu}<\lambda<\gamma_{4,i},\ \ 1\le i\le n,\ 1\le\nu\le J-1, \tag{5.23}$$

where

$$\gamma_{3,i,\nu}=\left[\underline{f}_{0,i}\,L_\nu\rho_i\sum_{\ell\in I_\nu\cap Z[0,N]}L_\nu H_i(\ell)a_i(\ell)\right]^{-1}$$

and

$$\gamma_{4,i}=\left[\overline{f}_{\infty,i}\sum_{\ell=0}^{N}H_i(\ell)b_i(\ell)\right]^{-1},$$

then $\lambda\in E$. Indeed,

$$(\gamma_{3,i,\nu},\gamma_{4,i})\subseteq E,\ \ 1\le i\le n,\ 1\le\nu\le J-1.$$

Moreover, if (C$_7$) holds, then

$$\left(\min_{\substack{1\le i\le n\\ 1\le \nu\le J-1}}\gamma_{3,i,\nu},\ \max_{1\le i\le n}\gamma_{4,i}\right)\subseteq E.$$

Proof. (iii) Here, the cone C in (3.5) is modified to that in the statement of Theorem 5.5(iii). The proof of (5.20) is similar to that in the proof of Theorem 3.3. To verify (5.21), let $1\le i\le n$ and $1\le\nu\le J-1$ be fixed. Using (3.6), (C$_2$), (C$_8$) and the fact that $\min_{k\in I_\nu\cap Z[0,N]}\theta_i u_i(k)\ge$ $\ge L_\nu\rho_i|u_i|_0=L_\nu\rho_i q_i$, we get

$$\begin{aligned}
q_i = |u_i|_0 &\ge \theta_i u_i(k_{\nu+1}-1)=\\
&= \theta_i\lambda\sum_{\ell=0}^{N}G_i(k_{\nu+1}-1,\ \ell)F_i(\ell,u(\ell))\ge\\
&\ge \theta_i\lambda\sum_{\ell\in Z[k_\nu,k_{\nu+1}-1]\cap Z[0,N]}G_i(k_{\nu+1}-1,\ \ell)(-1)^{\gamma_\nu}P_i(\ell,u(\ell))\ge\\
&\ge \lambda\sum_{\ell\in Z[k_\nu,k_{\nu+1}-1]\cap Z[0,N]}(-1)^{\gamma_\nu}G_i(k_{\nu+1}-1,\ \ell)a_i(\ell)f_i(u(\ell))\ge\\
&\ge \lambda\sum_{\ell\in I_\nu\cap Z[0,N]}L_\nu H_i(\ell)a_i(\ell)f_i(u(\ell))\ge\\
&\ge \lambda\sum_{\ell\in I_\nu\cap Z[0,N]}L_\nu H_i(\ell)a_i(\ell)f_i(L_\nu\rho_1q_1,L_\nu\rho_2q_2,\ldots,L_\nu\rho_nq_n)
\end{aligned}$$

which reduces to (5.21).

(v) Let λ satisfy (5.22) and let $\varepsilon_{i\nu} > 0,\ 1 \le i \le n,\ 1 \le \nu \le J-1$, be such that

$$\left[(\underline{f}_{\infty,i} - \varepsilon_{i\nu})L_\nu\rho_i \sum_{\ell\in I_\nu\cap Z[0,N]} L_\nu H_i(\ell)a_i(\ell)\right]^{-1} \le \lambda \le \left[(\overline{f}_{0,i} + \varepsilon_{i\nu})\sum_{\ell=0}^{N} H_i(\ell)b_i(\ell)\right]^{-1}, \quad 1 \le i \le n,\ 1 \le \nu \le J-1. \tag{5.24}$$

First, we can choose $w > 0$ so that for $1 \le i \le n$ and $1 \le \nu \le J-1$,

$$f_i(u) \le (\overline{f}_{0,i} + \varepsilon_{i\nu})|u_i|,\ 0 < |u_i| \le w. \tag{5.25}$$

As in the proof of Theorem 4.1, it now follows that $\|Su\| \le \|u\|$ for $u \in C \cap \partial\Omega_1$ where $\Omega_1 = \{u \in B \mid \|u\| < w\}$.

Next, pick $T > w > 0$ such that for $1 \le i \le n$ and $1 \le \nu \le J-1$,

$$f_i(u) \ge (\underline{f}_{\infty,i} - \varepsilon_{i\nu})|u_i|,\ |u_i| \ge T. \tag{5.26}$$

Let $u \in C$ be such that

$$\|u\| = T' \equiv \max_{\substack{1 \le j \le n \\ 1 \le \nu \le J-1}} \frac{T}{L_\nu\rho_j} \qquad (> w).$$

Suppose $\|u\| = |u_z|_0$ for some $z \in \{1,2,\ldots,n\}$. Let $\nu \in \{1,2,\ldots,J-1\}$ be fixed. Then, for $\ell \in I_\nu \cap Z[0,N]$ we have

$$|u_z(\ell)| \ge L_\nu\rho_z|u_z|_0 = L_\nu\rho_z\|u\| \ge L_\nu\rho_z\frac{T}{L_\nu\rho_z} = T,$$

which, in view of (5.26), yields

$$f_z(u(\ell)) \ge (\underline{f}_{\infty,z} - \varepsilon_{z\nu})|u_z(\ell)|, \qquad l \in I_\nu \cap Z[0,N]. \tag{5.27}$$

Using (3.6), (C_2), (5.27) and (5.24), we find

$$|Su_z(k_{\nu+1}-1)| = \theta_z Su_z(k_{\nu+1}-1) \geq$$

$$\geq \theta_z \lambda \sum_{\ell\in Z[k_\nu,k_{\nu+1}-1]\cap Z[0,N]} G_z(k_{\nu+1}-1,\ \ell)(-1)^{\gamma_\nu} P_z(\ell,u(\ell)) \geq$$

$$\geq \lambda \sum_{\ell\in Z[k_\nu,k_{\nu+1}-1]\cap Z[0,N]} (-1)^{\gamma_\nu} G_z(k_{\nu+1}-1,\ \ell) a_z(\ell) f_z(u(\ell)) \geq$$

$$\geq \lambda \sum_{\ell\in I_\nu\cap Z[0,N]} L_\nu H_z(\ell) a_z(\ell) f_z(u(\ell)) \geq$$

$$\geq \lambda \sum_{\ell\in I_\nu\cap Z[0,N]} L_\nu H_z(\ell) a_z(\ell)(\underline{f}_{\infty,z} - \varepsilon_{z\nu})|u_z(\ell)| \geq$$

$$\geq \lambda \sum_{\ell\in I_\nu\cap Z[0,N]} L_\nu H_z(\ell) a_z(\ell)(\underline{f}_{\infty,z} - \varepsilon_{z\nu}) L_\nu \rho_z |u_z|_0 =$$

$$= \lambda \sum_{\ell\in I_\nu\cap Z[0,N]} L_\nu H_z(\ell) a_z(\ell)(\underline{f}_{\infty,z} - \varepsilon_{z\nu}) L_\nu \rho_z \|u\| \ \geq \ \|u\|.$$

Therefore, $|Su_z|_0 \geq \|u\|$ and this leads to $\|Su\| \geq \|u\|$. Setting $\Omega_2 = \{u \in B \mid \|u\| < T'\}$, we have $\|Su\| \geq \|u\|$ for $u \in C \cap \partial\Omega_2$.

The rest of the proof is similar to that of Theorem 4.1.

(vi) The proof is similar to that of Theorem 4.2 with analogous modification as in the proof of Theorem 5.5(v).

Case 5.6. Sturm – Liouville boundary-value problem. Consider the system of Sturm – Liouville boundary-value problems

$$\Delta^m u_i(k) + \lambda P_i(k,u(k)) = 0,\ k \in Z[0,N],$$

$$\Delta^j u_i(0) = 0,\ 0 \leq j \leq m-3, \tag{5.28}$$

$$\zeta_i \Delta^{m-2} u_i(0) - \eta_i \Delta^{m-1} u_i(0) = 0, \quad \gamma_i \Delta^{m-2} u_i(N+1) + \delta_i \Delta^{m-1} u_i(N+1) = 0,$$

where $i = 1, 2, \ldots, n$. It is assumed that $m \geq 2,\ N \geq m-1$, and for each $1 \leq i \leq n$, $P_i : Z[0,N] \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

$$\zeta_i > 0, \qquad \gamma_i > 0, \qquad \eta_i \geq 0, \qquad \delta_i \geq \gamma_i, \qquad \psi_i \equiv \zeta_i\gamma_i(N+1) + \zeta_i\delta_i + \eta_i\gamma_i > 0.$$

Let $h_i(k,\ell)$ be the Green's function of the boundary-value problem

$$-\Delta^m y(k) = 0,\ k \in Z[0,N],$$
$$\Delta^j y(0) = 0,\ 0 \le j \le m-3,$$
$$\zeta_i \Delta^{m-2} y(0) - \eta_i \Delta^{m-1} y(0) = 0, \qquad \gamma_i \Delta^{m-2} y(N+1) + \delta_i \Delta^{m-1} y(N+1) = 0.$$

It can be verified that [14]

$$G_i(k,\ell) = \Delta^{m-2} h_i(k,\ell) \ \text{(w.r.t. } k) \tag{5.29}$$

is the Green's function of the boundary-value problem

$$-\Delta^2 w(k) = 0,\ k \in Z[0,N],$$

$$\zeta_i w(0) - \eta_i \Delta w(0) = 0, \qquad \gamma_i w(N+1) + \delta_i \Delta w(N+1) = 0.$$

Further, it is known that [14]

(a) $G_i(k,\ell) = \dfrac{1}{\psi_i}\begin{cases} [\eta_i + \zeta_i(\ell+1)][\delta_i + \gamma_i(N+1-k)], & \ell \in Z[0,k-1]; \\ (\eta_i + \zeta_i k)[\delta_i + \gamma_i(N-\ell)], & \ell \in Z[k,N]; \end{cases}$

(b) $G_i(k,\ell) \ge 0,\ \ (k,\ell) \in Z[0,N+2] \times Z[0,N]$;

(c) for $(k,\ell) \in Z[1,N] \times Z[0,N]$, we have

$$G_i(k,\ell) \ge A_i\, G_i(\ell,\ell)$$

where

$$A_i = \frac{(\eta_i + \zeta_i)(\delta_i + \gamma_i)}{(\eta_i + \zeta_i N)(\delta_i + \gamma_i N)};$$

d) for $(k,\ell) \in Z[0,N+2] \times Z[0,N]$, we have

$$G_i(k,\ell) \le B_i\, G_i(\ell,\ell)$$

where

$$B_i = \begin{cases} \dfrac{\eta_i + \zeta_i}{\eta_i}, & \eta_i > 0; \\ 2, & \eta_i = 0. \end{cases}$$

In the context of Section 3, let the Banach space

$$B = \left\{ u = (u_1, u_2, \ldots, u_n) \in (C(Z[0,N+m]))^n \;\middle|\; \Delta^j u_i(0) = 0,\ 0 \le j \le m-3,\ 1 \le i \le n \right\} \tag{5.30}$$

be equipped with norm

$$\|u\| = \max_{1 \le i \le n}\ \max_{k \in Z[0,N+2]} |\Delta^{m-2} u_i(k)| = \max_{1 \le i \le n} |u_i|_0 \tag{5.31}$$

where we denote $|u_i|_0 = \max_{k\in Z[0,N+2]} |\Delta^{m-2}u_i(k)|,\ 1 \le i \le n$. Further, define the cone C in B as

$$C = \left\{ u = (u_1, u_2, \dots, u_n) \in B \;\middle|\; \text{for each } 1 \le i \le n,\ \theta_i\Delta^{m-2}u_i(k) \ge 0 \text{ for } k \in Z[0, N+2], \right.$$

$$\left. \text{and } \min_{k\in Z[1,N]} \theta_i\Delta^{m-2}u_i(k) \ge M_i|u_i|_0 \right\} \tag{5.32}$$

where $M_i = \dfrac{A_i}{B_i} \in (0,1),\ 1 \le i \le n$. It can be shown that S maps C into C.

Lemma 5.1 [14].
(a) Let $u \in B$. For $0 \le j \le m-2$, we have

$$|\Delta^j u_i(k)| \le \frac{k^{(m-2-j)}}{(m-2-j)!}\,|u_i|_0, \qquad k \in Z[0, N+m-j],\ 1 \le i \le n. \tag{5.33}$$

In particular,

$$|u_i(k)| \le \frac{(N+m)^{(m-2)}}{(m-2)!}\,\|u\|, \qquad k \in Z[0, N+m],\ 1 \le i \le n. \tag{5.34}$$

(b) Let $u \in C$. For $0 \le j \le m-2$, we have

$$\theta_i\Delta^j u_i(k) \ge 0, \qquad k \in Z[0, N+m-j],\ 1 \le i \le n, \tag{5.35}$$

and

$$\theta_i\Delta^j u_i(k) \ge \frac{(k-1)^{(m-2-j)}}{(m-2-j)!}\,M_i\rho_i|u_i|_0, \qquad k \in Z[1, N+m-2-j],\ 1 \le i \le n. \tag{5.36}$$

In particular,

$$\theta_i u_i(k) \ge M_i\rho_i|u_i|_0, \qquad k \in Z[m-1, N+m-2],\ 1 \le i \le n. \tag{5.37}$$

Hence, if $u = (u_1, u_2, \dots, u_n) \in C$ is a solution of (5.28), then it follows from (5.35) that u is a constant-sign solution. Clearly, u is a solution of the system (5.28) if and only if u is a fixed point of the operator $S : B \to B$ defined by (3.3) where

$$Su_i(k) = \lambda\sum_{\ell=0}^{N} h_i(k,\ell)P_i(\ell, u(\ell)),\ \ k \in Z[0, N+m],\ 1 \le i \le n, \tag{5.38}$$

or equivalently

$$\Delta^{m-2}(Su_i)(k) = \lambda\sum_{\ell=0}^{N} G_i(k,\ell)P_i(\ell, u(\ell)),\ \ k \in Z[0, N+2],\ 1 \le i \le n. \tag{5.39}$$

Now, in the context of Section 3, let

$$g_i(k,\ell) = G_i(k,\ell), \qquad I = Z[0,N+2], \qquad Z[a,b] = Z[1,N],$$
$$M_i = \frac{A_i}{B_i} \qquad \text{and} \qquad H_i(\ell) = B_i\, G_i(\ell,\ell). \tag{5.40}$$

Then, noting (a) – (d), we see that (C_1) – (C_3) are fulfilled.

The results in Sections 3 and 4 together with Lemma 5.1 lead to the following theorem, which improves and extends the earlier work of [14, 21, 22] (for $n = 1$) — not only do we consider a more general P_i, our method is also more general.

Theorem 5.6. *Let $E = \{\lambda \mid \lambda > 0$ such that (5.28) has a constant-sign solution$\}$. With g_i, a, b, M_i and H_i given in (5.40), the statements (i), (ii), (iv)–(vii) of Theorem 5.1 hold. Moreover, we have the following:*

(iii) (Theorem 3.3). Let (C_4) – (C_6) and (C_8) hold. Suppose that $\lambda \in E$ and $u \in C$ (see (5.32)) is a corresponding eigenfunction. Let $q_i = |u_i|_0$, $1 \le i \le n$. Then, for each $1 \le i \le n$, we have

$$\lambda \ge q_i \left[f_i\left(\frac{N^{(m-2)}q_1}{(m-2)!}, \frac{N^{(m-2)}q_2}{(m-2)!}, \ldots, \frac{N^{(m-2)}q_n}{(m-2)!} \right) \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1}$$

and

$$\lambda \le q_i \left[f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n) \sum_{\ell=m-1}^{N} M_i H_i(\ell) a_i(\ell) \right]^{-1}.$$

Proof. (iii) For each $1 \le i \le n$, let $k_i^* \in I$ be such that

$$q_i = |u_i|_0 = \theta_i \Delta^{m-2} u_i(k_i^*),\ 1 \le i \le n.$$

Then, applying (C_3), (C_8) and (5.33) gives

$$\begin{aligned} q_i \;=\; & \theta_i \Delta^{m-2} u_i(k_i^*) = \theta_i \Delta^{m-2}(Su_i)(k_i^*) = \\ & = \theta_i \lambda \sum_{\ell=0}^{N} G_i(k_i^*,\ell) P_i(\ell, u(\ell)) \le \\ & \le \lambda \sum_{\ell=0}^{N} G_i(k_i^*,\ell) b_i(\ell) f_i(u(\ell)) \le \\ & \le \lambda \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i\left(\frac{N^{(m-2)}q_1}{(m-2)!}, \frac{N^{(m-2)}q_2}{(m-2)!}, \ldots, \frac{N^{(m-2)}q_n}{(m-2)!} \right) \end{aligned}$$

from which the first inequality is immediate.

Next, we use (C_2), (C_8) and (5.37) to get

$$
\begin{aligned}
q_i &= |u_i|_0 \geq \\
&\geq \theta_i \Delta^{m-2} u_i(m-1) = \\
&= \theta_i \lambda \sum_{\ell=0}^{N} G_i(m-1,\ell) P_i(\ell, u(\ell)) \geq \\
&\geq \lambda \sum_{\ell=0}^{N} G_i(m-1,\ell) a_i(\ell) f_i(u(\ell)) \geq \\
&\geq \lambda \sum_{\ell=m-1}^{N} M_i H_i(\ell) a_i(\ell) f_i(u(\ell)) \geq \\
&\geq \lambda \sum_{\ell=m-1}^{N} M_i H_i(\ell) a_i(\ell) f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n)
\end{aligned}
$$

which reduces to the second inequality.

6. Characterization of E for (1.2). This section extends the results in Section 3 to the system of difference equations (1.2) on the infinite set of $\mathbb{N} = \{0, 1, \ldots\}$. To begin, let the Banach space $B = (C(\mathbb{N}))^n$ be equipped with norm

$$\|u\| = \max_{1 \leq i \leq n} \max_{k \in \mathbb{N}} |u_i(k)| = \max_{1 \leq i \leq n} |u_i|_0 \tag{6.1}$$

where we let $|u_i|_0 = \max_{k \in \mathbb{N}} |u_i(k)|, \ 1 \leq i \leq n.$

We shall seek a solution $u = (u_1, u_2, \ldots, u_n)$ of (1.2) in $(C_l(\mathbb{N}))^n$ where

$$(C_l(\mathbb{N}))^n = \left\{ u \in (C(\mathbb{N}))^n \ \middle| \ \lim_{k \to \infty} u_i(k) \text{ exists, } 1 \leq i \leq n \right\}. \tag{6.2}$$

For the purpose of clarity, we shall list the conditions that are needed later. Note that in these conditions $\theta_i \in \{1, -1\}, \ 1 \leq i \leq n$ are fixed.

$(C_1)_\infty$ For each $1 \leq i \leq n$, assume that

$$g_i^k(\ell) \equiv g_i(k,\ell) \geq 0, \ \ (k,\ell) \in \mathbb{N} \times \mathbb{N},$$

$$\sum_{\ell=0}^{\infty} g_i^k(\ell) < \infty, \ k \in \mathbb{N} \text{ (i.e., } g_i^k(\ell) \in l^1(\mathbb{N}), \ k \in \mathbb{N}),$$

there exists $\tilde{g}_i \in l^1(\mathbb{N})$ such that $\lim\limits_{k\to\infty} \sum\limits_{\ell=0}^{\infty} |g_i^k(\ell) - \tilde{g}_i(\ell)| = 0$ (i.e., $g_i^k \to \tilde{g}_i$ in $l^1(\mathbb{N})$ as $k \to \infty$), $P_i : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

for each $r > 0$, there exists $M_{r,i}$ such that for $k \in \mathbb{N}$ and $|u_j| \le r,\ 1 \le j \le n,\ |P_i(k,u)| \le \le M_{r,i}$.

$(C_2)_\infty$ For each $1 \le i \le n$, there exists a constant $M_i \in (0,1)$, a continuous function $H_i : \mathbb{N} \to [0,\infty)$, and an interval $Z[a,b] \subseteq \mathbb{N}$ such that

$$g_i(k,\ell) \ge M_i H_i(\ell) \ge 0,\ \ (k,\ell) \in Z[a,b] \times \mathbb{N}.$$

$(C_3)_\infty$ For each $1 \le i \le n$,

$$g_i(k,\ell) \le H_i(\ell),\ \ (k,\ell) \in \mathbb{N} \times \mathbb{N}.$$

$(C_4)_\infty$ Let $\tilde{K}$ and K be as in Section 3 with $B = (C(\mathbb{N}))^n$. For each $1 \le i \le n$, assume that

$$\theta_i P_i(\ell,u) \ge 0,\ \ u \in \tilde{K},\ \ell \in \mathbb{N} \qquad \text{and} \qquad \theta_i P_i(\ell,u) > 0,\ \ u \in K,\ \ell \in \mathbb{N}.$$

$(C_5)_\infty$ For each $1 \le i \le n$, there exist continuous functions $f_i,\ a_i,\ b_i$ with $f_i : \mathbb{R}^n \to [0,\infty)$ and $a_i, b_i : \mathbb{N} \to [0,\infty)$ such that

$$a_i(\ell) \le \frac{\theta_i P_i(\ell,u)}{f_i(u)} \le b_i(\ell),\ \ u \in \tilde{K},\ \ell \in \mathbb{N}.$$

$(C_6)_\infty$ For each $1 \le i \le n$, the function a_i is not identically zero on any nondegenerate subinterval of $\mathbb{N}$, and there exists a number $0 < \rho_i \le 1$ such that

$$a_i(\ell) \ge \rho_i b_i(\ell),\ \ell \in \mathbb{N}.$$

$(C_7)_\infty$ For each $1 \le i,j \le n$, if $|u_j| \le |v_j|$, then

$$\theta_i P_i(\ell, u_1, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_n) \le \theta_i P_i(\ell, u_1, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_n),\ \ell \in \mathbb{N}.$$

$(C_8)_\infty$ For each $1 \le i,j \le n$, if $|u_j| \le |v_j|$, then

$$f_i(u_1, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_n) \le f_i(u_1, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_n).$$

Assume $(C_1)_\infty$ holds. Let the operator $S : (C_l(\mathbb{N}))^n \to (C_l(\mathbb{N}))^n$ be defined by

$$Su(k) = \left(Su_1(k), Su_2(k), \ldots, Su_n(k)\right),\ k \in \mathbb{N}, \tag{6.3}$$

where

$$Su_i(k) = \lambda \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u(\ell)), \qquad k \in \mathbb{N}, \quad 1 \le i \le n. \tag{6.4}$$

Clearly, a fixed point of the operator S is a solution of the system (1.2). We shall show that S maps $(C_l(\mathbb{N}))^n$ into itself. Let $u \in (C_l(\mathbb{N}))^n$ and $i \in \{1, 2, \ldots, n\}$ be fixed. We need to show that $\lim_{k\to\infty} Su_i(k)$ exists. Fix $r > 0$. Then, it follows from (C1)$_\infty$ that

$$\left|\sum_{\ell=0}^{\infty}[g_i(k,\ell) - \tilde{g}_i(\ell)]P_i(\ell, u(\ell))\right| \le \sum_{\ell=0}^{\infty}|g_i(k,\ell) - \tilde{g}_i(\ell)|M_{r,i} \to 0$$

as $k \to \infty$. Therefore, as $k \to \infty$ we have

$$Su_i(k) = \lambda\sum_{\ell=0}^{\infty} g_i(k,\ell)P_i(\ell, u(\ell)) \to \lambda\sum_{\ell=0}^{\infty}\tilde{g}_i(\ell)P_i(\ell, u(\ell)).$$

Hence, S maps $(C_l(\mathbb{N}))^n$ into $(C_l(\mathbb{N}))^n$ if (C1)$_\infty$ holds.

Next, we define a cone in B as

$$C = \left\{u \in (C_l(\mathbb{N}))^n \,\middle|\, \text{for each } 1 \le i \le n,\ \theta_i u_i(k) \ge 0 \text{ for } k \in \mathbb{N}, \right.$$

$$\left. \text{and } \min_{k\in Z[a,b]} \theta_i u_i(k) \ge M_i\rho_i|u_i|_0 \right\} \tag{6.5}$$

where M_i and ρ_i are defined in $(C_2)_\infty$ and $(C_6)_\infty$ respectively. Note that $C \subseteq \tilde{K}$. A fixed point of S obtained in C will be a *constant-sign solution* of the system (1.2). For $R > 0$, let

$$C(R) = \{u \in C \mid \|u\| \le R\}.$$

If $(C_1)_\infty$, $(C_4)_\infty$ and $(C_5)_\infty$ hold, then it is clear from (6.4) that for $u \in \tilde{K}$,

$$\lambda\sum_{\ell=0}^{\infty} g_i(k,\ell)a_i(\ell)f_i(u(\ell)) \le \theta_i Su_i(k) \le \lambda\sum_{\ell=0}^{\infty} g_i(k,\ell)b_i(\ell)f_i(u(\ell)), \quad k \in \mathbb{N},\ 1 \le i \le n. \tag{6.6}$$

Lemma 6.1. *Let $(C_1)_\infty$ hold. Then, the operator S is continuous and completely continuous.*

Proof. As in [10] (Chapter 5), $(C_1)_\infty$ ensures that S is continuous and completely continuous.

Lemma 6.2. *Let $(C_1)_\infty - (C_6)_\infty$ hold. Then, the operator S maps C into itself.*

Proof. The proof is similar to that of Lemma 3.2, with the intervals $Z[0, N]$ and I replaced by $\mathbb{N}$.

Theorem 6.1. *Let $(C_1)_\infty - (C_6)_\infty$ hold and let $H_i b_i \in l^1(\mathbb{N})$, $1 \le i \le n$. Then, there exists $c > 0$ such that the interval $(0, c] \subseteq E$.*

Proof. Let $R > 0$ be given. Define

$$c = R\left\{\left[\max_{1\le m\le n}\ \sup_{\substack{|u_j| \le R \\ 1 \le j \le n}} f_m(u_1, u_2, \ldots, u_n)\right]\sum_{\ell=0}^{\infty} H_i(\ell)b_i(\ell)\right\}^{-1}. \tag{6.7}$$

Let $\lambda \in (0, c]$. Using an argument similar to that in the proof of Theorem 3.1 yields $S(C(R)) \subseteq C(R)$. Applying Lemma 6.1 and Schauder's fixed point theorem, we see that S has a fixed point in $C(R)$. Clearly, this fixed point is a constant-sign solution of (1.2) and therefore λ is an eigenvalue of (1.2). Since $\lambda \in (0, c]$ is arbitrary, we have proved that the interval $(0, c] \subseteq E$.

Theorem 6.2. *Let $(C_1)_\infty$, $(C_4)_\infty$ and $(C_7)_\infty$ hold. Suppose that $\lambda^* \in E$. Then, for any $\lambda \in (0, \lambda^*)$, we have $\lambda \in E$, i.e., $(0, \lambda^*] \subseteq E$.*

Proof. Let $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ be the eigenfunction corresponding to the eigenvalue λ^*, i.e.,

$$u_i^*(k) = \lambda^* \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u^*(\ell)), \ \ k \in \mathbb{N}, \ 1 \le i \le n. \tag{6.8}$$

Define

$$K^* = \left\{ u \in (C_l(\mathbb{N}))^n \ \middle|\ \text{for each } 1 \le i \le n, \ 0 \le \theta_i u_i(k) \le \theta_i u_i^*(k), \ k \in \mathbb{N} \right\}.$$

For $u \in K^*$ and $\lambda \in (0, \lambda^*)$, an application of $(C1)_\infty$, $(C_4)_\infty$, $(C_7)_\infty$ and (6.8) gives

$$\begin{aligned} \theta_i S u_i(k) \ &= \ \theta_i \left[\lambda \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u(\ell)) \right] \ \le \ \theta_i \left[\lambda^* \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u^*(\ell)) \right] = \\ &= \ \theta_i u_i^*(k), \ \ k \in \mathbb{N}, \ 1 \le i \le n. \end{aligned}$$

This immediately implies that S maps K^* into K^*. Coupling with Lemma 6.1, Schauder's fixed point theorem guarantees that S has a fixed point in K^*, which is a constant-sign solution of (1.2). Hence, λ is an eigenvalue, i.e., $\lambda \in E$.

Corollary 6.1. *Let $(C_1)_\infty$, $(C_4)_\infty$ and $(C_7)_\infty$ hold. If $E \ne \emptyset$, then E is an interval.*

Proof. The argument is similar to that in the proof of Corollary 3.1, where Theorem 6.2 (instead of Theorem 3.2) is used.

We shall now establish conditions under which E is a bounded or an unbounded interval. For this, we need the following result.

Theorem 6.3. *Let $(C_1)_\infty - (C_6)_\infty$ and $(C_8)_\infty$ hold and let $H_i b_i \in l^1(\mathbb{N})$, $1 \le i \le n$. Suppose that λ is an eigenvalue of (1.2) and $u \in C$ is a corresponding eigenfunction. Let $q_i = |u_i|_0$, $1 \le \le i \le n$. Then, for each $1 \le i \le n$, we have*

$$\lambda \ge \frac{q_i}{f_i(q_1, q_2, \ldots, q_n)} \left[\sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \right]^{-1} \tag{6.9}$$

and

$$\lambda \le \frac{q_i}{f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n)} \left[\sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}. \tag{6.10}$$

Proof. The proof is similar to that of Theorem 3.3, with the intervals $Z[0,N]$ and I replaced by $\mathbb{N}$.

Theorem 6.4. *Let $(C_1)_\infty-(C_8)_\infty$ hold and let $H_ib_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. For each $1 \le i \le n$, let $F_i^B,\ F_i^0$ and F_i^∞ be defined as in Theorem 3.4.*

(a) If $f_i \in F_i^B$ for each $1 \le i \le n$, then $E = (0,c)$ or $(0,c]$ for some $c \in (0,\infty)$.

(b) If $f_i \in F_i^0$ for each $1 \le i \le n$, then $E = (0,c]$ for some $c \in (0,\infty)$.

(c) If $f_i \in F_i^\infty$ for each $1 \le i \le n$, then $E = (0,\infty)$.

Proof. (a) This is immediate from (6.10) and Corollary 6.1.

(b) The argument is similar to that in the proof of Theorem 3.4, with

$$\tilde{K}_i = \left\{ y \in C(\mathbb{N}) \;\middle|\; \lim_{k\to\infty} y(k) \text{ exists and } \theta_i y(k) \ge 0,\ k \in \mathbb{N} \right\}.$$

(c) Let $\lambda > 0$ be fixed. Choose $\varepsilon > 0$ so that

$$\lambda \max_{1\le i\le n} \sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \le \frac{1}{\varepsilon}. \tag{6.11}$$

The rest of the proof is similar to that of Theorem 3.4, with the intervals $Z[0,N]$ and I replaced by $\mathbb{N}$.

7. Subintervals of *E* for (1.2). For each $f_i,\ 1 \le i \le n$, introduced in $(C_5)_\infty$, we shall define

$$\overline{f}_{0,i} = \limsup_{\max_{1\le j\le n}|u_j|\to 0} \frac{f_i(u_1,u_2,\ldots,u_n)}{|u_i|}, \qquad \underline{f}_{0,i} = \liminf_{\max_{1\le j\le n}|u_j|\to 0} \frac{f_i(u_1,u_2,\ldots,u_n)}{|u_i|},$$

$$\overline{f}_{\infty,i} = \limsup_{\min_{1\le j\le n}|u_j|\to \infty} \frac{f_i(u_1,u_2,\ldots,u_n)}{|u_i|} \quad \text{and} \quad \underline{f}_{\infty,i} = \liminf_{\min_{1\le j\le n}|u_j|\to \infty} \frac{f_i(u_1,u_2,\ldots,u_n)}{|u_i|}.$$

Theorem 7.1. *Let $(C_1)_\infty$–$(C_6)_\infty$ hold and let $H_ib_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. If λ satisfies*

$$\hat{\gamma}_{1,i} < \lambda < \hat{\gamma}_{2,i},\ 1 \le i \le n, \tag{7.1}$$

where

$$\hat{\gamma}_{1,i} = \left[\underline{f}_{\infty,i}\, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\hat{\gamma}_{2,i} = \left[\overline{f}_{0,i} \sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $\lambda \in E$.

Proof. The proof is similar to that of Theorem 4.1, with the intervals $Z[0,N]$ and I replaced by $\mathbb{N}$.

The following corollary is immediate from Theorem 7.1.

Corollary 7.1. *Let* $(C_1)_\infty - (C_6)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. *Then,*

$$(\hat{\gamma}_{1,i}, \hat{\gamma}_{2,i}) \subseteq E,\ 1 \le i \le n,$$

where $\hat{\gamma}_{1,i}$ *and* $\hat{\gamma}_{2,i}$ *are defined in Theorem 7.1.*

Corollary 7.2. *Let* $(C_1)_\infty - (C_7)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. *Then,*

$$\left(\min_{1\le i\le n} \hat{\gamma}_{1,i},\ \max_{1\le i\le n} \hat{\gamma}_{2,i} \right) \subseteq E$$

where $\hat{\gamma}_{1,i}$ *and* $\hat{\gamma}_{2,i}$ *are defined in Theorem 7.1.*

Proof. This is immediate from Corollaries 7.1 and 6.1.

Theorem 7.2. *Let* $(C_1)_\infty - (C_6)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. If λ *satisfies*

$$\hat{\gamma}_{3,i} < \lambda < \hat{\gamma}_{4,i},\ 1 \le i \le n, \tag{7.2}$$

where

$$\hat{\gamma}_{3,i} = \left[\underline{f}_{0,i}\, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\hat{\gamma}_{4,i} = \left[\overline{f}_{\infty,i} \sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $\lambda \in E$.

Proof. The proof is similar to that of Theorem 4.2, with the intervals $Z[0,N]$ and I replaced by $\mathbb{N}$.

Theorem 7.2 leads to the following corollary.

Corollary 7.3. *Let* $(C_1)_\infty - (C_6)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. *Then,*

$$(\hat{\gamma}_{3,i}, \hat{\gamma}_{4,i}) \subseteq E,\ 1 \le i \le n,$$

where $\hat{\gamma}_{3,i}$ *and* $\hat{\gamma}_{4,i}$ *are defined in Theorem 7.2.*

Corollary 7.4. *Let* $(C_1)_\infty - (C_7)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. *Then,*

$$\left(\min_{1\le i\le n} \hat{\gamma}_{3,i},\ \max_{1\le i\le n} \hat{\gamma}_{4,i} \right) \subseteq E$$

where $\hat{\gamma}_{3,i}$ *and* $\hat{\gamma}_{4,i}$ *are defined in Theorem 7.2.*

Proof. This is immediate from Corollaries 7.3 and 6.1.

Remark 7.1. For a fixed $i \in \{1, 2, \ldots, n\}$, if f_i is *superlinear* (i.e., $\overline{f}_{0,i} = 0$ and $\underline{f}_{\infty,i} = \infty$) or *sublinear* (i.e., $\underline{f}_{0,i} = \infty$ and $\overline{f}_{\infty,i} = 0$), then we conclude from Corollaries 7.1 and 7.3 that $E = (0, \infty)$, i.e., (1.2) has a constant-sign solution for any $\lambda > 0$.

8. Characterization of *E* for (1.3). Let the Banach space $B = (C(I))^n$ be equipped with norm $\|\cdot\|$ as given in (3.2). Define the operator $S : B \to B$ by (3.3) where

$$Su_i(k) = \lambda_i \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u(\ell)), \;\; k \in I, \; 1 \le i \le n. \tag{8.1}$$

Clearly, a fixed point of the operator S is a solution of the system (1.3).

Next, with the conditions $(C_1)-(C_8)$ stated as in Section 3 and the cone C defined as in (3.5), it is obvious that a fixed point of S obtained in C or $\tilde{K}$ will be a *constant-sign solution* of the system (1.3).

If $(C_1), (C_4)$ and (C_5) hold, then it is clear from (8.1) that for $u \in \tilde{K}$,

$$\lambda_i \sum_{\ell=0}^{N} g_i(k,\ell) a_i(\ell) f_i(u(\ell)) \le \theta_i Su_i(k) \le \lambda_i \sum_{\ell=0}^{N} g_i(k,\ell) b_i(\ell) f_i(u(\ell)), \;\; k \in I, \; 1 \le i \le n. \tag{8.2}$$

Using similar arguments as in Section 3, we obtain the following results.

Lemma 8.1. *Let (C_1) hold. Then, the operator S is continuous and completely continuous.*

Lemma 8.2. *Let $(C_1)-(C_6)$ hold. Then, the operator S maps C into itself.*

Theorem 8.1. *Let $(C_1)-(C_6)$ hold. Then, there exist $c_i > 0, \; 1 \le i \le n$, such that*

$$(0, c_1] \times (0, c_2] \times \ldots \times (0, c_n] \subseteq E.$$

Proof. Let $R > 0$ be given. For each $1 \le i \le n$, define

$$c_i = R \left\{ \left[\max_{1 \le m \le n} \sup_{\substack{|u_j| \le R \\ 1 \le j \le n}} f_m(u_1, u_2, \ldots, u_n) \right] \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right\}^{-1}.$$

Let $\lambda_i \in (0, c_i], \; 1 \le i \le n$. Using a similar technique as in the proof of Theorem 3.1, we can show that $S(C(R)) \subseteq C(R)$. Also, from Lemma 8.1 the operator S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in $C(R)$. Clearly, this fixed point is a constant-sign solution of (1.3) and therefore $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an eigenvalue of (1.3). Since $\lambda_i \in (0, c_i]$ is arbitrary, we have proved that $(0, c_1] \times (0, c_2] \times \ldots \times (0, c_n] \subseteq E$.

Theorem 8.2. *Let $(C_1), (C_4)$ and (C_7) hold. Suppose that $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \in E$. Then, for any $\lambda_i \in (0, \lambda_i^*), \; 1 \le i \le n$, we have $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in E$, i.e.,*

$$(0, \lambda_1^*] \times (0, \lambda_2^*] \times \ldots \times (0, \lambda_n^*] \subseteq E.$$

Proof. Let $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ be the eigenfunction corresponding to the eigenvalue $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$. Thus, we have

$$u_i^*(k) = \lambda_i^* \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u^*(\ell)), \ \ k \in I, \ 1 \le i \le n.$$

Define K^* as in the proof of Theorem 3.2. For $u \in K^*$ and $\lambda_i \in (0, \lambda_i^*), \ 1 \le i \le n$, it follows that

$$\begin{aligned} \theta_i S u_i(k) \ = \ \theta_i \left[\lambda_i \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u(\ell)) \right] \ &\le \ \theta_i \left[\lambda_i^* \sum_{\ell=0}^{N} g_i(k,\ell) P_i(\ell, u^*(\ell)) \right] = \\ &= \ \theta_i u_i^*(k), \ \ k \in I, \ 1 \le i \le n. \end{aligned}$$

Hence, we have shown that $S(K^*) \subseteq K^*$. Moreover, from Lemma 8.1 the operator S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in K^*, which is a constant-sign solution of (1.3). Hence, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an eigenvalue of (1.3).

Theorem 8.3. *Let $(C_1)-(C_6)$ and (C_8) hold. Suppose that $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an eigenvalue of (1.3) and $u \in C$ is a corresponding eigenfunction. Let $q_i = |u_i|_0, \ 1 \le i \le n$. Then, for each $1 \le i \le n$, we have*

$$\lambda_i \ge \frac{q_i}{f_i(q_1, q_2, \ldots, q_n)} \left[\sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1} \tag{8.3}$$

and

$$\lambda_i \le \frac{q_i}{f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n)} \left[\sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}. \tag{8.4}$$

Theorem 8.4. *Let $(C_1)-(C_6)$ and (C_8) hold. For each $1 \le i \le n$, define F_i^∞ as in Theorem 3.4. If $f_i \in F_i^\infty$ for each $1 \le i \le n$, then $E = (0,\infty)^n$.*

Proof. Fix $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in (0,\infty)^n$. Choose $\varepsilon > 0$ so that for each $1 \le i \le n$,

$$\lambda_i \max_{1 \le j \le n} \sum_{\ell=0}^{N} H_j(\ell) b_j(\ell) \le \frac{1}{\varepsilon}. \tag{8.5}$$

By definition, if $f_i \in F_i^\infty, \ 1 \le i \le n$, then there exists $R = R(\varepsilon) > 0$ such that the following holds for each $1 \le i \le n$:

$$f_i(u_1, u_2, \ldots, u_n) < \varepsilon |u_i|, \ \ |u_j| \ge R, \ 1 \le j \le n. \tag{8.6}$$

We shall prove that $S(C(R)) \subseteq C(R)$. To begin, let $u \in C(R)$. By Lemma 8.2, we have $Su \in C$. Thus, it remains to show that $\|Su\| \leq R$. Using (8.2), $(C_3), (C_8)$, (8.6) and (8.5), we find for $k \in I$ and $1 \leq i \leq n$,

$$|Su_i(k)| = \theta_i Su_i(k) \leq$$

$$\leq \lambda_i \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) f_i(u(\ell)) \leq$$

$$\leq \lambda_i f_i(R, R, \ldots, R) \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \leq$$

$$\leq \lambda_i (\varepsilon R) \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \ \leq \ R.$$

It follows that $\|Su\| \leq R$ and hence $S(C(R)) \subseteq C(R)$. From Lemma 8.1 the operator S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in $C(R)$. Clearly, this fixed point is a constant-sign solution of (1.3) and therefore $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an eigenvalue of (1.3). Since $\lambda \in (0, \infty)^n$ is arbitrary, we have proved that $E = (0, \infty)^n$.

9. Subintervals of E for (1.3). Define $\overline{f}_{0,i}$, $\underline{f}_{0,i}$, $\overline{f}_{\infty,i}$ and $\underline{f}_{\infty,i}$ as in Section 4. Using similar arguments as in Section 4, we obtain the following results.

Theorem 9.1. *Let $(C_1)-(C_6)$ hold. For each $1 \leq i \leq n$, if λ_i satisfies*

$$\gamma_{1,i} < \lambda_i < \gamma_{2,i}, \tag{9.1}$$

where

$$\gamma_{1,i} = \left[\underline{f}_{\infty,i} \, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\gamma_{2,i} = \left[\overline{f}_{0,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in E$.

Corollary 9.1. *Let $(C_1)-(C_6)$ hold. Then,*

$$(\gamma_{1,1}, \gamma_{2,1}) \times (\gamma_{1,2}, \gamma_{2,2}) \times \ldots \times (\gamma_{1,n}, \gamma_{2,n}) \subseteq E$$

where $\gamma_{1,i}$ and $\gamma_{2,i}$ are defined in Theorem 9.1.

Theorem 9.2. *Let* $(C_1)-(C_6)$ *hold. For each* $1 \le i \le n$, *if* λ_i *satisfies*

$$\gamma_{3,i} < \lambda_i < \gamma_{4,i} \tag{9.2}$$

where

$$\gamma_{3,i} = \left[\underline{f}_{0,i} \, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\gamma_{4,i} = \left[\overline{f}_{\infty,i} \sum_{\ell=0}^{N} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in E$.

Corollary 9.2. *Let* $(C_1)-(C_6)$ *hold. Then,*

$$(\gamma_{3,1}, \gamma_{4,1}) \times (\gamma_{3,2}, \gamma_{4,2}) \times \ldots \times (\gamma_{3,n}, \gamma_{4,n}) \subseteq E$$

where $\gamma_{3,i}$ *and* $\gamma_{4,i}$ *are defined in Theorem 9.2.*

Remark 9.1. For each $1 \le i \le n$, if f_i is *superlinear* (i.e., $\overline{f}_{0,i} = 0$ and $\underline{f}_{\infty,i} = \infty$) or *sublinear* (i.e., $\underline{f}_{0,i} = \infty$ and $\overline{f}_{\infty,i} = 0$), then we conclude from Corollaries 9.1 and 9.2 that $E = (0, \infty)^n$, i.e., (1.3) has a constant-sign solution for any $\lambda_i > 0$, $1 \le i \le n$.

10. Characterization of *E* for (1.4). Let the Banach space $B = (C(\mathbb{N}))^n$ be equipped with norm $\|\cdot\|$ as given in (6.1). With $(C_l(\mathbb{N}))^n$ given in (6.2), define the operator $S : (C_l(\mathbb{N}))^n \to \to (C_l(\mathbb{N}))^n$ by (6.3) where

$$Su_i(k) = \lambda_i \sum_{\ell=0}^{\infty} g_i(k,\ell) P_i(\ell, u(\ell)), \quad k \in \mathbb{N}, \ 1 \le i \le n. \tag{10.1}$$

Clearly, a fixed point of the operator S is a solution of the system (1.4).

Next, with the conditions $(C_1)_\infty - (C_8)_\infty$ stated as in Section 6 and the cone C defined as in (6.5), it is obvious that a fixed point of S obtained in C will be a *constant-sign solution* of the system (1.4).

If $(C_1)_\infty$, $(C_4)_\infty$ and $(C_5)_\infty$ hold, then it is clear from (10.1) that for $u \in \tilde{K}$,

$$\lambda_i \sum_{\ell=0}^{\infty} g_i(k,\ell) a_i(\ell) f_i(u(\ell)) \le \theta_i S u_i(k) \le \lambda_i \sum_{\ell=0}^{\infty} g_i(k,\ell) b_i(\ell) f_i(u(\ell)), \quad k \in \mathbb{N}, \ 1 \le i \le n. \tag{10.2}$$

Using similar arguments as in Section 6, we obtain the following results.

Lemma 10.1. *Let* $(C_1)_\infty$ *hold. Then, the operator* S *is continuous and completely continuous.*

Lemma 10.2. *Let* $(C_1)_\infty - (C_6)_\infty$ *hold. Then, the operator* S *maps* C *into itself.*

Theorem 10.1. *Let $(C_1)_\infty - (C_6)_\infty$ hold and let $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. Then, there exist $c_i > 0,\ 1 \le i \le n$, such that*

$$(0, c_1] \times (0, c_2] \times \ldots \times (0, c_n] \subseteq E.$$

Proof. Let $R > 0$ be given. For each $1 \le i \le n$, define

$$c_i = R \left\{ \left[\max_{1 \le m \le n} \sup_{\substack{|u_j| \le R \\ 1 \le j \le n}} f_m(u_1, u_2, \ldots, u_n) \right] \sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \right\}^{-1}.$$

The rest of the proof is similar to that of Theorem 8.1.

Theorem 10.2. *Let $(C_1)_\infty$, $(C_4)_\infty$ and $(C_7)_\infty$ hold. Suppose that $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \in E$. Then, for any $\lambda_i \in (0, \lambda_i^*),\ 1 \le i \le n$, we have $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in E$, i.e.,*

$$(0, \lambda_1^*] \times (0, \lambda_2^*] \times \ldots \times (0, \lambda_n^*] \subseteq E.$$

Proof. The proof is similar to that of Theorem 8.2, with K^* defined as in Theorem 6.2.

Theorem 10.3. *Let $(C_1)_\infty - (C_6)_\infty$ and $(C_8)_\infty$ hold and let $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. Suppose that $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an eigenvalue of (1.4) and $u \in C$ is a corresponding eigenfuncti-on. Let $q_i = |u_i|_0,\ 1 \le i \le n$. Then, for each $1 \le i \le n$, we have*

$$\lambda_i \ge \frac{q_i}{f_i(q_1, q_2, \ldots, q_n)} \left[\sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \right]^{-1} \tag{10.3}$$

and

$$\lambda_i \le \frac{q_i}{f_i(M_1\rho_1 q_1, M_2\rho_2 q_2, \ldots, M_n\rho_n q_n)} \left[\sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}. \tag{10.4}$$

Theorem 10.4. *Let $(C_1)_\infty - (C_6)_\infty$ and $(C_8)_\infty$ hold and let $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. For each $1 \le i \le n$, define F_i^∞ as in Theorem 3.4. If $f_i \in F_i^\infty$ for each $1 \le i \le n$, then $E = (0, \infty)^n$.*

The proof is similar to that of Theorem 8.4, where the intervals $Z[0, N]$ and I are replaced by $\mathbb{N}$, and Lemmas 10.1 and 10.2 are used instead of Lemmas 8.1 and 8.2.

11. Subintervals of *E* for (1.4). Define $\overline{f}_{0,i}$, $\underline{f}_{0,i}$, $\overline{f}_{\infty,i}$ and $\underline{f}_{\infty,i}$ as in Section 7. Using similar arguments as in Section 7, we obtain the following results.

Theorem 11.1. *Let $(C_1)_\infty - (C_6)_\infty$ hold and let $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. For each $1 \le i \le$ $\le n$, if λ_i satisfies*

$$\hat{\gamma}_{1,i} < \lambda_i < \hat{\gamma}_{2,i} \tag{11.1}$$

where

$$\hat{\gamma}_{1,i} = \left[\underline{f}_{\infty,i} \, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\hat{\gamma}_{2,i} = \left[\overline{f}_{0,i} \sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in E$.

Corollary 11.1. *Let* $(C_1)_\infty - (C_6)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. *Then,*

$$(\hat{\gamma}_{1,1}, \hat{\gamma}_{2,1}) \times (\hat{\gamma}_{1,2}, \hat{\gamma}_{2,2}) \times \ldots \times (\hat{\gamma}_{1,n}, \hat{\gamma}_{2,n}) \subseteq E$$

where $\hat{\gamma}_{1,i}$ *and* $\hat{\gamma}_{2,i}$ *are defined in Theorem 11.1.*

Theorem 11.2. *Let* $(C_1)_\infty - (C_6)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. *For each* $1 \le i \le$ $\le n$, *if* λ_i *satisfies*

$$\hat{\gamma}_{3,i} < \lambda_i < \hat{\gamma}_{4,i} \tag{11.2}$$

where

$$\hat{\gamma}_{3,i} = \left[\underline{f}_{0,i} \, M_i \rho_i \sum_{\ell=a}^{b} M_i H_i(\ell) a_i(\ell) \right]^{-1}$$

and

$$\hat{\gamma}_{4,i} = \left[\overline{f}_{\infty,i} \sum_{\ell=0}^{\infty} H_i(\ell) b_i(\ell) \right]^{-1},$$

then $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in E$.

Corollary 11.1. *Let* $(C_1)_\infty - (C_6)_\infty$ *hold and let* $H_i b_i \in l^1(\mathbb{N}),\ 1 \le i \le n$. *Then,*

$$(\hat{\gamma}_{3,1}, \hat{\gamma}_{4,1}) \times (\hat{\gamma}_{3,2}, \hat{\gamma}_{4,2}) \times \ldots \times (\hat{\gamma}_{3,n}, \hat{\gamma}_{4,n}) \subseteq E$$

where $\hat{\gamma}_{3,i}$ *and* $\hat{\gamma}_{4,i}$ *are defined in Theorem 11.2.*

Remark 11.1. For each $1 \le i \le n$, if f_i is *superlinear* (i.e., $\overline{f}_{0,i} = 0$ and $\underline{f}_{\infty,i} = \infty$) or *sublinear* (i.e., $\underline{f}_{0,i} = \infty$ and $\overline{f}_{\infty,i} = 0$), then we conclude from Corollaries 11.1 and 11.2 that $E = (0,\infty)^n$, i.e., (1.4) has a constant-sign solution for any $\lambda_i > 0,\ 1 \le i \le n$.

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