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THE DOMAIN OF DEPENDENCE INEQUALITY AND ASYMPTOTIC STABILITY FOR A VISCOELASTIC SOLID*

ОБЛАСТЬ ЗАЛЕЖНОСТІ НЕРІВНОСТІ ТА АСИМПТОТИЧНОЇ СТІЙКОСТІ ДЛЯ В'ЯЗКОЕЛАСТИЧНОГО ТВЕРДОГО ТІЛА

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The existence, uniqueness and asymptotic stability is shown for the integrodifferential system of the viscoelasticity. Moteover a domain of dependence theorem is proved by using the properties of the free energy related with such a system. This theorem provides a finite signal speed and then the hyperbolicity of the integrodifferential system.

Присвячена питанням існування, единості та асимптотичної стійкості розв'язків в'язкоеластичної системи.

1. Introduction. It was shown in [1] that the thermodynamic restrictions imposed on the constitutive equation of the linear viscoelasticity:

$$T(x,t) = G_0(x)\nabla u(x,t) + \int_0^\infty G'(x,s)\nabla u(x,t-s)ds \tag{1}$$

imply existence and uniqueness for the evolutive problem of the linear viscoelasticity subject to boundary Dirichlet conditions. It was also demonstrated that the null solution is attractive under the same restrictions.

Our aim is to present a domain of dependence inequality for solutions to the dynamic equations of linear viscoelasticity. Some results of this topic have been diven in [2, 3], but a general result for viscoelastic motions has not yet proved, because in those papers the maximum propagation speed of disturbances depends on time. In the present paper a different method of approach is adopted, in fact our result relies the properties of Helmholtz free energy potential. These potentials for materials with memory have been deeply studied in [4] and a main result is an explicit expression of the maximal Helmholtz free energy only under the assumption that the constitutive equation (1) obeys to the requirements following by the Second Law of thermodynamics. For this reason, the maximal free energy is used here to prove the domain of dependence inequality, which provides the minimal speed with which energy propagates in Ω.

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Other free energies are also exhibited, but under additional assumptions on the constitutive equation (1).

Our method can be extended to any free energy potential, and used to study non-linear problems in viscoelasticity. Our results show that the hyperbolicity of the dynamic equations of linear viscoelasticity is closely connected with the existence of a free energy.

In this paper we consider the evolutive problem subject to boundary conditions of elastic type and the existence, uniqueness and stability theorems presented in the first part extend the corresponding results of [1]. They are obtained by using the Fourier time-transform method, which implies an information about the solution of the evolution problem in the space time-domain $\Omega \times \mathbf{R}$ from time-harmonic solutions with fixed frequency.

2. Formulation of the problem. The dynamical problem for a continuous linear viscoelastic solid in a smooth bounded domain $\Omega \subset \mathbf{R}^3$, with elastic boundary conditions, is

$$\begin{aligned} &\ddot{u}(x,t) = \nabla T(x,t) + f(x,t), && x \in \Omega, \quad t > 0, \\ &T(\sigma,t)n(\sigma) + \alpha(\sigma)u(\sigma,t) = 0, && \sigma \in \partial\Omega, \quad t > 0, \\ &u(x,\tau) = u^0(x,-\tau), && x \in \Omega, \quad \tau \le 0, \end{aligned} \tag{2}$$

where u denotes the displacement vector, T the stress tensor, f the body force, $u^t(x,s) = u(x,\, t-s)$ the history of the displacement vector, u^0 the initial history, n the outward normal on $\partial\Omega$ and the scalar function $\alpha \in L^2(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfies

$$\alpha(\sigma) \ge \alpha_m > 0 \quad \text{a.e. in } \partial\Omega. \tag{3}$$

The stress-strain relation of the linear viscoelasticity (1) is characterized by the instantaneous elastic modulus G_0 and the Boltzmann G'. We assume that G_0 and G' are symmetric fourth-order tensors*, $G_0 \in C(\overline{\Omega})$ and

$$G' \in L^1(\mathbf{R}^+;\overline{\Omega}) \cap L^2(\mathbf{R}^+;\Omega). \tag{4}$$

The relaxation function

$$G(x,t) = G_0(x) + \int_0^t G'(x,s)ds$$

is continuous in $\overline{\Omega} \times \mathbf{R}^+$, differentiable in $\Omega \times \mathbf{R}^{++}$ and is well-defined along with

$$G_\infty(x) = \lim_{t\to\infty} G(x,t) = G_0(x) + \int_0^\infty G'(x,s)ds.$$

The body is a solid, so we require that C_∞ is uniformly positive definite in Ω, i.e.

$$0 < g_{\infty m}\|A\|^2 \le \inf_{x\in\Omega} AG_\infty(x)A, \quad A \in \mathrm{Sym}\backslash\{0\}. \tag{5}$$

*Throughout this paper Lin is the set of all second-order tensor, Sym the subset of the symmetric second-order tensor and sym denotes the symmetric part of a tensor. A fourth-order tensor G is symmetric if $GA = GA^T$ and $A \cdot GB = B{\cdot}GA$ $A,\, B \in$ Lin.

Let $f \in L^2(\mathbf{R})$, we denote by $\hat{f}$ the Fourier transform: $\hat{f}(\omega) = \int\limits_{\mathrm{R}} \exp(-i\omega s) f(s) ds$. For causal functions, i. e. functions defined on R^+, identified with functions on R which vanish on $(-\infty, 0)$, $\hat{f} = f_c - i f_s$, where f_c and f_s are the half range Fourier sine and cosine transforms:

$$f_c(\omega) = \int\limits_{\mathrm{R}+} f(s) \cos \omega s ds, \quad f_s(\omega) = \int\limits_{\mathrm{R}+} f(s) \sin \omega s ds.$$

A consequence of the second law of thermodynamics for cyclic processes [5] is that $-G'_s(x,\omega)$ is uniformly positive definite in Ω for any $\omega > 0$, i. e., there exists a continuous function g_m: $\mathbf{R}^{++} \to \mathbf{R}^{++}$ such that:

$$g_m(\omega)\|A\|^2 \leq -\inf_{x\in\Omega} AG'_s(x,\omega)A, \quad A \in \mathrm{Sym}. \tag{6}$$

Inequality (6) gives as a consequence that $G'(x,0) = \lim\limits_{\omega\to\infty} \omega G'_s\,(x,\omega)$ is uniformly negative semidefinite in Ω. We require the more restrictive property of definetess, i.e. there exists $g'_0 > 0$ such that:

$$g'_0\|A\|^2 \leq -\inf_{x\in\Omega} AG'(x,0)A, \quad A \in \mathrm{Sym}. \tag{7}$$

Moreover thermodynamic requirements assure that $G_0 - G_\infty$ and thus, G_0 are uniformly positive definite, i.e.

$$0 < g_{0_m}\|A\|^2 \leq \inf_{x\in\Omega} AG_0(x)A, \quad A \in \mathrm{Sym}\backslash\{0\}. \tag{8}$$

At least G_0, G_∞ and $G'_s(\cdot,\omega)$ are uniformly bounded in Ω, i.e. for every $A \in \mathrm{Sym}$

$$\begin{gathered}\sup_{x\in\Omega} G_0(x)A\,A \leq g_{0_M}\|A\|^2, \quad \sup_{x\in\Omega} G_\infty(x)A\,A \leq g_{\infty_M}\|A\|^2, \\ \sup_{x\in\Omega} G'_s(x,\omega)A\,A \leq g_M(\omega)\|A\|^2\end{gathered} \tag{9}$$

with g_{0_M}, g_{∞_M} and $g_M(\omega) < \infty$.

Taking in account the constitutive equation (1) we can rewrite system (2) in the form

$$\begin{gathered}(\ddot{x},t) = \nabla[G_0(x)\nabla u(x,t) + [G' * \nabla u](x,t) + \nabla T_0(x,t)] + f(x,t), \quad x \in \Omega, \ t > 0, \\ [G_0(\sigma)\nabla u(\sigma,t) + [G' * \nabla u](\sigma,t) + T_0(\sigma,t)]n(\sigma) + \alpha(\sigma)u(\sigma,t) = 0, \quad \sigma \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,\end{gathered} \tag{10}$$

where $[G' * \nabla u](x,t) = \int_0^t G'(x,s)\nabla u^t(x,s)ds$, and

$$u_0(x) = u^0(x,0), \quad v_0(x) = -\frac{\partial}{\partial s}u^0(x,s)|_{s=0}, \quad T_0(x,t) = \int\limits_t^\infty G'(\sigma,s)\nabla u^0(\sigma,s-t)ds.$$

System (10) is an integro-differential mixed problem with radiation boundary condition $(10)_2$.

In order to give a variational formulation of problem (10) we introduce the space

$$\mathcal{H}(\Omega,\mathbf{R}^+) = H^1(\mathbf{R}^+, L^2(\Omega)) \cap L^2(\mathbf{R}^+, H^1(\Omega))$$

which is a Hilbert space with the inner product

$$[\varphi_1, \varphi_2] = \int_{\mathbf{R}^+}\int_{\Omega}\{\nabla\varphi_1(x,t)\nabla\varphi_2(x,t) + \dot{\varphi}_1(x,t)\dot{\varphi}_2(x,t)\}dx\,dt +$$

$$+\int_{\mathbf{R}^+}\int_{\partial\Omega}\varphi_1(x,\sigma)\varphi_2(x,t)d\sigma\,dt. \tag{11}$$

This one is equivalent to the usual inner product

$$\int_{\mathbf{R}^+}\int_{\Omega}\{\nabla\varphi_1(x,t)\nabla\varphi_2(x,t) + \varphi_1(x,t)\varphi_2(x,t) + \dot{\varphi}_1(x,t)\dot{\varphi}_2(x,t)\}dx\,dt,$$

since the following estimate holds for functions $f \in H^1(\Omega)$ with Ω bounded, regular open subset of $\mathbf{R}^3$ [6]:

$$||f||^2_{\Omega} \leq K_1||\nabla f||^2_{\Omega} + K_2||f||^2_{\partial\Omega}, \tag{12}$$

where K_1 and K_2 are constants depending on Ω.

Definition 1. *A function $u \in \mathcal{H}(\Omega, \mathbf{R}^+)$ is a weak solution of the initial boundary-value problem (10) with data f, $\nabla \cdot T_0 \in L^2(\mathbf{R}^+, L^2\ (\Omega))$, $u_0 \in H^1(\Omega)$ and $v_0 \in L^2(\Omega)$, if $u(x,0) =$ $= u_0(x)$ almost everywhere in Ω and*

$$\int_{\mathbf{R}^+}\int_{\Omega}\left\{\left[G_0(x)\nabla u(x,t) + \int_0^t G'(x,s)\nabla u^t(x,s)ds\right]\nabla\phi(x,t) -\right.$$

$$\left. -\dot{u}(x,t)\dot{\phi}(x,t)\right\}dx\,dt + \int_{\mathbf{R}^+}\int_{\partial\Omega}\alpha(\sigma)u(\sigma,t)\phi(\sigma,t)d\sigma\,dt =$$

$$= \int_{\Omega} v_0(x)\phi(x,0)dz + \int_{\mathbf{R}^+}\int_{\Omega}[f(x,t)\phi(x,t) - T_0(x,t)\nabla\phi(x,t)]dx\,dt \tag{13}$$

for all $\phi \in \mathcal{H}(\Omega, \mathbf{R}^+)$.

In the next section we prove the following theorem:

Theorem 1. *For any linear viscoelastic solid obeying to (1), with relaxation function G satisfying the consitutive assumptions (4) – (9), the evolutive problem (10), with satisfing (3), $f \in L^2\ (\mathbf{R}^+, L^2\ (\Omega))$, $T_0 \in H^1_0\ (\mathbf{R}^+, H^1_0\ (\Omega))$, and initial data equal to zero, has one and only one weak solution $y \in \mathcal{H}(\Omega, \mathbf{R}^+)$.*

3. Transform problem. Let $\mathcal{H}_{\mathcal{F}}(\Omega, \mathbf{R})$ be the space of the Fourier transforms of functions of $\mathcal{H}(\Omega, \mathbf{R}^+)$. For causal time functions we have

$$\hat{\varphi} \in L^2(\mathbf{R}, H^1(\Omega)), \quad i\omega\hat{\varphi} - \varphi_2 \in L^2(\mathbf{R}, L^2(\Omega)), \quad \text{with } \varphi_0(x) = \lim_{t\to 0^+}\varphi(x,t).$$

Plancherel's theorem for the Fourier transform applied to (11) defines naturally the following inner product on $\mathcal{H}_{\mathcal{F}}(\Omega, \mathbf{R})$:

$$[\hat\varphi_1,\hat\varphi_2]_{\mathcal F}=\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\Omega}\{\nabla\hat\varphi_1(x,\omega)\nabla\hat\varphi_2^*(x,\omega)+[i\omega\hat\varphi_1(x,\omega)-\varphi_{1_0}(x)]\times$$

$$\times[i\omega\hat\varphi_2(x,\omega)-\varphi_{2_0}(x)]^*\}dx\,dt+\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\partial\Omega}\hat\varphi_1(\sigma,\omega)\hat\varphi_2^*(\sigma,\omega)d\sigma\,d\omega,$$

where $\hat\varphi^*$ denotes complex conjugate of $\hat\varphi$, and a natural isomorphism is defined between $\mathcal H$ $(\Omega,\mathbf R^+)$ and $\mathcal H_{\mathcal F}$ $(\Omega,\mathbf R)$ [7]. Let a be the following sesquilinear form on $\mathcal H_{\mathcal F}$ $(\Omega,\mathbf R)$:

$$a(\hat u,\hat\varphi)=\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\Omega}-[i\omega\hat u(x,\omega)-u(x,0)][i\omega\hat\varphi(x,\omega)-\varphi(x,0)]^*dx\,d\omega+$$

$$+\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\Omega}[G_0(x)+\widehat{G}'(x,\omega)]\nabla\hat u(x,\omega)\nabla\hat\varphi^*(x,\omega)dx\,d\omega+$$

$$+\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\partial\Omega}\alpha\hat u(\sigma,\omega)\hat\varphi^*(\sigma,\omega)d\sigma\,d\omega-\int\limits_{\Omega}\dot u(x,0)\varphi(x,0)dx. \tag{14}$$

Plancherel's theorem applied to (13) gives:

$$a(\hat u,\hat\varphi)=\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\Omega}[\hat f(x,\omega)\hat\varphi^*(x,\omega)-\widehat{T}_0(x,\omega)\nabla\hat\varphi^*(x,\omega)]dx\,d\omega, \tag{15}$$

and $\mathcal H_{\mathcal F}(\Omega,\mathbf R)$ is the natural space in which one must find the Fourier transform of the weak solution for the problem (10). Therefore we are able to prove the following lemma:

Lemma 1. *A function $\hat u\in\mathcal H_{\mathcal F}(\Omega,\mathbf R)$ is the Fourier transform of a weak solution of the initial boundary-value problem (10) in the sense of Definition 1 if and only if equality (15) holds for all $\hat\varphi\in\mathcal H_{\mathcal F}(\Omega,\mathbf R)$* .

Due to the properties of causal time functions, the following equalities hold:

$$\hat\varphi(x,0)=\frac{1}{\pi}\int\limits_{\mathbf R}[i\omega\hat\varphi(x,\omega)-\varphi(x,0)]d\omega,$$

$$\varphi(x,0)=\frac{1}{\pi}\int\limits_{\mathbf R}\hat\varphi(x,\omega)\,d\omega,$$

and the sesquilinear form a becomes

$$a(\hat u,\hat\varphi)=\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\Omega}[\omega^2\hat u(x,\omega)+\dot u(x,0)+i\omega u(x,0)]\hat\varphi^*(x,\omega)dx\,d\omega+$$

$$+\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\Omega}[G_0(x)+\widehat{G}'(x,\omega)]\nabla\hat u(x,\omega)\nabla\hat\varphi^*(x,\omega)dx\,d\omega+$$

$$+\frac{1}{2\pi}\int\limits_{\mathbf R}\int\limits_{\partial\Omega}\alpha\hat u(\sigma,\omega)\hat\varphi^*(\sigma,\omega)d\sigma\,d\omega. \tag{16}$$

Substituing (16) in (15) and taking $\dot{\varphi}(x,\omega) = \varphi_1(x)\varphi_2(\omega)$, with $\varphi_1 \in H^1(\Omega)$ and $\varphi_2 \in L^2(\mathbf{R})$, by the arbitrariness choice of φ_2 it follows that, for almost all $\omega \in \mathbf{R}$, the following identity holds:

$$\int_\Omega [\omega^2 \hat{u}(x,\omega) + \dot{u}_0(x) + i\omega u_0(x)]\hat{\varphi}_1^*(x)dx +$$

$$+ \int_\Omega [G_0(x) + \widehat{G}'(x,\omega)]\nabla\hat{u}(x,\omega)\nabla\hat{\varphi}_1^*(x)dx +$$

$$+ \int_{\partial\Omega} \alpha\hat{u}(\sigma,\omega)\hat{\varphi}_1^*(\sigma)d\sigma = \int_\Omega \Big[\hat{f}(x,\omega)\varphi_1^*(x) - \widehat{T}_0(x,\omega)\nabla\varphi_1^*(x)\Big]dx, \tag{17}$$

for every $\varphi_1 \in H^1(\Omega)$. But identity (17) means that $\hat{u}(\cdot,\omega)$ is a generalized solution in $H^1(\Omega)$ for the elliptic problem

$$-\omega^2\hat{u}(x,\omega) - \nabla\Big\{\Big[G_0(x) + \widehat{G}'(x,\omega)\Big]\nabla\hat{u}(x,\omega)\Big\} =$$

$$= \dot{u}_0(x) + i\omega u_0(x) + \hat{f}(x,\omega) + \nabla\widehat{T}_0(x,\omega), \quad x \in \Omega, \tag{18}$$

$$\Big[G_0(\sigma) + \widehat{G}'(\sigma,\omega)\Big]\nabla\hat{u}(\sigma,\omega)n(\sigma) + \alpha(\sigma)\hat{u}(\sigma,\omega) = -T_0(\sigma,\omega)n(\sigma), \quad \sigma \in \partial\Omega.$$

Under the hypotheses of Theorem 1 on initial data, problem (18) becomes

$$-\omega^2\hat{u}(x,\omega) - \nabla\Big\{\Big[G_0(x) + \widehat{G}'(x,\omega)\Big]\nabla\hat{u}(x,\omega)\Big\} = \hat{f}(x,\omega) + \nabla\widehat{T}_0(x,\omega), \quad x \in \Omega,$$

$$\Big[G_0(\sigma) + \widehat{G}'(\sigma,\omega)\Big]\nabla\hat{u}(\sigma,\omega)n(\sigma) + \alpha(\sigma)\hat{u}(\sigma,\omega) = 0, \quad \sigma \in \partial\Omega. \tag{19}$$

Remark 1. The hypotheses of boundeness and positive definetess for G_∞ and $G'_s(\cdot,\omega)$ give

$$g_m(\omega)\|\nabla\hat{u}(\omega)\|^2 \le -\int_\Omega G'_s(x,\omega)\nabla\hat{u}(x,\omega)\nabla\hat{u}^*(x,\omega)dx \le g_M(\omega)\|\nabla\hat{u}(\omega)\|^2, \quad \omega > 0,$$

$$g_{\infty_m}\|\nabla\hat{u}(0)\|^2 \le \int_\Omega G_\infty\nabla\hat{u}(x,0)\nabla\hat{u}^*(x,0)dx \le g_{\infty_M}\|\nabla\hat{u}(0)\|^2, \quad \omega = 0,$$

so that problem (19) is Fredholm solvable in $H^1(\Omega)$ for every source in $L^2(\Omega)$ (see Theorem 4.1 [16, p. 186]) and, as a consequence of Fredholm's Theorems, the existence theorem follows from the uniqueness theorem.

Theorem 2 (uniqueness). *For every $\omega \in \mathbf{R}$ problem (19) has almost one solution $\hat{u}(\cdot,\omega) \in \mathcal{H}^1(\Omega)$.*

Proof. To prove the uniqueness is equivalent to prove that for every $\omega \in \mathbf{R}$ the problem

$$-\omega^2\hat{u}(x,\omega) - \nabla\Big\{\Big[G_0(x) + \widehat{G}'(x,\omega)\Big]\nabla\hat{u}(x,\omega)\Big\} = 0, \quad x \in \Omega,$$

$$\Big[G_0(\sigma)+\widehat{G}'(\sigma,\omega)\Big]\nabla\hat{u}(\sigma,\omega)n(\sigma)+\alpha(\sigma)\hat{u}(\sigma,\omega)=0,\quad \sigma\in\partial\Omega,$$

has only the trivial solution. In this case (17) becomes

$$\int_\Omega\{\omega^2\hat{u}(x,\omega)\hat{\varphi}^*(x)-[G_0(x)+\widehat{G}'(x,\omega)]\nabla\hat{u}(x,\omega)\nabla\hat{\varphi}^*(x,\omega)\}dx+$$

$$+\int_{\partial\Omega}\alpha(\sigma)\hat{u}(\sigma,\omega)\hat{\varphi}^*(\sigma)d\sigma=0. \tag{20}$$

If $\omega=0$, then (20) for $\hat{\varphi}=\hat{u}(\cdot\,,0)$ gives

$$\int_\Omega G_\infty(x)\nabla\hat{u}(x,0)\nabla\hat{u}^*(x,0)dx+\int_{\partial\Omega}\alpha(\sigma)\,|\hat{u}(\sigma,0)|^2d\sigma=0.$$

The symmetry and positive definetess of G_∞, with the positivity of α, yield

$$||\mathrm{sym}\,\nabla\hat{u}(\cdot\,,0)||=0,\quad ||\hat{u}(\cdot\,,0)||_{\partial\omega}=0.$$

Then $\hat{u}(\cdot\,,0)\in H_0^1(\Omega)$ and Korn's inequality [8] yields $||\hat{u}(\cdot\,,0)||=0$.
If $\omega\neq 0$, for $\hat{\varphi}=\hat{u}(\cdot\,,0)$, the immaginary part of (20) gives

$$\int_\Omega G_s'(x,\omega)\nabla\hat{u}(x,\omega)\nabla\hat{u}^*(x,\omega)dx=0.$$

Then assumption (6) ensures $||\mathrm{sym}\,\nabla\hat{u}(\cdot\,,\omega)||=0$. Hence, for $\hat{f}=0$ and $\widehat{T}_0=0$, (20) gives

$$\int_\Omega\omega^2\hat{u}(x,\omega)\hat{\varphi}^*(x)dx=0\quad\forall\varphi\in C_0^\infty(\Omega),$$

and this is equivalent to $||\hat{u}(\cdot\,,\omega)||=0$.

Remark 2. Theorem 1 and Remark 1 assure that the differential operator $\mathcal{T}(\omega)$ defined by system (19) is an isomorphism of $H^1(\Omega)$ onto $L^2(\Omega)$. Since $\mathcal{T}(\omega)$ is a continuous function of ω, then the inverse operator $\mathcal{T}^{-1}(\omega)$ is a continuous function of ω (see Lemma 44.1 of [7]).

The previous remark leads to

Theorem 3. *For every $\omega\in\mathbf{R}$ problem (19) has one and only one solution $\hat{u}(\cdot\,,\omega)\in\mathcal{H}^1(\Omega)$. Besides, the following inequality holds*

$$||\hat{u}_x(\omega)||+||i\omega\hat{u}(\omega)||+||\hat{u}(\omega)||_{\partial\Omega}\le\mathcal{A}(\omega)\left[||\hat{f}(\omega)||+||\omega\widehat{T}_0(\omega)||\right], \tag{21}$$

with $\mathcal{A}\in L^\infty(\mathbf{R})$.

Proof. If $\hat{u}$ is a solution of problem (19), then the following equality holds:

$$\int_\omega\Big\{-|\omega\hat{u}(x,\omega)|^2+\Big[G_0(x)+\widehat{G}'(x,\omega)\Big]\nabla\hat{u}(x,\omega)\nabla\hat{u}^*(x,\omega)\Big\}dx+$$

$$+\int_{\partial\Omega}\alpha(\sigma)\,|\hat{u}(\sigma,\omega)|^2d\sigma=\int_\omega\Big\{\hat{f}(x,\omega)\hat{u}^*(x,\omega)-\widehat{T}_0(x,\omega)\nabla\hat{u}^*(x,\omega)\Big\}dx. \tag{22}$$

The proof is divided in three parts: as first wq consider ω close to 0.

Since $G_0(x) + \widehat{G}'(x, \cdot)$ is a continuous function of ω and

$$\lim_{\omega\to 0}\{G_0(x) + \widehat{G}'(x,\omega)\} = \lim_{\omega\to 0}\{G_0(x) + \widehat{G}'_c(x,\omega)\} = G_\infty(x),$$

there exists ω_1 such that if $|\omega| < \omega_1$

$$\inf_{x\in\Omega} ||G_0(x) + \widehat{G}'(x,\omega)|| \geq \inf_{x\in\Omega} ||G_0(x) + \widehat{G}'_c(x,\omega)|| \geq \frac{1}{2} g_{\infty_m} > 0. \tag{23}$$

Moreover, if we recall (12), it is possibile to find ω_2 such that if $|\omega| < \omega_2 \leq \omega_1$, then

$$||\omega\hat{u}(\omega)||^2 \leq \omega_2^2 \left[K_1||\nabla\hat{u}(\omega)||^2 + K_2||\hat{u}(\omega)||^2_{\partial\Omega}\right] \leq \frac{1}{4} g_{\infty_m}||\nabla\hat{u}(\omega)||^2 + \frac{1}{2}\alpha_m||\hat{u}(\omega)||^2_{\partial\Omega}. \tag{24}$$

Then, if $|\omega| < \omega_2$, the real part of (22), (23) and (24) yield

$$\frac{1}{4} g_{\infty_m}||\nabla\hat{u}(\omega)||^2 + \frac{1}{2}\alpha_m||\hat{u}(\omega)||^2_{\partial\Omega} \leq \int_\Omega [G_0(x) + G'_c(x,\omega)]\nabla\hat{u}(x,\omega)\nabla\hat{u}^*(x,\omega)dx +$$

$$+ \int_{\partial\Omega} \alpha(\sigma)\,|\hat{u}(\sigma,\omega)|^2 d\sigma - \omega^2||\hat{u}||^2 \leq ||\hat{f}(\omega)||\,||\hat{u}(\omega)|| + ||\widehat{T}_0(\omega)||\,||\nabla\hat{u}(\omega)|| \leq$$

$$\leq \left[K_1||\hat{f}(\omega)|| + ||\widehat{T}_0(\omega)||\right]\,||\nabla\hat{u}(\omega)|| + K_2||\hat{f}(\omega)||\,||\hat{u}(\omega)||_{\partial\Omega}, \tag{25}$$

and, with straightforward calculations, (25) leads to

$$||\nabla\hat{u}(\omega)|| + ||\hat{u}(\omega)||_{\partial\Omega} \leq \lambda[||\hat{f}(\omega)|| + ||\widehat{T}_0||] \quad \forall\,|\omega| < \omega_2, \tag{26}$$

where λ depends on g_{∞_m}, α_m, K_1 and K_2.

Inequality (21), for $|\omega| < \omega_2$, follows from (26) and from the classical inequality [16]

$$||\hat{u}||^2_{\partial\Omega} \leq ||\nabla\hat{u}||^2 + K_3||\hat{u}||^2, \tag{27}$$

with the constant K_3 depending on Ω, which holds for functions of $\hat{u} \in H^1(\Omega)$, with Ω bounded and regular domain.

Now we consider ω close to ∞. The immaginary and real parts of (22) give

$$\begin{gathered} g_m(\omega)||\nabla\hat{u}(\omega)||^2 \leq ||\hat{f}(\omega)||\,||\hat{u}(\omega)|| + ||\widehat{T}_0(\omega)||\,||\nabla\hat{u}(\omega)||, \\ ||\omega\hat{u}(\omega)||^2 \leq \beta||\nabla\hat{u}(\omega)||^2 + \alpha_M||\hat{u}(\omega)||^2_{\partial\Omega} + ||\hat{f}(\omega)||\,||\hat{u}(\omega)|| + ||\widehat{T}_0(\omega)||\,||\nabla\hat{u}(\omega)||, \end{gathered} \tag{28}$$

with

$$\alpha_M = \operatorname*{ess\,sup}_{\sigma\in\partial\Omega} \alpha(\sigma), \quad \text{and} \quad \beta = \sup_{(x,\omega)\in\Omega\times\mathbf{R}} |G_0(x) + G'_c(x,\omega)| < \infty,$$

where $|\cdot|$ denotes a norm in the finite dimensional space Lin{Sym, Sym}. Inequalities (27), (28) give

$$||\nabla\hat{u}(\omega)||^2 \leq \frac{1}{\omega g_m(\omega)}||\hat{f}(\omega)||\,||\omega u(\omega)|| + \frac{1}{g(\omega)}||\widehat{T}_0(\omega)||\,||\nabla\hat{u}(\omega)||,$$

$$\|\omega\hat{u}(\omega)\|^2 \leq \frac{\alpha_M + \beta + g_m(\omega)}{g_m(\omega)}\left[\frac{1}{\omega}\|\hat{f}(\omega)\|\,\|\omega u(\omega)\| + \|\widehat{T}_0(\omega)\|\,\|\nabla\hat{u}(\omega)\|\right] + \tag{29}$$

$$+\alpha_M K_3\|\hat{u}(\omega)\|^2$$

and, for $\omega^2 > 2\alpha_M K_3$, $(29)_2$ yields

$$\frac{1}{2}\|\omega\hat{u}(\omega)\|^2 \leq \frac{\alpha_M + \beta + g_m(\omega)}{g_m(\omega)}\left[\frac{1}{\omega}\|\hat{f}(\omega)\|\,\|\omega u(\omega)\| + \|\widehat{T}_0(\omega)\|\,\|\nabla\hat{u}(\omega)\|\right] \tag{30}$$

and, with straightforward calculations, inequalities (29) and (30) lead to

$$\|\nabla\hat{u}(\omega)\| + \|\omega\hat{u}(\omega)\| \leq 2\frac{2\alpha_M + 2\beta + 2g_m(\omega) + 1}{\omega g_m(\omega)}\left[\|\hat{f}(\omega)\| + \|\omega\widehat{T}_0(\omega)\|\right]. \tag{31}$$

The positive definetess of G_0' assures that for $|\omega| \geq \omega_3 > \sqrt{2\alpha_M K_3}$

$$2\frac{2\alpha_M + 2\beta + 2g_m(\omega) + 1}{\omega g_m(\omega)} < 4\frac{2\alpha_M + 2\beta + 3}{g_0'} \tag{32}$$

and, for $|\omega| > \omega_3$, (21) follows from (31), (32) and (27).

At last, the continuity of the inverse operator $\mathcal{T}^{-1}(\omega)$, assures that inequality (21) holds in the compact set: $\omega_2 \leq |\omega| \leq \omega_3$.

Proof [Theorem 1]. Theorem 3 and the hypotheses on the data give

$$\int_R [\|\hat{u}_x(\omega)\|^2 + \|i\omega\hat{u}(\omega)\|^2 + \|\hat{u}(\omega)\|^2_{\partial\Omega}]d\omega \leq \int_{\mathbf{R}} \mathcal{A}^2(\omega)[\|\hat{f}(\omega)\| + \|\omega\widehat{T}_0(\omega)\|]^2 d\omega < \infty,$$

then $\hat{u} \in \mathcal{H}_{\mathcal{F}}(\omega;\mathbf{R})$, and the isomorphism between $\mathcal{H}(\Omega;\mathbf{R}^+)$ and $\mathcal{H}_{\mathcal{F}}(\Omega;\mathbf{R}^+)$ guarantees that $\hat{u}$ is the Fourier transform of the solution $u \in \mathcal{H}\ (\Omega;\mathbf{R}^+)$ of problem (10).

4. Thermodynamic restrictions and domain of dependence inequality. In this section we recall some recent results [4] on thermodynamic potentials which allow us to define the maximal free energy potential as consequences of the requirements on the constitutive equation (1). Subsequently we prove that the energy propagates through the space with finite speed, showing a priori domain of dependence inequality for the evolution problem (2), with initial past history u^0 which has finite maximal free energy.

First we recall further properties of the relaxation function: reguirements (4), (8) and (6) assure that $\omega G_s'\ (x,\cdot) \in L^\infty(\mathbf{R})$ and $G_s'(x,\cdot)$ belong to $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ for every $x \in \Omega$ (see [9], theorem 6.5d), and Fourier inversion formula gives

$$G_\infty(x) - G_0(x) = \frac{2}{\pi}\int_{\mathbf{R}+} \frac{G_s'(x,\omega)}{\omega} d\omega \tag{33}$$

then the boundedness of the left-hand side of (33) and (6) yield

$$\frac{G_s'(x,\omega)}{\omega} \in L^1(\mathbf{R}) \qquad \forall x \in \Omega.$$

In the standart constitutive equation of linear viscoelasticity the stress tensor is a functional of the strain history $E^t = \mathrm{sym}\,\nabla u^t$ and the symmetry of the relaxation function allows one to replace E with ∇u in $(1)_2$. In this section we define thermodynamic potentials as functionals of E.

It is natural to call an admissible history of the deformation gradient, any history for which the stress tensor T is bounded. The constitutive equation (1) allows us to give the following.

Definition 2. *A measurable function $E^t(x,\cdot)$: $\mathbf{R}^+ \to$ Sym is an admissible history if*

$$\left| \int_{\mathbf{R}^+} G'(x,s) E^t(x,s) ds \right| < \infty. \tag{34}$$

We observe that any admissible history may be considerer as a linear continuous function on the space

$$\mathcal{F} = \{V : \mathbf{R}^+ \to \mathrm{Lin}(\mathrm{Sym},\mathrm{Sym}); \quad V = \alpha G' + W, \quad \alpha \in \mathbf{R}, \quad W \in C_0^\infty(\mathbf{R}^+)\},$$

where W has values in Lin (Sym, Sym). We can take the set of admissible histories as large as possible by letting this set equal to $\mathcal{F}'$ (space of all continuous functionals on $\mathcal{F}$). After a straightforward calculation $\mathcal{F}'$ turns out to be the set of histories $E^t \in \mathcal{D}'$ (dual space of C_0^∞ $(\mathbf{R}^+)$) such that (34) holds.

It is possible to give a definition of the Fourier transform of arbitrary distributions in $\mathcal{D}'$ with the use of the Parseval's identity, just as it was for the tempered distributions [10].

Let $\mathcal{Z}$ be space of testing function of rapid decreasing whose Fourier transform are in C_0^∞ $(\mathbf{R}^+)$, and a distribution f belongs to $\mathcal{D}'$, then we call the Fourier transform of the distribution $\hat{f} \in \mathcal{Z}'$ (dual of $\mathcal{Z}$) so that $< \hat{f}, \phi > = < f, \hat{\phi} >$ for every $\hat{\phi} \in \mathcal{Z}$, where $< \cdot\, , \cdot >$ denotes the duality bracket.

This definition extends merely the definition of ordinary Fourier transform, in the sense that the Fourier transform of functions belonging to L^1 or L^2 is a special case of the generalized Fourier transform.

Let ${}_r\mathcal{F}'$ denote the admissible set of all past-histories ${}_rE^t(x,\cdot)$ which are obtained from histories $E^t(x,\cdot) \in \mathcal{F}'$ by restriction to $\mathbf{R}^{++}$. As a consequence we have: $\mathcal{F}'$ =Sym$\times\, {}_r\mathcal{F}'$ and $E^t(x,\cdot) = (E(x,t), {}_rE^t(x,\cdot))$.

Definition 3. *A free energy, relative to the constitutive equation (1) is a functional Ψ : $\mathcal{G} \subset$ Sym$\times_r\mathcal{F}' \to \mathbf{R}^+$ endowed with the following properties:*

(i) *the set $\mathcal{G} \subset \mathcal{F}'$ of admissible histories is such that if $E_0^t \in \mathcal{G}$, then each history E^{t+s}, $s > 0$, for which E^{t+s} $(s+\tau) = E_0^t(\tau)$, is an element of $\mathcal{G}$;*

(ii) Ψ *is continuous and differentiable with respect to the first argument and*

$$T(x,t) = \partial_{E(x,t)} \Psi(E(x,t) {}_rE^t(x,\cdot));$$

(iii) *for each $E^t \in \mathcal{G}$ and $s \geq 0$, such that $\dfrac{\partial}{\partial s}E(t+s)$ is continuous, $\Psi(E^{t+s}$ $(x,\cdot))$ is differentiable with respect to s and satisfies the inequality*

$$\frac{\partial}{\partial s}\Psi(E^{t+s}(x,\cdot)) \leq T(E^{t+s}(x,\cdot))\frac{\partial}{\partial s}E(t+s);$$

(iv) Ψ *is minimal on constant histories i.e. $\Psi(E^\dagger) \leq \Psi(E^t)$, where $E^\dagger$ is the history $E^t(x,s) \stackrel{s}{\equiv} E(x,t)$, and $\Psi(E^t(x,\cdot)) = \dfrac{1}{2}G_\infty(x)$ $E(x,t)\cdot$ $E(x,t)$, if and only if $E^t = E^\dagger$.*

Let $\mathcal{G}_0 = \{E^t : \mathbf{R}^+ \to L^2(\Omega);\ \|E^t(\cdot)\| \in L^2(\mathbf{R}^+)\}$. Since $G'(x,\cdot)$ and $G'_s(x,\cdot)$ belong to $L^1(\mathbf{R}^+) \cap L^2\ (\mathbf{R}^+)$, for every $E^t \in \mathcal{G}_0$, by use of the Plancherel's theorem, the constitutive equation for the stress tensor may be written as

$$T(x,t) = G_\infty(x)E(x,t) + \frac{2}{\pi}\int\limits_{\mathbf{R}^+} G'_s(x,\omega)\left[E^t_s(x,\omega) - \frac{E(x,t)}{\omega}\right]d\omega \tag{35}$$

and the thermodynamic requirements of the constitutive equation allow one to give the following (see [4]).

Theorem 4. *If the relaxation function G satisfies the constitutive assumptions, then for every history $E^t \in \mathcal{G}_0$, the functional*

$$\Psi_M(E^t(x,\cdot)) = \frac{1}{2}G_\infty(x)E(x,t)E(x,t) -$$
$$-\frac{1}{2\pi}\int\limits_{\mathbf{R}^+} \omega G'_s(x,\omega)\left[E^t_s(x,\omega) - \frac{E(x,t)}{\omega}\right]\left[E^t_s(x,\omega) - \frac{E(x,t)}{\omega}\right]^* d\omega$$

is a free energy density.

Proposition 1. *Ψ_M defines a norm, namely*

$$|E^t(x,\cdot)|^2_M \overset{\text{def}}{\equiv} 2\psi_M(E^t(x,\cdot)) \tag{36}$$

and the space $\mathcal{H}_M$, obtained as completion of $\mathcal{G}_0$ relative to this norm, is a Banach space. Moreover the stress tensor T is well-defined and is continuous on $\mathcal{H}_M$, in the sense that

$$|T(x,t)|^2 \le [\,|G_0(x) - G_\infty(x)| + |G_\infty(x)|]\,|E^t(x,\cdot)|^2_M. \tag{37}$$

Proof. As introduction, we remember that for any symmetric and positive definite tensor A, we can define the symmetric and positive definite tensor $\sqrt{A}$, so that $\sqrt{A}\,\sqrt{A} = A$ and $|\sqrt{A}| = \sqrt{|A|}$.

Since G_∞ and $\omega G'_s$ are positive definite, expression (35) of the stress tensor yields

$$|T(x,t)|^2 \le (1+\alpha)|G_\infty(x)|E(x,t)G_\infty(x)E(x,t) - \left(1+\frac{1}{\alpha}\right)\frac{2}{\pi}\left|\int\limits_{\mathbf{R}^+} -\frac{G'_s(x,\omega)}{\omega}d\omega\right| \times$$
$$\times\frac{1}{\pi}\int\limits_{\mathbf{R}^+} \omega G'_s(x,\omega)\left[E^t_s(x,\omega) - \frac{E(x,t)}{\omega}\right]\left[E^t_s(x,\omega) - \frac{E(x,t)}{\omega}\right]^* d\omega.$$

Choosing $\alpha = \dfrac{|G_0(x) - G_\infty(x)|}{|G_\infty(x)|}$, (33) and (36) give (37).

Remark 3. For every history $E^t \in \mathcal{H}_M$, a straightforward calculation yields (see [4])

$$\dot{\Psi}_M(E^t(x,\cdot)) = T(x,t)\dot{E}(x,t). \tag{38}$$

We call total maximal mechanical energy at time t, related to the maximal free energy Ψ_M, the function

$$e_M(t) = \int\limits_{\Omega}\left[\frac{1}{2}\,|\dot{u}(x,t)|^2 + \Psi_M(\nabla u^t(x,\cdot))\right]dx + \frac{1}{2}\int\limits_{\partial\Omega}\alpha(\sigma)\,|u(\sigma,t)|^2 d\sigma.$$

Proposition 2. *For any history u^t, such that $E^t \in \mathcal{H}_M$, a positive constant k exists, such that,*

$$||\dot{u}(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2 \leq k e_M(t). \tag{39}$$

Proof. The properties of definetess of G_∞ and G'_s assure that Ψ_M satisfies

$$\int_\Omega \Psi_M(E^t(x,\cdot))dx \geq \frac{1}{2} g_{\infty_m} ||\text{sym}\, \nabla u(t)||^2$$

so that

$$||\dot{u}(t)||^2 + g_{\infty_m} ||\text{sym}\, \nabla u(t)||^2 + \alpha_m ||u(t)||^2_{\partial\Omega} \leq 2e_M(t). \tag{40}$$

Finally, the Korn's inequality applied to (40) yields (39).

In order to define the class of solutions of the evolutive problem in linear viscoelasticity which saticfies the domain of dependence inequality, we give the following definition:

Definition 4. *A function $u \in \mathcal{H}(\Omega, 0, \tau)$ is a weak solution of the initial boundary-value problem (2) in the space time domain $\Omega \times (0, \tau)$ with initial history u^0 so that sym $\nabla u^0 \in \Phi_0$ and $f \in L^2(\Omega,0,\tau)$ if $u(x,0) = u_0(x,0)$ almost everywhere in Ω and*

$$\int_0^\tau \int_\Omega \Bigg\{ \Big[G_0(x) \nabla u(x,t) + \int_0^\infty G'(x,s) u^t(x,s) ds \Big] \nabla \phi(x,t) -$$

$$- \dot{u}(x,t) \dot{\phi}(x,t) \Bigg\} dx dt + \int_0^\tau \int_{\partial\Omega} \alpha(\sigma) u(\sigma,t) \phi(\sigma,t) d\sigma dt =$$

$$= \int_\Omega v_0(x) \phi(x,0) dx + \int_0^\tau \int_\Omega f(x,t) \phi(x,t) dx dt$$

for every $\phi(x,t) \in \mathcal{H}(\Omega, 0, \tau)$, where $v_0 = -\frac{\partial}{\partial s} u^0(x,s)|_{s=0}$.

Proposition 3. *Let $u \in \mathcal{H}(\Omega, 0, \tau)$ be a weak solution of (2), then*

$$E^t(x,s) = \begin{cases} \text{sym}\nabla u(s, t-s) & \text{if} \quad t > s; \\ \\ \text{sym}\nabla u^0(s, t-s) & \text{if} \quad t \leq s \end{cases}$$

belongs to Φ_0 and

$$\operatorname*{ess\,sup}_{(x,t) \in \Omega \times (0,\tau)} \frac{2|T(x,t)\dot{u}(x,t)|}{|\dot{u}(x,t)|^2 + |(E^t(x,\cdot)|^2_M} = \gamma(\tau) \leq \sqrt{|G_0 - G_\infty| + |G_\infty|}. \tag{41}$$

Proof. A classical algebric inequality and (37) heads to

$$2|T(x,t)\dot{u}(x,t)| \leq \frac{1}{\beta} [|G_0(x) - G_\infty(x)| + |G_\infty(x)|] \, |(E^t(x,\cdot)|^2_M + \beta |\dot{u}(x,t)|^2, \tag{42}$$

with $\beta > 0$. If we choose $\beta^2 = |G_0(x) - G_\infty(x)| + |G_\infty(x)|$, (42) gives (41).

Theorem 5. (Domain of dependence inequality). *Any weak solution of (2) satisfies*

$$\int\limits_{\Omega\cap S_r(x_0)} [|\dot u(x,\tau)|^2 + |(E^r(x,\cdot)|^2_M]dx + \int\limits_{\partial\Omega\cap S_r(x_0)} \alpha(\sigma)|u(\sigma,\tau)|^2 d\sigma \le$$

$$\le \int\limits_{\Omega\cap S_{r+\gamma r}(x_0)} [|\dot u(x,0)|^2 + |(E^0(x,\cdot)|^2_M]dx + \int\limits_{\partial\Omega\cap S_{r+\gamma r}(x_0)} \alpha(\sigma)|u(\sigma,0)|^2 d\sigma +$$

$$+2 \int\limits_{\Omega\cap S_{r+\gamma(r-t)}(x_0)} f(x,t)\dot u(x,t)dx\,dt, \tag{43}$$

where $S_r(x_0) = \{x : |x - x_0| < r\}$ *and* γ *is defined in (41).*

Proof. Let $u : [0,\tau] \to H^1(\Omega)$ be a weak solution of (2), and ϕ a scalar function such that $\phi \in C_0^\infty(\mathbf{R}^3 \times \mathbf{R})$. We introduce

$$e_\phi(t) = \int\limits_\Omega \left[\frac{1}{2}|\dot u(x,t)|^2 + \Psi_M(E^t(x,\cdot)\right]\phi(x,t)dx + \frac{1}{2}\int\limits_{\partial\Omega} \alpha(\sigma)|\dot u(\sigma,t)|^2 d\sigma.$$

The first derivative of e_ϕ and (38) give, after the integration by parts

$$\frac{d}{dt}e_\phi(t) = \int\limits_\Omega [\ddot u(x,t) - \nabla T(E^t(x,\cdot)]\dot u(x,t)\phi(x,t)\,dx +$$

$$+\int\limits_\Omega \left\{\left[\frac{1}{2}|\dot u(x,t)|^2 + \Psi_M(E^t(x,\cdot)\right]\dot\phi(x,t) - T(E^t(x,\cdot)\dot u(x,t)\nabla\phi(x,t)\right\}dx +$$

$$+\frac{1}{2}\int\limits_{\partial\Omega} \alpha(\sigma)|\dot u(\sigma,t)|^2\dot\phi(\sigma,t)d\sigma. \tag{44}$$

The function ϕ is now specialised to have the form

$$\phi(x,t) = \phi_\delta(|x - x_0| - r - \gamma(\tau - t))$$

where γ is defined by (42), $\phi_\delta \in C^\infty(\mathbf{R})$ and

$$\phi_\delta(s) = \begin{cases} 1 & \text{if } s \le -\delta; \\ 0 & \text{if } s > \delta; \end{cases} \qquad 0 \le \phi_\delta(s) \le 1; \quad \phi_\delta'(s) \le 0, \quad \forall s \in \mathbf{R},$$

so that

$$\nabla\psi(x,t) = \nabla|x - x_0|\phi_\delta', \qquad \dot\psi(x,t) = \gamma\phi_s'.$$

Recalling that u is a solution of (2), with this choice of ψ, (44) becomes

$$\frac{d}{dt}e_\phi(t) = \int\limits_\Omega f(x,t)\dot u(x,t)\phi_\delta(|x - x_0| - r - \gamma(\tau - t))dx +$$

$$+\int\limits_\Omega \left\{\gamma\left[\frac{1}{2}|\dot u(x,t)|^2 + \Psi_M(E^t(x,\cdot))\right] - T(E^t(x,\cdot)\dot u(x,t)\nabla(|x - x_0|)\right\} \times$$

$$\times\phi_\delta'(|x-x_0|-r-\gamma(\tau-t))dx+$$

$$+\frac{1}{2}\int\limits_{\partial\Omega}\alpha(\sigma)|\dot u(\sigma,t)|^2\gamma\phi_\delta'(|\sigma-x_0|-r-\gamma(\tau-t))d\sigma\le$$

$$\le\int\limits_{\Omega}f(x,t)\dot u(x,t)\phi_\delta(|x-x_0|-r-\gamma(\tau-t))dx, \tag{45}$$

and a time integration of (45) gives

$$e_\phi(\tau)-e_\phi(t)\le\int\limits_0^\tau\int\limits_\Omega f(x,t)\dot u(x,t)\phi_\delta(|x-x_0|-r-\gamma(\tau-t))dx\,dt. \tag{46}$$

Finally, since ϕ_δ tends boundedly to the characteristic function for $S_{r+\gamma(\tau-t)}(x_0)$ as $\delta\to0$, the passage to the limit in (46) gives (43).

The free energy Ψ_M exists for any relaxation function satisfying the minimal set of properties which have been required to have agreement with thermodynamic principles. On the other hand, in the class of linear viscoelastic materials there is not a unique free energy functional, and for relaxation functions G with further properties is possible to give an explicit representation of other free energy functionals as shown in the following examples.

Example 1. The relaxation function G is compatible with thermodynamic principles, moreover G' is negative definite and G'' is positive semidefinite. Under these hypothesis the functional

$$\Psi_G(E^t(x,\cdot))=\frac{1}{2}G_\infty(x)E^t(x,t)\cdot E^t(x,t)-$$

$$-\frac{1}{2}\int\limits_{\mathbf{R}^+}G'(x,s)[E^t(x,s)-E(x,t)]\cdot[E^t(x,s)-E(x,t)]ds$$

in the "Graffi – Volterra" free energy density [18].

Proposition 4. *Ψ_G defines a norm, namely $|E^t(x,\cdot)|_G^2\stackrel{\text{def}}{\equiv}2\psi_G(E^t(x,\cdot))$ and the space $\mathcal{H}_G$, obtained as a completion of $\mathcal{G}_0$ relative to this norm, is a Banach space. Moreover the stress tensor T is well-defined and is continuous on $\mathcal{H}_G$, in the sense that*

$$|T(x,t)|^2\le[|G_0(x)-G_\infty(x)|+|G_\infty(x)|]|E^t(x,\cdot)|_G^2. \tag{47}$$

Proof. Since G_∞ and $-G'$ are positive definite, (1) yields

$$|T(x,t)|^2\le(1+\beta)|G_\infty(x)|E(x,t)G_\infty(x)E(x,t)-\left(1+\frac{1}{\beta}\right)\left|\int\limits_{\mathbf{R}^+}-G'(x,s)ds\right|\times$$

$$\times\int\limits_{\mathbf{R}^+}G'(x,s)[E^t(x,s)-E(x,t)][E^t(x,s)-E(x,t)]ds, \tag{48}$$

with $\beta>0$. If we choose $\beta=\dfrac{|G_0(x)-G_\infty(x)|}{|G_\infty(x)|}$, (48) gives (47).

Inequality (47) leads to the domain of dependence inequality for weak solutions of (2) with initial history $u^0 \in \mathcal{H}_G$.

In the next example an unidimensional relaxation function is considered:

Example 2. The relaxation function G is given by a sum of exponential functions:

$$G(x,s) = G_\infty(x) + \sum_{k=1}^{n} A_k(x)e^{-a_k s}$$

with a_k, $k = 1, 2, \ldots, N$, positive constants, G_∞ and A_k, $k = 1, 2, \ldots, N$ positive functions belonging to $C(\overline{\Omega} \cap C^1(\Omega))$.

Viscoelastic materials with relaxation functions of this kind have been studied in [12] and [13] (with a defferent approach) and the following free energy functional is defined

$$\Psi_e(E^t(x,\cdot)) = \frac{1}{2}G_\infty(x)E(x,t)E(x,t)+$$
$$+\frac{1}{2}\int\limits_{\mathbf{R}^+}\int\limits_{\mathbf{R}^+} G''(x, s_1+s_2)[E^t(x,s_1) - E(x,t)][E^t(x,s_2) - E(x,t)]ds_1 ds_2.$$

Proposition 5. *Ψ_e defines a norm, namely $|E^t(x,\cdot)|_e^2 \stackrel{\text{def}}{\equiv} 2\psi_e(E^t(x,\cdot))$ and the space $\mathcal{H}_e$, obtained as a completion of $\mathcal{G}_0$ relative to this norm, is a Banach space. Moreover the stress tensor T is well-defined and is continuous on $\mathcal{H}_e$, in the sense that*

$$|T(x,t)|^2 \leq G_0(x)|E^t(x,\cdot)|_e^2. \tag{49}$$

Proof. The constitutive equation (1) for the stress tensor can be rewritten

$$T(x,t) = G_\infty(x,t)E(x,t) + \sum_{k=1}^{n} a_k A_k(x)E_k(x,t),$$

where $E_k(x,t) = \int\limits_{\mathbf{R}} e^{-a_k s}[E^t(x,s_2) - E(x,t)]ds$, and the following inequality holds

$$|T(x,t)|^2 \leq \left(G_\infty(x) + \sum_{k=1}^{n} A_k(x)\right)\left(G_\infty(x)E^2(x,t) + \sum_{k=1}^{n} a_k^2 A_k(x)E_k^2(x,t)\right), \tag{50}$$

and (49) is a consequence of (50) and of the following relations:

$$G_\infty(x) + \sum_{k=1}^{n} A_k(x) = G_0(x); \quad \sum_{k=1}^{n} \alpha_k^2 A_k(x)e^{\alpha_k(s_1+s_2)} = G''(x, s_1+s_2).$$

Remark 4. For the unidimensional models of linear viscoelasticity

$$|G_0(x) - G_\infty(x)| + |G_\infty(x)| = G_0(x)$$

and (37), (47) becomes

$$|T(x,t)|^2 \leq G_0(x)\,|E^t(x,\cdot)|_M^2; \quad |T(x,t)|^2 \leq G_0(x)\,|E^t(x,\cdot)|_G^2.$$

The spaces $\mathcal{H}_G$ and $\mathcal{H}_e$ are larger than $\mathcal{H}_M$ because they include, for example, bounded periodic histories which are not in $\mathcal{H}_M$, so that for relaxation functions defined in Examples 1 and 2, the domain of dependence inequality for the evolutive problem (2) can be proved for a lager class of initial data. However it is important to observe that for initial histories belonging to $\mathcal{G}_0$, the best estimate of the speed of propagation is given by defined in (41), because Ψ_M is the maximal free energy.

5. Hyperbolicity. Inequality (41) assures that the ratio $\dfrac{2|T(x,t)\dot{u}(x,t)|}{|\dot{u}(x,t)|^2+|(E^t(x,\cdot)|_M^2}$ is bounded by the constant $\sqrt{|G_0-G_\infty|+|G_\infty|}$ which depends neither on the time t nor on the solution u. This property gives the domain of dependence inequality (Theorem 5) and sufficient conditions for hyperbolicity of problem (2). In order to prove the last assertion, we recall the definition of hyperbolicity for differential operators of type:

$$\begin{aligned} &Lu(x,t)=0 \quad \text{in } \Omega\times(0,\tau_0),\\ &u(x,0)=u_0(x) \quad \text{in } \Omega. \end{aligned} \tag{51}$$

Definition 5. *The operator L is hyperbolic if for every smooth initial data u_0 satisfying $u_0(x)=0$ for $x\in\Omega\setminus S_r(x_0)$, problem (51) has a unique smooth solution $u(x,t)$ having finite signal speed, i.e. there exists a positive scalar constant c such that u at time t vanishes outside the set $S_{r+ct}\cap\Omega$ (see [14] and [15]).*

The above definition can be extended to an integrodifferential system by substituting the initial data u_0 with the history of u at time $t=0$, $u^{t=0}(x,s)=u^0(x,s)$.

Theorem 6. *Under the hypotheses of Theorem 5 problem (2), with source $f=0$ in $\Omega\times(0,\tau_0)$ and initial history $u^0(x,s)=0$ in $\Omega\setminus S_r(x_0)$, has a unique solution u which at time t vanishes outside the set $\Omega\cap S_{r+ct}\ (x_0)$, with $c=\sqrt{|G_0-G_\infty|+|G_\infty|}$.*

Proof. Let u be a weak solution of (2) and $\bar{x}\in\Omega\setminus(S_{2r+ct}(x_0)\cap\Omega)$. The conditions on the source and the initial history with (43) lead to

$$\int\limits_{\Omega\cap S_r(\bar{x})}[|\dot{u}(x,t)|^2+|(E^t(x,\cdot)|_M^2]dx+\int\limits_{\partial\Omega\cap S_r(\bar{x})}\alpha(\sigma)|u(\sigma,t)|^2d\sigma=0. \tag{52}$$

By virtue of (39), the equality (52) assures that $u(x,t)=0$ in $\Omega\cap S_r(\bar{x})$.

Since $\bar{x}$ is an arbitrary point of the set $\Omega\setminus(S_{2r+ct}(x_0)\cap\Omega)$, we can conclude that $u(x,t)=0$ outside the set $S_{r+ct}\ (x_0)\cap\Omega$.

Remark 5. Theorem 6 emphasizes the importance of the existence of an upper bound for the propagation speed of the disturbances. In fact without this upper bound we cannot obtain the hyperbolicity of the integro differential problem as a consequence of the domain of dependence inequality.

1. *Fabrizio M., Lazzari B.* On existence and the asymptotic stability of solutions for linear viscoelastic solids //Arch. Ration. Mech. and Anal. - 1991. - **116**. - P. 139–152.
2. *Luca I.* Domain of influence and uniqctiess in viscothermoelasticity of integral type //Continuum Mech. Thermodyn. - 1989. - **1**. - P 213–226.
3. *Carbonaro B.* The finite speed of propagation of disturbance in an unbounded linearly viscoelastic material //Boll. Unione mat. ital. - 1993. - **7B**. - P. 875–891.
4. *Fabrizio M., Giorgi C., Morro A.* Free energies and dissipation properties for systems with memory //Arch. Ration. Mech. and Anal. - 1994. - **125**. - P. 341–373.

5. *Fabrizio M., Morro A.* Viscoelastic relaxation function compatible with thermodynamics //J. Elast. – 1988. – **19**. – P. 63–75.

6. *Showalter R.E.* Hilbert space methods for differential equation. – London etc.: Pitman, 1977.

7. *Treves F.* Basic linear partial differential equation. – New York: Acad. press, 1975.

8. *Duvant G., Lions J.* Inequalities in mechanics and physics. – Berlin etc.: Springer, 1976.

9. *Baggett L., Fulks W.* Fourier analysis. – Api Anjou press, Boulder, 1979.

10. *Zemanian A.H.* Distribution theory and transform analysis. – McGraw-Hill Book Company, 1965.

11. *Noll W.* A new theory for simple material //Arch. Ration. Mech. and Anal. – 1977. – **66**. – P. 341–373.

12. *Graffi D., Fabrizio M.* Sulla nozione di stato per materiali viscoelastici di tipo "rate" //Atti. Accad. naz. Lincei Rend. fis. – 1989. – **83**, N$^{\underline{o}}$8. – P. 201–208.

13. *Graffi D., Fabrizio M.* Non unicita dell'energia libera per materiali viscoelastici //Ibid. – P. 209–214.

14. *Hersch R.* How to classify differential polynomials //Amer. Math. Mon. – 1973. – **80**. – P. 641–654.

15. *Davis P.L.* Hyperbolic integrodifferential equations //Proc. Amer. Math. Soc. – 1975. – **47**. – P. 155–160.

16. *Ladyzhenskaya O.A.* The boundary value problem of mathematical physics. – New York etc.: Springer, 1985.

17. *Mikhailov V.P.* Partial differential equations. – Moscow: Mir, 1978.

18. *Graffi D.* Sull'espressione dell'energia libera nei materiali visco-elastici lineari //Ann. mat. pura ed appl. – 1974. – **98**. – P. 273–279.

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