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ON STURM – LIOUVILLE AND THOMAS – FERMI SINGULAR BOUNDARY-VALUE PROBLEMS

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In this paper we investigate the existence of a positive solution of a second order singular Sturm – Liouville boundary-value problem, by constructing upper and lower solutions and combined them with properties of the consequent mapping. Applications to well known Emden – Fowler and Thomas – Fermi boundary-value problems are also presented. Further we generalize some of O'Regan's results, allowing constants in the boundary conditions to be negative.

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1. Introduction

Singular boundary-value problem of the form

$$y'' + q(t)y^{-\sigma} = h(t), \quad 0 < t < 1, \tag{1.1}$$

$$y(0) = a, \qquad y(1) = b, \tag{1.2}$$

$$y(0) = a, \qquad y'(1) = c, \tag{1.3}$$

$$y(0) = a, \qquad \alpha y'(1) + \beta y(1) = \gamma, \tag{1.4}$$

where $\sigma > 0$, $q, h \in C(0,1)$ and $q > 0$ on $(0,1)$, $a \geq 0, b \geq 0$, $c \geq 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$, appears in the study of many physical models. Problems (1.1), (1.2) (respectively (1.1) – (1.3) or (1.1) – (1.4)) may be singular because q and h are allowed to have a suitable singularity at $t = 0$ or $t = 1$ and moreover a, b and c can be equal to zero.

Equation (1.1) with $\sigma > 0$ is known as the generalized *Emden – Fowler* equation with negative exponent and arises frequently in applied mathematics (see [1] and the references cited therein). Thomas [2] and Fermi [3] in order to determine the electrical potential in an atom, derived the also singular equation

$$y'' = t^{-\frac{1}{2}} y^{\frac{3}{2}} \tag{1.5}$$

associated to next boundary conditions:

$$\begin{array}{ll} y(0) = 1,\ by'(1) = y(1) & \text{(neutral atom of Bohr radius 1)}, \\ y(0) = 1,\ y(1) = 0 & \text{(ionized atom)}, \\ y(0) = 1,\ \lim\limits_{t\to\infty} y(t) = 0 & \text{(isolated neutral atom)}. \end{array} \tag{1.6}$$

In the book of O'Regan [4] the study has been extended to the more general Sturm – Liouville boundary-value problem

$$\frac{1}{p(t)}(p(t)y'(t))' + q(t)f(t, y(t), p(t)y'(t)) = 0, \quad 0 < t < 1, \tag{1.7}$$

$$\begin{array}{ll} -\alpha y(0) + \beta \lim\limits_{t\to 0+} p(t)y'(t) = 0, & \alpha > 0,\ \beta \geq 0, \\ \gamma y(1) + \delta \lim\limits_{t\to 1-} p(t)y'(t) = 0, & \gamma \geq 0,\ \delta \geq 0 \text{ and } \gamma^2 + \delta^2 > 0, \end{array} \tag{1.8}$$

and the nonlinear function f may be singular at the point $y = 0$ but not at $t = 0$ or $t = 1$. (He pointed out there, that there is no restriction in assuming homogeneous boundary data.)

His approach relied on the Leray – Schauder alternative for compact maps. More precise existence results for (1.7), (1.8) were given (Theorem 5.1 and Corollary 5.1) for the case where $f \in C([0,1] \times (0,\infty) \times \mathbb{R}, \mathbb{R})$, under the assumptions:

$$p \in C[0,1] \cap C^1(0,1), \text{ with } p > 0 \text{ on } (0,1) \text{ and } \int_0^1 \frac{1}{p(s)} ds < \infty, \tag{1.9}$$

$$q \in C(0,1) \quad \text{with} \quad q > 0 \quad \text{on} \quad (0,1) \quad \text{and} \quad \int_0^1 p(x)q(x)dx < \infty. \tag{1.10}$$

It holds $f(t,u,v) = g(t,u,v) + h(t,u,v)$ where g and h satisfy

$$\begin{array}{l} |h(t,u,v)| \leq K\{|u|^\gamma + |v|^\tau + 1\},\ 0 < \gamma, \tau < 1, \\ ug(t,u,v) \geq c\,|u|^2 + d\,|uv|\,,\ d \leq 0 \text{ and} \\ |g(t,u,v)| \leq A(t,u)\,|v|^2 + B(t,u), \\ \text{where } A \text{ and } B \text{ are bounded on bounded sets} \end{array} \tag{1.11}$$

and finally

$$p(t)\sqrt{q(t)} \text{ is bounded on } [0,1]. \tag{1.12}$$

We must notice that (1.11) replaces in some sense the (necessary) growth rate condition of Nagumo.

In this paper, we examine the same existence problem (1.7), (1.8) but we use the method of upper and lower solutions combined with properties of the consequent mapping. More precisely, we assume here only that

A) there exist $m, K > 0$ and $n = 2k+1,\ k = 1, 2, \ldots$, such that

$$\left|p(t)q(t)f(t, y(t), p(t)y'(t))\right| \leq \frac{K}{p(t)}\left|p(t)y'(t))\right|^{n+1},\ \text{ if }\ \left|p(t)y'(t))\right| \geq m. \tag{1.13}$$

We observed that this condition *leads to the construction of an "upper" and "lower" solution of* (1.7). Moreover we introduce the new concept of *strong pair* of upper and lower solutions and so (1.13) returns some bounds for the derivative $p(t)y'(t)$ of any solution which stays between lower and upper solutions, ensuring in this way that the consequent map is well-defined. The properties of the last map give occasion of existence results.

In this paper, further of an existence result we give an estimation for the derivative y', assuming a rather mild growth rate on f (see (1.13)) and so we don't use the Nagumo type condition (1.11) at all. Further our main purpose is to extend the validity of existence results and/or for some cases when constants $\alpha,\ \beta,\ \gamma$ and δ *are negative* (see Remark 4.1).

Furthermore motivated by the Thomas Fermi equation (1.5), we establish an existence result to generalized Bohr boundary-value problem (1.7), (1.8), where

$$y(0) = a \quad \text{and} \quad \left(p(1)\int_0^1 \frac{ds}{p(s)}\right) y'(1) + y(1) = 0, \tag{1.14}$$

under rather general assumptions (see Theorem 5.1). A similar problem has been studied by O'Regan (see [4], Theorem 5.3) for the case when f is *independent of its last argument*.

The paper is organized as follows. In Section 2 we introduce notations, definitions and preliminary facts, which are used throughout this paper. The basic existence result in given in Section 3. Finally, two applications are given, the first for a singular second order Sturm – Liouville boundary-value problem in Section 4 and the second for a generalized Bohr boundary-value problem in Section 5.

2. Preliminaries

Consider the initial value problem (IVP for short)

$$\frac{1}{p(t)}(p(t)x'(t))' + q(t)f(t, x(t), p(t)x'(t)) = 0,\ 0 < t < 1, \tag{2.1}$$

$$(\tau, x(\tau), p(\tau)x'(\tau)) = (\tau, \xi, \eta) = P \in [0,1] \times \mathbb{R}^2 = \Omega, \tag{2.2}$$

and let $\mathcal{X}(P)$ be the set of all (noncontinuable) solutions of IVP (2.1), (2.2), that is $x(\tau) = \xi$ and $\lim\limits_{t\to\tau} p(t)x'(t) = \eta$.

Definition 2.1. *Consider two functions* α *and* $\beta \in C[0,1] \cap C^2(0,1)$ *with* $p\alpha'$ *and* $p\beta' \in C^1[0,1]$ *and such that* $\alpha(t) \leq \beta(t),\ t \in [0,1]$.

Then α is called lower solution to (2.1) if

$$\frac{1}{p(t)}(p(t)\alpha'(t))' + q(t)f(t,\alpha(t),p(t)\alpha'(t)) > 0, \quad 0 < t < 1,$$

and β is called upper solution to (2.1) if

$$\frac{1}{p(t)}(p(t)\beta'(t))' + q(t)f(t,\beta(t),p(t)\beta'(t)) < 0, \quad 0 < t < 1.$$

Further (α,β) is a strong pair of lower and upper solutions, iff

$$\beta'(t) \le \alpha'(t), \ 0 \le t \le 1,$$

where if $\lim\limits_{t\to 0+} p(t) = 0$ and/or $\lim\limits_{t\to 1-} p(t) = 0$, it may be

$$\lim_{t\to 0+} \beta'(t) = \beta'(0) = -\infty \text{ and/or } \lim_{t\to 1-} \beta'(t) = \beta'(1) = -\infty$$

and similarly

$$\lim_{t\to 0+} \alpha'(t) = \alpha'(0) = +\infty \text{ and/or } \lim_{t\to 1-} \alpha'(t) = \alpha'(1) = +\infty,$$

respectively.

Lemma 2.1. *Let (α,β) be a strong pair of lower and upper solutions p and q satisfying (1.9), (1.10) and further the function $f(t,.,px')$ be nonincreasing for any $(t,px') \in [0,1]\times\mathbb{R}$.*

Then for any solution $x = x(t)$ of (2.1) with

$$\beta'(0) \le x'(0) \le \alpha'(0) \ \textit{ and } \ \alpha(t) \le x(t) \le \beta(t), \ t \in [0,1]\cap \mathrm{Dom}\,(x),$$

the following inequality folds true:

$$\beta'(t) \le x'(t) \le \alpha'(t), \ t \in [0,1]\cap \mathrm{Dom}\,(x).$$

Moreover any such solution $x = x(t)$ can be defined on $[0,1]$ and

$$\left|p(t)x'(t)\right| \le M = \max_{t\in[0,1]}\{\left|p(t)\alpha'(t)\right|, \ \left|p(t)\beta'(t)\right|\}.$$

Proof. Let's suppose that there are $s,\tau \in (0,1)$ such that

$$\alpha(t) \le x(t) \le \beta(t), \ t \in [0,\tau], \ \beta'(t) \le x'(t) \le \alpha'(t), \ t \in [0,s],$$

and $\beta'(t) > x'(t), \quad t \in (s,\tau]$. Then since β is an upper solution to (2.1), by the monotonicity of f,

$$\begin{aligned}\frac{1}{p(s)}(p(s)\beta'(s))'(s) &< -q(s)f(s,\beta(s),p(s)\beta'(s))\\ &= -q(s)f(s,\beta(s),p(s)x'(s))\\ &\le -q(s)f(s,x(s),p(s)x'(s)) = \frac{1}{p(s)}(p(s)x'(s))'(s).\end{aligned}$$

Consequently we get

$$\beta''(s) < x''(s) \text{ and } \beta'(s) = x'(s)$$

and thus inequality $\beta'(t) > x'(t), \quad t \in (s, \tau]$ cannot be true.

By a similar argument we can easily prove that $x'(t) \leq \alpha'(t), t \in [0,1] \cap \mathrm{Dom}\,(x)$. The last conclusion follows by the well-known theorem of extensibility of solutions of (2.1), since these and their derivatives px' are bounded.

The lemma is proved.

Consider now a subset $\hat{\omega}$ of Ω such that

$$\hat{\omega}^{\circ} \neq \emptyset \quad \text{and} \quad \Omega - \overline{\hat{\omega}} \neq \emptyset,$$

and let

$$\hat{\omega}(\tau) := \{(t,x,y) \in \hat{\omega} : t = \tau\}$$

be its *cross-section* at $t = \tau$.

A point $P = (\tau, \xi, \eta) \in \partial\hat{\omega}$ is a point of *egress* of $\hat{\omega}$ (with respect to the system (2.1)), iff for any solution $x \in \mathcal{X}(P)$ there exists $\varepsilon > 0$ such that the *graph* of the restriction $x|[\tau - \varepsilon, \tau]$ is in $\hat{\omega}^{\circ}$, i.e.,

$$G(x|[\tau - \varepsilon, \tau); P) := \{(t, x(t), x'(t)) : \tau - \varepsilon \leq t < \tau\} \subseteq \hat{\omega}^{\circ}.$$

If moreover for all solutions $x \in \mathcal{X}(P)$ there is $\varepsilon > 0$ such that

$$G(x|(\tau, \tau + \varepsilon]; P) \subseteq \Omega - \overline{\hat{\omega}},$$

then P is called a *strict egress point* (see [5] or [6]). As usual the set of egress (strict egress) points of $\hat{\omega}$ will be denoted by $\hat{\omega}^{e}$ (respectively $\hat{\omega}^{se}$).

Remark 2.1. Let now consider the set

$$\omega := \{(t, x, x') : 0 \leq t \leq 1, \ \alpha(t) \leq x \leq \beta(t), \ x' \in \mathbb{R}\},$$

and let

$$Q'_{\alpha} = Q'_{\alpha}\,[0,1] := \{(t, x, x') \in \partial\omega : 0 \leq t \leq 1, \ x = \alpha(t) \text{ and } x' \leq \alpha'(t)\} \text{ and}$$
$$Q'_{\beta} = Q'_{\beta}\,[0,1] := \{(t, x, x') \in \partial\omega : 0 \leq t \leq 1, \ x = \beta(t) \text{ and } x' \geq \beta'(t)\}.$$

Then we can easily see [7] that every egress point of ω is a strict egress one and to be more specific we have

$$\omega^{e} = \omega^{se} = Q'_{\alpha} \cup Q'_{\beta} \cup \omega(1). \tag{2.3}$$

A point $P = (\tau, \xi, \eta) \in \omega^e$ is a *consequent* of another one $P_0 = (t_0, x_0, x_0') \in \omega^\circ,\ t_0 \leq \tau$, iff there exists a solution $x \in \mathcal{X}(P_0, P) := \mathcal{X}(P_0) \cap \mathcal{X}(P)$ such that

$$G(x|(t_0, \tau); P_0, P) \subseteq \omega^\circ.$$

The set of consequent points of $P_0 \in \omega^\circ$ will denoted by $\mathcal{K}(P_0)$ and the so defined (set-valued) mapping

$$\mathcal{K} : S(\omega) \subseteq \omega^\circ \to \omega^e, \quad S(\omega) = \{Q \in \omega^\circ : \mathcal{K}(Q) \neq 0\}$$

will be referred as the consequent mapping.

Consider a set-valued mapping F, which maps the points of a topological space X into compact subsets of another one Y. F is *upper semi-continuous* (u.s.c.) at $x_0 \in X$ iff for any open subset V in Y with $F(x) \subseteq Y$, there exists a neighbourhood U of x_0 such that $F(x) \subseteq V$, for every $x \in U$.

The next lemmas (see [8]) give sufficient conditions for the upper semicontinuity of the consequent mapping and some useful properties for a class of u.s.c. mappings. We notice that the consequent mapping is included in this class [9].

Lemma 2.2. *If $P \in S(\omega)$ and every solution $x \in \mathcal{X}(P)$ egresses strictly from ω, then the consequent mapping $\mathcal{K}$ is u.s.c. at the point P and furthermore the image $\mathcal{K}(P)$ is a continuum in $\partial\omega$.*

Lemma 2.3. *Let X and Y be metric spaces and let $F : X \to 2^Y$ be an u.s.c. mapping. If A is a continuum subset of X, such that for every $x \in A$ the image $F(x)$ is a continuum, then the image $F(A) = \cup\{F(x) : x \in A\}$ is also a continuum subset of Y.*

Notice that, if an IVP has a unique solution, then the consequent map is simply continuous.

We shall need another Lemma (see [9] or [10], Chapter V, Theorem 2).

Lemma 2.4. *If A is an arbitrary proper subset of a continuum C and if S is a connected component of A, then*

$$clS \cap cl(C \backslash A) \neq \emptyset, \quad \textit{i.e. } \bar{S} \cap \partial A \neq \emptyset.$$

3. Existence Results

Theorem 3.1. *Suppose that assumptions (1.9), (1.10) are fulfilled and further that there is a constant $M > 0$ such that*

$$|f(t, u, v)| \leq M, \quad 0 \leq t \leq 1,\ (u, v) \in \mathbb{R}^2. \tag{3.1}$$

Then for any $y_0 \in [\alpha(0), \beta(0)]$, the equation (2.1) has a global solution $y = y(t),\ 0 \leq t \leq 1$ (i.e. $[0, 1] = \mathrm{Dom}\,(x)$) with $y(0) = y_0$. (Here $\alpha(0), \beta(0)$ are the values at $t = 0$ of an upper solution $\alpha(t)$ and a lower solution $\beta(t)$ of (2.1) respectively.)

Proof. Consider the Banach space

$$K^1[0,1] = \{u \in C[0,1] : \ pu' \in C[0,1] \text{ with norm } \|u\|_1\},$$

where

$$\|u\|_1 = \max\{\|u\|, \ \|pu'\|\}$$

and $\|u\|$ is the usual sup-norm of u on $[0,1]$. Let also

$$T = K^1[0,1] \to K^1[0,1]$$

be an operator defined by

$$(Ty)(t) := y(0) - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)f(x, y(x), p(x)y'(x))dxds.$$

We claim that T has a fixed point in

$$K_0^1[0,1] \ = \{u \in K^1[0,1] : u(0) \in [\alpha(0), \beta(0)]\}.$$

To prove the compactness of T, we notice that there exist K_0 and K_1 such that

$$|(Ty)(t)| \leq |y(0)| + M \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds \leq K_0$$

and

$$\left|p(t)(Ty)'(t)\right| \leq M \int_0^t p(x)q(x)dx \leq K_1,$$

i.e.,

$$\|(Ty)\|_1 \leq K = \max\{K_0, K_1\}, \ y \in K_0^1[0,1].$$

Furthermore $TK_0^1[0,1]$ is an equicontinuous family, since

$$\begin{aligned} \left|(Ty)(t) - (Ty)(t')\right| &< \left| \int_{t'}^t \frac{1}{p(s)} \int_0^s p(x)q(x)f(x, y(x), p(x)y'(x))dxds \right| \\ &\leq M \left|\phi(t) - \phi(t')\right| \end{aligned}$$

and

$$\left|p(t)(Ty)'(t) - p(t)(Ty)(t')\right| < \left|\int_{t'}^{t} p(x)q(x)f(x, y(x), p(x)y'(x))dx\right|$$

$$\leq M \left|\phi^*(t) - \phi^*(t')\right|, \ y \in K_0^1[0,1],$$

because the functions

$$\phi(t) = \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds \ \text{ and } \ \phi^*(t) = \int_0^t p(x)q(x)dx$$

are absolutely continuous. Finally by an application of the standard Schauder fixed point theorem, we get a solution $y \in \mathcal{X}(P), \ P = (0, y_0, 0)$, i.e., $\mathcal{X}(P) \neq \emptyset$ and any $y \in \mathcal{X}(P)$ is defined over the entire interval $[0, 1]$.

The theorem is proved.

Let now V be any subset of Ω and J a subset of the interval $[0, 1]$. Consider the cross-section of V,

$$V(J) = \{(t, y, y') \in V : t \in J\},$$

over the interval J (J may be a single-point set). Next assume that

$$0 \leq \alpha(t) \leq \beta(t), \ t \in [0,1] \tag{3.2}$$

and consider a line $E_0 = \{(0, y, y') \in \omega(0) : y = y_0 \in [\alpha(0), \beta(0)]\}$, where we recall that $\omega = \{(t, y, y') : 0 \leq t \leq 1, \ \alpha(t) \leq y \leq \beta(t), \ y' \in \mathbb{R}\}$.

Theorem 3.2. *Assume that hypotheses of Theorem 3.1 are still holding. Then there exist points* $P_1 = (0, y_0, y_1') \in E_0$ *and* $P_2 = (0, y_0, y_2') \in E_0$ *such that any solution* $y \in \mathcal{X}(P_1)$ *egresses from* ω *through the surface*

$$S_\alpha := \{(t, y, y') \in \overline{\omega} : y = \alpha(t)\}$$

and similarly any solution $y \in \mathcal{X}(P_2)$ *egresses from* ω *through the surface*

$$S_\beta := \{(t, y, y') \in \overline{\omega} : y = \beta(t)\}.$$

Proof. For any $x \in \mathcal{X}(P_2)$ we have

$$y(t) = y_2' \int_0^t \frac{1}{p(x)} dx + y_0 - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)f(x, y(x), p(x)y'(x))dxds.$$

Consequently by the conditions (3.1) and (3.2) we get

$$y(t) \geq y_2' \int_0^t \frac{1}{p(x)}dx - |y_0| - M \left| \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds \right|.$$

Thus, if we choose y_2' large enough we get $y(t) > \beta(t)$ for some $0 < t \leq 1$, given that (1.9), (1.10) hold true.

The theorem is proved.

We are ready to state and prove a general existence result. Let $E_0 \subseteq \omega(0)$ and $E_1 \subseteq \omega(1)$ be two continua.

Theorem 3.3. *Suppose that there is a strong pair (α, β) of upper and lower solutions and assumptions (1.9), (1.10) and (3.1) are fulfilled. Assume further that the function $f(t, ., px')$ is nonincreasing for any $(t, px') \in [0,1] \times \mathbb{R}$. Moreover*

$$\begin{aligned} \mathcal{K}(E_0) \cap Q'_\alpha[0,1] \neq \emptyset \quad &\text{and} \quad \mathcal{K}(E_0) \cap Q'_\beta[0,1] \neq \emptyset, \\ E_1 \cap Q'_\alpha(1) = \emptyset \quad &\text{and} \quad E_1 \cap Q'_\beta(1) = \emptyset. \end{aligned} \tag{3.3}$$

Then the boundary-value problem

$$\frac{1}{p(t)}(p(t)y'(t))' + q(t)f(t, y(t), p(t)y'(t)) = 0, \quad 0 < t < 1,$$

$$y \in \mathcal{X}(E_0) \cap \mathcal{X}(E_1)$$

has at least a solution $y = y(t)$ such that

$$\alpha(t) \leq y(t) \leq \beta(t) \text{ and } \beta'(t) \leq y'(t) \leq \alpha'(t), \quad 0 \leq t \leq 1.$$

Proof. By Lemma 2.1, Remark 2.1 and Theorem 3.1, any solution $y \in \mathcal{X}(E_0)$ is defined on entire the interval $[0,1]$. Thus the consequent mapping $\mathcal{K}$ is well defined and in view of Lemma 2.2 its image $\mathcal{K}(E_0)$ is a continuum subset of $\partial\omega$. Thus

$$S_\alpha(1) \cap \mathcal{K}(E_0) \neq \emptyset \text{ and } S_\beta(1) \cap \mathcal{K}(E_0) \neq \emptyset.$$

Since now both the sets $S_\alpha(1) \cap \mathcal{K}(E_0)$ and $S_\beta(1) \cap \mathcal{K}(E_0)$ are clearly compact by Lemma 2.4, we conclude that there is a connected component E_1^* of $\omega(1) \cap \mathcal{K}(E_0)$ such that

$$E_1^* \cap S_\alpha(1) \neq \emptyset \text{ and } E_1^* \cap S_\beta(1) \neq \emptyset,$$

i.e.,

$$E_1^* \cap Q'_\alpha(1) \neq \emptyset \text{ and } E_1^* \cap Q'_\beta(1) \neq \emptyset.$$

Consequently in view of (2.3), by assumptions (3.3) we get

$$E_1^* \cap E_1 \neq \emptyset,$$

that is, there exists a solution $y \in \mathcal{X}(E_0) \cap \mathcal{X}(E_1)$ such that

$$\alpha(t) \leq y(t) \leq \beta(t) \text{ and } \beta'(t) \leq y'(t) \leq \alpha'(t), \quad 0 \leq t \leq 1,$$

where the last estimates follow by the definition of the consequent mapping $\mathcal{K}$ and Lemma 2.1.

4. Sturm – Liouville Boundary-Value Problems

In this section an existence result for the singular Sturm – Liouville boundary-value problem

$$\frac{1}{p(t)}(p(t)y'(t))' + q(t)f(t, y(t), p(t)y'(t)) = 0, \quad 0 < t < 1, \tag{4.1}$$

$$\begin{aligned} &-\alpha y(0) + \beta \lim_{t\to 0+} p(t)y'(t) = c, \quad \alpha \neq 0,\ \beta > 0, \\ &\gamma y(1) + \delta \lim_{t\to 1-} p(t)y'(t) = d, \qquad \gamma \neq 0,\ \delta > 0 \end{aligned} \tag{4.2}$$

will be given, where *we do not assume that α and/or γ are necessarily positive.* The singularity of the nonlinear function f may occurs at $y = 0$ and the functions $1/p, q$ and $1/q$ may be singular at $t = 0$ or/and $t = 1$.

Assume the next conditions:

(A_1) $q \in C(0,1)$ with $q > 0$ on $(0,1)$;

(A_2) $p \in C[0,1] \cap C^1(0,1)$ with $p > 0$ on $(0,1)$,

$$\int_0^1 \frac{dt}{p(t)} < \infty \quad \text{and} \quad \int_0^1 p(t)q(t)dt < \infty;$$

(A_3) $f : [0,1] \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and further the function $f(t,.,px')$ is nonincreasing for any $(t,px') \in [0,1] \times \mathbb{R}$.

Theorem 4.1. *Suppose that conditions $(A_1) - (A_3)$ are satisfied. Moreover assume:*

(A_4) *There exist constants $K > 0$, $0 < a_0 < b_0$, a natural number $n = 2k+1,\ k = 1, 2, \ldots$, and further*

$$0 < m < \frac{1}{\sqrt[n]{Kn \displaystyle\int_0^1 \frac{dt}{p(t)}}}$$

such that

$$|p(t)q(t)f(t,u,v)| < \frac{K}{p(t)}\,|v|^{n+1}\,,\ 0 < t < 1,\ a_0 \le u \le b_0, \textit{and}\ \ |v| \ge m, \tag{4.3}$$

and finally

(A_5) *The constants of the Sturm – Liouville boundary conditions satisfy*

$$\begin{gathered} -\alpha b_0 - \beta m \le c \le \beta m - \alpha a_0,\ if\ p_0 > 0, \\ \gamma\widehat{\alpha}_0 + \delta\widehat{m}_0 < d < \ \gamma\widehat{b}_0 + \delta\widehat{m}_1,\ if\ p_1 > 0,\ and \\ a_0 < -\frac{c}{\alpha} < b_0, if\ p_0 = 0\ and\ \widehat{\alpha}_0 < \frac{d}{\gamma} < \widehat{b}_0,\ if\ p_1 = 0. \end{gathered} \tag{4.4}$$

Then the boundary-value problem (4.1), (4.2) has a solution $y \in C[0,1] \cap C^1(0,1)$ *(with* $px' \in AC([0,1])$*) such that*

$$\alpha(t) \le y(t) \le \beta(t), \quad 0 \le t \le 1.$$

Furthermore its derivative satisfies also the estimate

$$\beta'(t) \le y'(t) \le \alpha'(t), \quad 0 \le t \le 1,$$

where the constants $\widehat{\alpha}_0,\ \widehat{b}_0,\ \widehat{m}_0,\ \widehat{m}_1,\ p_0,\ p_1$ *and the functions* $\alpha(t)$ *and* $\beta(t),\ 0 \le t \le 1$*, will be defined below.*

Remark 4.1. For $\alpha = 0 = c$ and $\delta = 0 = d$, the boundary-value problem (4.1), (4.2) has also a solution $y \in C[0,1] \cap C^1(0,1)$ as above and this has been proved in a recent paper of the authors (see [7]).

By (4.4) we can see that the slope α/β of the line

$$E_0 = \{(t,y,y') \in \omega(0) : -\alpha y(0) + \beta p_0 y'(0) = c\}$$

must satisfy $\alpha/\beta \in [-(\beta m + c)/(\beta b_0), (\beta m - c)/(\beta a_0)]$ and so it may accept negative values and this extends the usual case of $\alpha, \beta \ge 0$. At the same time, since

$$-\frac{\gamma}{\delta} \in \left(\frac{\delta\hat{m} - d}{\delta a_0}, \frac{\delta\hat{m} - d}{\delta b_0}\right),$$

the slope $-\gamma/\delta$ of the line

$$E_1 = \{(t,y,y') \in \omega(1) : \gamma y(1) + \delta p_1 y'(1) = d\}$$

may accept positive values and this also extends the usual case of $\gamma, \delta \ge 0$.

Proof. Returning to the proof of the theorem, we shall construct a strong pair (α, β) of upper and lower solutions of (4.1). Consider the scalar differential equation

$$(p(t)y'(t))' = \frac{K}{p(t)}(p(t)y'(t))^{n+1},\ t \in [0,1]. \tag{4.5}$$

Then a solution of (4.5) is

$$\alpha(t) = a_0 + \frac{1}{K(n-1)m_0^{n-1}}\left\{1 - [1 - Knm_0^n P(t)]^{\frac{n-1}{n}}\right\},\ 0 \le t \le 1,$$

where $P(t) = \int\limits_0^t \frac{ds}{p(s)} > 0$, $\frac{1}{\sqrt[n]{KnP(1)}} \ge m_0 \ge m > 0,\ 0 \le t \le 1$, and so

$$\alpha'(t) = \frac{m_0}{p(t)\sqrt[n]{1 - Knm_0^n P(t)}} \ge \frac{m_0}{p(t)} \ge \frac{m}{p(t)} > 0. \tag{4.6}$$

We choose now $0 < a_0 < b_0$ so that

$$0 < \alpha(t) < b_0,\ 0 \le t \le 1,$$

and so by (4.3), for each $0 \le t \le 1$ we get

$$(p(t)\alpha'(t))' = \frac{K}{p(t)}(p(t)\alpha'(t))^{n+1} > -p(t)q(t)f(t, \alpha(t), p(t)\alpha'(t)),$$

i.e., the function $x = \alpha(t)$ is a lower solution of (4.1).

Similarly the map

$$\beta(t) = b_0 - \frac{1}{K(n-1)m_1^{n-1}}\left\{1 - [1 + Knm_1^n P(t)]^{\frac{n-1}{n}}\right\},\ 0 \le t \le 1,$$

is a solution of

$$(p(t)y'(t))' = -\frac{K}{p(t)}(p(t)y'(t))^{n+1}$$

such that for all $0 \le t \le 1$,

$$\beta'(t) = \frac{m_1}{p(t)\sqrt[n]{1 + Knm_1^n P(t)}} \le \frac{m_1}{p(t)} \le -\frac{m}{p(t)} < 0. \tag{4.7}$$

Thus by (4.3), for each $0 \le t \le 1$ we get (we recall that $n = 2k+1$)

$$(p(t)\beta'(t))' = -\frac{K}{p(t)}(p(t)\beta'(t))^{n+1} < -p(t)q(t)f(t, \beta(t), p(t)\beta'(t)),$$

i.e., the function $x = \beta(t)$ is an upper solution of (4.1).

Now by (4.7) and since we may choose $0 < \alpha(t) < b_0,\ 0 \le t \le 1$, we get

$$0 < \alpha(t) < \beta(t),\ 0 \le t \le 1,$$

and moreover by (4.6) and (4.7),

$$\beta'(t) < 0 < \alpha'(t),\ 0 \le t \le 1,$$

that is the pair of upper and lower solution (α, β) is a strong one.

Consider the cross-section

$$\omega(0) = \{(t, x, x') : t = 0,\ \alpha(t) \le x \le \beta(t),\ x' \in \mathbb{R}\},$$

and recall that the egress points of $\omega(\tau),\ 0 \le \tau \le 1$, consist of the union $Q'_\alpha(\tau) \cup Q'_\beta(\tau)$, where

$$Q'_\alpha(\tau) := \{(\tau, \alpha(\tau), x') : x' \le \alpha'(\tau)\} \quad \text{and}$$
$$Q'_\beta(\tau) := \{(\tau, \beta(\tau), x') : x' \ge \beta'(\tau)\},$$

and notice that (set $p_0 := \lim\limits_{t \to 0+} p(t)$ and $p_1 := \lim\limits_{t \to 1-} p(t)$)

$$\alpha'(0) = \frac{m_0}{p_0} > 0,\ \beta'(0) = \frac{m_1}{p_0} < 0 \ \text{and}$$
$$\alpha'(1) := \frac{\widehat{m}_0}{p_1} > 0,\ \beta'(1) := \frac{\widehat{m}_1}{p_1} < 0.$$

We examine now two cases.

Both limits p_0 and p_1 exist and, of course, are positive.

Then the condition $-\alpha y(0) + \beta \lim\limits_{t \to 0+} p(t) y'(0) = c$ reduces to

$$y'(0) = \frac{c + \alpha y(0)}{\beta p_0}$$

and thus, if $y(0) = \alpha(0) = a_0$, we get by assumption (4.4) and (4.6)

$$y'(0) = \frac{c + \alpha a_0}{\beta p_0} \le \frac{m}{p_0} \le \frac{m_0}{p_0} = \alpha'(0)$$

and similarly, if $y(0) = \beta(0) = b_0$, we have

$$y'(0) = \frac{c + \alpha b_0}{\beta p_0} \ge -\frac{m}{p_0} \ge \frac{m_1}{p_0} = \beta'(0).$$

Consequently we obtain

$$Q'_\alpha(0) \cap E_0 \neq \emptyset \ \text{and}\ Q'_\beta(0) \cap E_0 \neq \emptyset \tag{4.8}$$

where recall that

$$E_0 = \{(t, y, y') \in \omega(0) : -\alpha y(0) + \beta p_0 y'(0) = c\}.$$

Now the condition $\gamma y(1) + \delta \lim_{t\to 1-} p(t)y'(t) = d$, which turns up to the cross-section $\omega(1)$ of ω reduces to

$$y'(1) = \frac{d - \gamma y(1)}{\delta p_1}$$

and thus, if $y(1) = \alpha(1) := \widehat{a}_0 > 0$, we get by assumption (4.4),

$$y'(1) = \frac{d - \gamma \widehat{a}_0}{\delta p_1} > \frac{\widehat{m}_0}{p_1} = \alpha'(1)$$

and similarly, if $y(1) = \beta(1) := \widehat{b}_0$, we have

$$y'(1) = \frac{d - \gamma \widehat{b}_0}{\delta p_1} < \frac{\widehat{m}_1}{p_1} = \beta'(1).$$

Consequently by (2.3), we now obtain

$$Q'_\alpha(1) \cap E_1 = \emptyset \text{ and } Q'_\beta(1) \cap E_1 = \emptyset \tag{4.9}$$

where

$$E_1 = \{(t, y, y') \in \omega(1) : \gamma y(1) + \delta p_1 y'(1) = d\}.$$

To show that (4.1), (4.2) has a solution, we will now examine the modified problem

$$(E) \quad \frac{1}{p(t)}(p(t)y'(t))' + q(t)F(t, y(t), p(t)y'(t)) = 0, \quad 0 < t < 1,$$

$$(C) \quad \begin{aligned} &-\alpha y(0) + \beta \lim_{t\to 0+} p(t)y'(t) = c, \\ &\gamma y(1) + \delta \lim_{t\to 1-} p(t)y'(t) = d, \end{aligned}$$

where

$$F(t,u,v) = \begin{cases} f(t,u,v), & \text{if } \alpha(t) \le u \le \beta(t),\ p(t)\beta'(t) \le v \le p(t)\alpha'(t), \\ f(t,\alpha(t),v), & \text{if } u \le \alpha(t),\ p(t)\beta'(t) \le v \le p(t)\alpha'(t), \\ f(t,\beta(t),v), & \text{if } u \ge \beta(t),\ p(t)\beta'(t) \le v \le p(t)\alpha'(t), \\ f(t,u,p(t)\beta'(t)), & \text{if } \alpha(t) \le u \le \beta(t),\ v \le p(t)\beta'(t), \\ f(t,u,p(t)\alpha'(t)), & \text{if } \alpha(t) \le u \le \beta(t),\ p(t)\alpha'(t) \le v, \\ f(t,\alpha(t),p(t)\beta'(t)), & \text{if } u \le \alpha(t),\ v \le p(t)\beta'(t), \\ f(t,\alpha(t),p(t)\alpha'(t)), & \text{if } u \le \alpha(t),\ p(t)\alpha'(t) \le v, \\ f(t,\beta(t),p(t)\beta'(t)), & \text{if } u \ge \beta(t),\ v \le p(t)\beta'(t), \\ f(t,\beta(t),p(t)\alpha'(t)), & \text{if } u \ge \beta(t),\ p(t)\alpha'(t) \le v. \end{cases}$$

Since $0 < \alpha(t) \leq \beta(t),\ 0 \leq t \leq 1$, we notice that the modification F is a bounded function and let M be one of its bounds.

Finally by an application of our general principle (Theorem 3.3), we get a (positive) solution $y = y(t)$ of the Sturm – Liouville boundary-value problem (E), (C), i.e.,

$$0 \leq \alpha(t) \leq y(t) \leq \beta(t), \quad 0 \leq t \leq 1,$$

which satisfies moreover the estimate

$$\beta'(t) \leq y'(t) \leq \alpha'(t), \quad 0 \leq t \leq 1.$$

Hence in view of definition of the modification F, $y = y(t)$ is a solution of the original equation (4.1).

Assume now that

$$\lim_{t \to 0+} p(t) = p_0 = 0 = p_1 = \lim_{t \to 1-} p(t).$$

Then, obviously, by (4.6) and (4.7), we get

$$\alpha'(0) = \alpha'(1) = +\infty, \quad \beta'(0) = \beta'(1) = -\infty.$$

Moreover by (4.4) and since for $\alpha \neq 0$

$$E_0 = \{(t, y, y') \in \omega(0) : -\alpha y(0) + \beta \lim_{t \to 0+} p(t) y'(0) = c\}$$

$$\supseteq \left\{(0, y, y') : y = -\frac{c}{\alpha} \in (a_0, b_0) \text{ and } y' \in \mathbb{R}\right\},$$

in view of Theorem 3.2, we may set

$$E_0 = \{(0, y, y') : y = -\frac{c}{\alpha} \in (a_0, b_0) \text{ and } y_1' \leq y' \leq y_2'\},$$

since we may choose a suitable interval (a_0, b_0) for the case when $\alpha c < 0$ (for $\alpha c \geq 0$, the problem remains open (if $c = 0$, then $\lim_{t \to 0+} y'(t) = \infty$)).

Remark 4.2. Now if $\alpha = 0 = c$, then $y'(0)$ is undefined and therefore we may restrict our consideration to the set

$$E_0 = \{(0, y, y') \in \omega(0) : y'(0) = 0\}$$

and in this case, clearly, (4.8) still holds. However for $\alpha = 0$ and $c \neq 0$, our existence problem remains open.

A similar situation holds also at the other end point $t = 1$. Precisely

$$E_1 = \{(t, y, y') \in \omega(1) : \gamma y(1) + \delta \lim_{t \to 1-} p(t) y'(1) = d\}$$

$$\supseteq \left\{(0, y, y') : y = \frac{d}{\gamma} \in [\widehat{a}_0, \hat{b}_0] \text{ and } y' \in \mathbb{R}\right\},$$

and (4.9) clearly is true (for $d/\gamma \in \{\widehat{a}_0, \widehat{b}_0\}$ we do not need to use (4.9) at all).

Then, if we use once again the boundary-value problem (E), (C), existence results follows via Theorem 3.3 and this ends the proof.

Remark 4.3. Let us notice (see Remark 4.2) that for the case

$$\alpha = c = 0, \ \lim_{t \to 0+} p(t) = p_0 = 0,$$

the BVP (4.1), (4.2) has a 1-parametric family of solutions satisfying the properties given in the above theorem.

Since for $p_0 = 0$, we have

$$E_0 = \{(t, y, y') \in \omega(0) : \alpha y(0) + \beta \lim_{t \to 0+} p(t) y'(0) = c\} = \{0\} \times [\alpha(0), \beta(0)] \times \mathbb{R}$$

and further

$$Q'_\alpha(0) = \{(0, \alpha(0), y') : y' \le \alpha'(0) = +\infty\}$$

and similarly

$$Q'_\beta(0) = \{(0, \beta(0), y') : y' \ge \beta'(0) = -\infty\},$$

we can choose, in place of E_0, any set

$$E_\lambda = \{(0, y, y') \in E_0 : y' = \lambda\}.$$

Then obviously

$$Q'_\alpha(0) \cap E_\lambda \ne \emptyset \ \text{ and } \ Q'_\beta(0) \cap E_\lambda \ne \emptyset.$$

Thus, by applying the previous Theorem we get the desired family $\{y_\lambda(t)\}$, $\lambda \in \mathbb{R}$, of solutions.

5. Thomas – Fermi Boundary-Value Problems

In this Section, motivated by the Thomas – Fermi equation (1.5), we consider the generalized Bohr's type boundary-value problem, namely,

$$\begin{gathered} \frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t, y(t), p(t)y'(t)), \quad 0 < t < 1, \\ y(0) = a \quad \text{and} \quad \left(p(1) \int_0^1 \frac{ds}{p(s)} \right) y'(1) + y(1) = 0. \end{gathered} \tag{5.1}$$

A similar problem has been studied by O'Regan in [4] (Theorem 5.3) for the case when f is independent of its last argument. We set $P^*(t) = p(t) \int_0^t \frac{ds}{p(s)}, \ t \in [0,1]$.

Theorem 5.1. *Let* $f : [0,1] \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ *be continuous. We assume:*

$$\begin{gathered} p \in C[0,1] \cap C^1(0,1], \ p > 0 \textit{ on } (0,1], \ p'(t) \geq 0, \ \ t \in (0,1), \\ \textit{and} \quad \int_0^1 \frac{1}{p(s)} ds < \infty, \end{gathered} \tag{5.2}$$

$$q \in C(0,1) \textit{ with } \quad q > 0 \textit{ on } (0,1) \textit{ and } \int_0^1 p(x)q(x)dx < \infty, \tag{5.3}$$

$$\begin{gathered} \textit{there exists a constant} \quad M \geq 0 \quad \textit{such that} \quad yf(t,y,0) > 0 \\ \textit{for} \quad |y| > M \quad \textit{and all} \quad t \in [0,1], \end{gathered} \tag{5.4}$$

$$\begin{gathered} \textit{there is} \quad \gamma \in \left(0, \frac{M}{P^*(1)}\right) \quad \textit{such that} \quad -M-\gamma < a < M+\gamma \\ \textit{and both functions} \quad f(t,.,v) \quad \textit{and} \quad f(t,u,.) \quad \textit{are nonincreasing for each} \\ t \in [0,1], \ u \in [-M-\gamma, \ M+\gamma] \textit{and} \quad v \in [-\gamma,\gamma]. \end{gathered} \tag{5.5}$$

Then (5.1) has at least a solution in $C[0,1] \cap C^2(0,1]$.

Proof. Consider the functions

$$\alpha(t) = -M - \gamma(1-t) \text{ and } \beta(t) = M + \gamma(1-t),\ t \in [0,1].$$

Then $\alpha(t)$ and $\beta(t),\ t \in [0,1]$, are lower and upper solutions of (5.1). Indeed, in view of (5.2) – (5.4) and since $f(t,u,.)$ is nonincreasing, we get

$$\begin{aligned}(p(t)\alpha'(t))' &- p(t)q(t)f(t,\alpha(t),p(t)\alpha'(t))\\ &= \gamma p'(t) - p(t)q(t)f(t,\alpha(t),\gamma p(t))\\ &\geq \gamma p'(t) - p(t)q(t)f(t,\alpha(t),0) \geq 0, \quad 0 < t < 1,\end{aligned}$$

because $\alpha(t) \leq -M,\quad 0 \leq t \leq 1$. Similarly we can prove that β is an upper solution. Furthermore (α,β) is a strong pair of lower and upper solutions. By using an adaptation F of f similar to the one given in Theorem 4.1, clearly it is not a restriction to assume that the function f is bounded on $[0,1] \times \mathbb{R}^2$.

If

$$E_0 = \{(0,y,y') : y(0) = a,\ y' \in \mathbb{R}\} \text{ and } E_1 = \{(0,y,y') : P^*(1)y' + y = 0\}$$

by (5.5) the result of Theorem 3.2 holds and so we get

$$\mathcal{K}(E_0) \cap Q'_\alpha \neq \emptyset \text{ and } \mathcal{K}(E_0) \cap Q'_\beta \neq \emptyset.$$

Further by (5.5) we can easily verify that

$$E_1 \cap Q'_\alpha(1) = \emptyset \text{ and } E_1 \cap Q'_\beta(1) = \emptyset.$$

Existence now follows by an argument similar to the one given in the proof of the previous theorem.

Corollary 5.1. *Under the assumptions (5.2), (5.3) and (5.5) of the previous theorem and*

$$f(t,0,0) = 0 \text{ and } yf(t,y,0) > 0,\ t \in [0,1],\ y \neq 0,$$

the BVP (5.1) accepts a positive (negative) solution if $\alpha > 0$ *(*$\alpha < 0$ *respectively).*

Proof. All that we need to notice is that for $\alpha > 0$ ($\alpha < 0$) we can choose $\alpha(t) = 0$ ($\beta(t) = 0$), $0 \leq t \leq 1$, because $f(t,0,0) = 0,\ 0 \leq t \leq 1$.

We finally consider the Thomas – Fermi type singular equation

$$y'' = -t^{-\frac{1}{2}}y^{\frac{3}{2}} := q(t)f(t,y) \tag{5.6}$$

associated to one of the boundary conditions:

$$y(0) = 1, \; y(b) = 0 \quad (b > 0), \tag{5.7}$$

$$y(0) = 1, \; \lim_{t \to \infty} y(t) = 0. \tag{5.8}$$

We notice at once that

$$\alpha(t) = 0, \; \beta(t) = 2, \; t \geq 0$$

is a strong pair of lower and upper solutions to (5.6). Considering a modification F of f, similar to the one given in the proof of Theorem 4.1, we may assume that f is bounded and $f(t, .)$ is nonincreasing. By Theorem 3.1, any solution $y = y(t)$ with $y(0) = 1$ is defined over the interval $[0, \infty)$. Assume that there is no solution $y \in \mathcal{X}(E_0)$ of (5.6) such that its graph

$$G(y|[0, \infty); E_0) = \{(t, y(t)) : t \geq 0\} \subseteq \omega.$$

Then any solution $y \in \mathcal{X}(E_0)$ egresses strictly from ω and further the set $\mathcal{K}(E_0)$ is a continuum. Now the sets

$$\mathcal{K}(E_0) \cap S_\alpha \text{ and } \mathcal{K}(E_0) \cap S_\beta$$

clearly consist of a closed partition of $\mathcal{K}(E_0)$, and this is a contradiction. Let $A := \mathcal{K}(E_0) \cap S_\alpha$ and let S be a connected component of A. By Lemma 2.4, S must be an unbounded set (otherwise we easily get a similar contradiction) and this means that there exists a solution $y \in \mathcal{X}(E_0)$ such that $\lim_{t \to \infty} y(t) = 0$.

Similarly we may get a solution of (5.6), satisfying the boundary condition (5.7).

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