UDC 517.5

NOTE ON QUATERNION LINEAR DYNAMICAL SYSTEMS*

ПРО КВАТЕРНІОННІ ЛІНІЙНІ ДИНАМІЧНІ СИСТЕМИ

N. Dilna

Inst. Math., Slovak Acad. Sci.
Stefánikova, 49, Bratislava, 81473, Slovakia
e-mail: dilna@mat.savba.sk

M. Fečkan

Comenius Univ. Bratislava
Mlynská dolina, Bratislava, 84248, Slovakia

Inst. Math., Slovak Acad. Sci.
Stefánikova, 49, Bratislava, 81473, Slovakia
e-mail: Michal.Feckan@fmph.uniba.sk

J. Wang

Guizhou Univ.
Guiyang, Guizhou, 550025, China
email: jrwang@gzu.edu.cn

Paper dedicated to 85 years from date of birthday
Professor Anatoliy Mykhailovych Samoilenko

Conditions on stability and unstability of quaternion linear dynamical systems, linear differential and linear fractional equations are established.

Знайдено умови стійкості та нестійкості кватерніонних лінійних динамічних систем, лінійних диференціальних рівнянь і лінійних різницевих рівнянь дробового порядку.

1. Introduction. Quaternion-valued equations are a new kind of equation with many applications in any life sciences, mainly in physics [1 – 15]. The concept of quaternions was originally invented by Hamilton and started in 1843. It extends complex numbers to four-dimensional space.

Quaternions have shown advantages over real-valued vectors in physics and engineering applications for their powerful modeling of rotation and orientation. Orientation can be defined as a set of parameters that relates the angular position of a frame to another reference frame. The complete rotation matrix for moving from the inertial frame to the body frame is given by the multiplication of three matrices where the rotation matrix is moving from vehicle-2 frame to the body frame, vehicle-1 frame to vehicle-2 frame, and inertial frame to vehicle-1 frame, respectively

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(more details are in [7, 16]). Quaternions provide an alternative measurement technique that is not limited by a gimbal lock phenomenon. The cause of such a phenomenon is when the pitch angle is 90 degrees [16].

Therefore, all CH Robotics attitude sensors [7] use quaternions, so the output is always valid even when Euler angles are not.

2. Problem formulation. We consider linear dynamical system

$$x_{n+1} = px_n + x_n q, \quad n \in \mathbb{N}_0, \tag{1}$$

with $p,\ q,\ x_n,\ x_{n+1}$ are quaternions (see Definition 1), $p, q, x_n, x_{n+1} \in \mathbb{H}$. We denote the set of quaternions by $\mathbb{H}$.

The main goal of our investigations is find conditions on stability and unstability (see Theorems 1, 2) also we establish that Eq. (1) is hyperbolic (see Theorem 3). Moreover we study conditions on stability if q and $p \in \mathbb{C}$ (see Theorems 4). All this results are in Section 5. Some properties of the elements in the space $\mathbb{H}$ are established in Section 4. We also consider continuous case in Section 6, i.e., linear differential equations on $\mathbb{H}$. More we discuss linear fractional difference and differential equations on $\mathbb{H}$ as well in Section 7. Theory is numerically illustrated on a concrete example in Section 8. Paper is finished with Section 9 by outlining a possible father investigation.

3. Notations and definitions.

Definition 1 [7]. *Quaternions are* 4*-vectors whose multiplication rules are governed by a simple noncommutative division algebra. We denote the quaternion* $q = (q_0, q_1, q_2, q_3)^T \in \mathbb{R}^4$ *by*

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \tag{2}$$

where $q_0,\ q_1,\ q_2,\ q_3$ *are real numbers an* $\mathbf{i},\ \mathbf{j},\ \mathbf{k}$ *satisfy the multiplication table formed by*

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}. \tag{3}$$

Let us define for the quaternions q from (2) and

$$p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}, \tag{4}$$

the inner product

$$\langle q, p\rangle := q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3, \tag{5}$$

and the modulus, by

$$\|q\| := (\langle q, q\rangle)^{\frac{1}{2}} = \left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right)^{\frac{1}{2}}. \tag{6}$$

The above inner product makes $\mathbb{H}$ a four-dimensional real Hilbert space. Given (2), we introduce the real part operator

$$\Re q := q_0 \tag{7}$$

as well as the vectorial or imaginary part by

$$\Im q := q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}. \tag{8}$$

And finally the conjugate is introduced by

$$\overline{q} := \Re q - \Im q = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}. \tag{9}$$

Definition 2. *A continuous linear system is exponentially stable if and only if the poles of its transfer function lie strictly within the unit circle centered on the origin of the complex plane. Exponential stability is a form of asymptotic stability.*

Definition 3. *If $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C_1 map and r is a fixed point then r is said to be a hyperbolic fixed point when the Jacobian matrix $DA(r)$ has no eigenvalues on the unit circle.*

Definition 4 [9]**.** *Neimark – Sacker bifurcation is the birth of a closed invariant curve from a fixed point in dynamical systems with discrete time (iterated maps), when the fixed point changes stability via a pair of complex eigenvalues with unit modulus. The bifurcation can be supercritical or subcritical, resulting in a stable or unstable (within an invariant two-dimensional manifold) closed invariant curve, respectively. When it happens in the Poincare map of a limit cycle, the bifurcation generates an invariant two-dimensional torus in the corresponding ODE.*

4. Auxiliary properties. In this section in Lemma 1 we introduce some well-known general properties of quaternions on the space $\mathbb{H}$ [3, 7].

Lemma 1. *The following statements are true:*

(i) $(\Im q)^2 = -\|\Im q\|^2$;

(ii) $\overline{qp} = \overline{p}\,\overline{q}$;

(iii) $q\overline{q} = \overline{q}q = \|q\|^2$;

(iv) $\|qp\| = \|pq\| = \|q\|\|p\|$;

(v) $\langle q, p\rangle = \dfrac{1}{2}\,(q\overline{p} + p\overline{q}) = \Re(q\overline{p}) = \Re(\overline{q}p)$;

(vi) $\langle q, pq\rangle = \|q\|^2\Re p$.

Proof. For readers convenience we present also the proof. To prove Lemma 1 we use (2) – (9).

(i)

$$
\begin{aligned}
(\Im q)^2 = (q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})^2 &= -q_1^2 - q_2^2 - q_3^2 + q_1q_2\mathbf{ij} + q_1q_3\mathbf{ik} + \\
&+ q_2q_1\mathbf{ji} + q_2q_3\mathbf{jk} + q_3q_1\mathbf{ki} + q_3q_2\mathbf{kj} = -q_1^2 - q_2^2 - q_3^2 + \\
&+ q_1q_2\mathbf{k} - q_1q_3\mathbf{j} - q_2q_1\mathbf{k} + q_2q_3\mathbf{i} - q_3q_1\mathbf{j} - q_3q_2\mathbf{i} = -q_1^2 - q_2^2 - q_3^2, \\
-\|\Im q\|^2 &= -(\langle \Im q, \Im q\rangle) = -\big(q_1^2 + q_2^2 + q_3^2\big);
\end{aligned}
$$

(ii)

$$
\begin{aligned}
qp &= (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) = \\
&= q_0p_0 + q_0p_1\mathbf{i} + q_0p_2\mathbf{j} + q_0p_3\mathbf{k} + q_1p_0\mathbf{i} + q_1p_1\mathbf{i}^2 + q_1p_2\mathbf{ij} + q_1p_3\mathbf{ik} + \\
&\quad + q_2p_0\mathbf{j} + q_2p_1\mathbf{ji} + q_2p_2\mathbf{j}^2 + q_2p_3\mathbf{jk} + q_3p_0\mathbf{k} + q_3p_1\mathbf{ki} + q_3p_2\mathbf{kj} + q_3p_3\mathbf{k}^2 = \\
&= q_0p_0 + q_0p_1\mathbf{i} + q_0p_2\mathbf{j} + q_0p_3\mathbf{k} + q_1p_0\mathbf{i} - q_1p_1 + q_1p_2\mathbf{k} - q_1p_3\mathbf{j} + \\
&\quad + q_2p_0\mathbf{j} - q_2p_1\mathbf{k} - q_2p_2 + q_2p_3\mathbf{i} + q_3p_0\mathbf{k} + q_3p_1\mathbf{j} - q_3p_2\mathbf{i} - q_3p_3 = \\
&= q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\mathbf{i} + \\
&\quad + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\mathbf{j} + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\mathbf{k}, \\
\overline{qp} &= q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 - (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\mathbf{i} -
\end{aligned}
$$

$$- (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\mathbf{j} - (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\mathbf{k},$$

$$\overline{p}\,\overline{q} = (p_0 - p_1\mathbf{i} - p_2\mathbf{j} - p_3\mathbf{k})(q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}) =$$

$$= p_0q_0 - p_0q_1\mathbf{i} - p_0q_2\mathbf{j} - p_0q_3\mathbf{k} - p_1q_0\mathbf{i} + p_1q_1\mathbf{i}^2 + p_1q_2\mathbf{ij} + p_1q_3\mathbf{ik}+$$

$$+ p_2q_0\mathbf{j} + p_2q_1\mathbf{ji} + p_2q_2\mathbf{j}^2 + p_2q_3\mathbf{jk} + p_3q_0\mathbf{k} + p_3q_1\mathbf{ki} + p_3q_2\mathbf{kj} + q_3p_3\mathbf{k}^2 =$$

$$= p_0q_0 - p_0q_1\mathbf{i} - p_0q_2\mathbf{j} - p_0q_3\mathbf{k} - p_1q_0\mathbf{i} - p_1q_1 + p_1q_2\mathbf{k} + p_1q_3\mathbf{j}-$$

$$- p_2q_0\mathbf{j} - p_2q_1\mathbf{k} - p_2q_2 + p_2q_3\mathbf{i} + p_3q_0\mathbf{k} - p_3q_1\mathbf{j} - p_3q_2\mathbf{i} - q_3p_3 =$$

$$= p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 - (p_1q_0 + p_0q_1 + p_3q_2 - p_2q_3)\mathbf{i}-$$

$$- (p_2q_0 - p_3q_1 + p_0q_2 + p_1q_3)\mathbf{j} - (p_3q_0 + p_2q_1 - p_1q_2 + p_0q_3)\mathbf{k};$$

(iii)

$$q\overline{q} = (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}) =$$

$$= q_0^2 - q_0q_1\mathbf{i} - q_0q_2\mathbf{j} - q_0q_3\mathbf{k} + q_1q_0\mathbf{i} - q_1^2\mathbf{i}^2 - q_1q_2\mathbf{ij} - q_1q_3\mathbf{ik}+$$

$$+ q_2q_0\mathbf{j} - q_2q_1\mathbf{ji} - q_2q_2\mathbf{j}^2 - q_2q_3\mathbf{jk} + q_3q_0\mathbf{k} - q_3q_1\mathbf{ki} - q_3q_2\mathbf{kj} - q_3p_3\mathbf{k}^2 =$$

$$= q_0^2 - q_0q_1\mathbf{i} - q_0q_2\mathbf{j} - q_0q_3\mathbf{k} + q_1q_0\mathbf{i} - q_1^2 - q_1q_2\mathbf{k} - q_1q_3\mathbf{j}+$$

$$+ q_2q_0\mathbf{j} + q_2q_1\mathbf{k} + q_2^2 + q_2q_3\mathbf{i} + q_3q_0\mathbf{k} + q_3q_1\mathbf{j} + q_3q_2\mathbf{i} + q_3^2 =$$

$$= q_0^2 + q_1^2 + q_2^2 + q_3^2,$$

$$\overline{q}q = (q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) =$$

$$= q_0^2 + q_0q_1\mathbf{i} + q_0q_2\mathbf{j} + q_0q_3\mathbf{k} - q_1q_0\mathbf{i} - q_1q_1\mathbf{i}^2 + q_1q_2\mathbf{ij} - q_1q_3\mathbf{ik}-$$

$$- q_2q_0\mathbf{j} - q_2q_1\mathbf{ji} - q_2q_2\mathbf{j}^2 - q_2q_3\mathbf{jk} - q_3q_0\mathbf{k} - q_3q_1\mathbf{ki} - q_3q_2\mathbf{kj} - q_3^2\mathbf{k}^2 =$$

$$= q_0^2 + q_0q_1\mathbf{i} + q_0q_2\mathbf{j} + q_0q_3\mathbf{k} - q_1q_0\mathbf{i} + q_2^2 - q_1q_2\mathbf{k} - q_1q_3\mathbf{j}-$$

$$- q_2q_0\mathbf{j} + q_2q_1\mathbf{k} + q_2^2 - q_2q_3\mathbf{i} - q_3q_0\mathbf{k} + q_3q_1\mathbf{j} + q_3q_2\mathbf{i} + q_2^2 =$$

$$= q_0^2 + q_1^2 + q_2^2 + q_3^2;$$

(iv)

$$\|qp\| = \left(\langle qp, qp\rangle\right)^{\frac{1}{2}} = \big((q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3)^2 + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)^2+$$

$$+ (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)^2 + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)^2\big)^{\frac{1}{2}} =$$

$$= \Big(q_0^2p_0^2 - q_0p_0q_1p_1 - q_0p_0q_2p_2 - q_0p_0q_3p_3 - q_1p_1q_0p_0 + q_1^2p_1^2 + q_1p_1q_2p_2+$$

$$+ q_1p_1q_3p_3 - q_2p_2p_0q_0 + q_2p_2q_1p_1 + q_2^2p_2^2 + q_2p_2q_3p_3 - q_3p_3q_0p_0 + q_3p_3q_1p_1+$$

$$+ q_3p_3q_2p_2 + q_3^2p_3^2 + q_0^2p_1^2 + q_0p_1q_1p_0 + q_0p_1q_2p_3 - q_0p_1q_3p_2 + q_1p_0q_0p_1+$$

$$+ q_1^2p_0^2 + q_1p_0q_2p_3 - q_1p_0q_3p_2 + q_2p_3q_0p_1 + q_2p_3q_1p_0 + q_2^2p_3^2 - q_2p_3q_3p_2-$$
$$- q_3p_2q_0p_1 - q_3p_2q_1p_0 - q_3p_2q_2p_3 + q_3^2p_2^2 + q_0^2p_2^2 - q_0p_2q_1p_3 + q_0p_2q_2p_0+$$
$$+ q_0p_2q_3p_1 - q_1p_3q_0p_2 + q_1^2p_3^2 - q_1p_3q_2p_0 - q_1p_3q_3p_1 + q_2p_0q_0p_2-$$
$$- q_2p_0q_1p_3 + q_2^2p_0^2 + q_2p_0q_3p_1 + q_3p_1q_0p_2 - q_3p_2q_1p_3 + q_3p_1q_2p_0+$$
$$+ q_3^2p_1^2 + q_0^2p_3^2 + q_0p_3q_1p_2 - q_0p_3q_2p_1 + q_0p_3q_3p_0 + q_1p_2q_0p_3 + q_1^2p_2^2-$$
$$- q_1p_2q_2p_1 + q_1p_2q_3p_0 - q_2p_1q_0p_3 - q_2p_1q_1p_2 + q_2^2p_1^2 - q_2p_1q_3p_0+$$
$$+ q_3p_0q_0p_3 + q_3p_0q_1p_2 - q_3p_0q_2p_1 + q_3^2p_0^2\Big)^{\frac{1}{2}} =$$
$$= \Big(q_0^2p_0^2 + q_1^2p_1^2 + q_2^2p_2^2 + q_3^2p_3^2 + q_0^2p_1^2 + q_1^2p_0^2 + q_2^2p_3^2 + q_3^2p_2^2+$$
$$+ q_0^2p_2^2 + q_1^2p_3^2 + q_2^2p_0^2 + q_3^2p_1^2 + q_0^2p_3^2 + q_1^2p_2^2 + q_2^2p_1^2 + q_3^2p_0^2\Big)^{\frac{1}{2}},$$
$$\|q\|\|p\| = (\langle q,q\rangle)^{\frac{1}{2}}(\langle p,p\rangle)^{\frac{1}{2}} = (q_0^2+q_1^2+q_2^2+q_3^2)^{\frac{1}{2}}(p_0^2+p_1^2+p_2^2+p_3^2)^{\frac{1}{2}} =$$
$$= \Big(q_0^2p_0^2 + q_0^2p_1^2 + q_0^2p_2^2 + q_0^2p_3^2 + q_1^2p_0^2 + q_1^2p_1^2 + q_1^2p_2^2+$$
$$+ q_1^2p_3^2 + q_2^2p_0^2 + q_2^2p_1^2 + q_2^2p_2^2 + q_2^2p_3^2 + q_3^2p_0^2 + q_3^2p_1^2 + q_3^2p_2^2 + q_3^2p_3^2\Big)^{\frac{1}{2}};$$

(v)

$$\langle p,q\rangle = q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3,$$
$$\frac{1}{2}(q\overline{p} + p\overline{q}) = \frac{1}{2}\Big((q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(p_0 - p_1\mathbf{i} - p_2\mathbf{j} - p_3\mathbf{k})+$$
$$+ (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k})\Big) =$$
$$= \frac{1}{2}\Big(q_0p_0 - q_0p_1\mathbf{i} - q_0p_2\mathbf{j} - q_0p_3\mathbf{k} + q_1p_0\mathbf{i} + q_1p_1 - q_1p_2\mathbf{k} + q_1p_3\mathbf{j}+$$
$$+ q_2p_0\mathbf{j} + q_2p_1\mathbf{k} + q_2p_2 - q_2p_3\mathbf{i} + q_3p_0\mathbf{k} - q_3p_1\mathbf{j} + q_3p_2\mathbf{i} + q_3p_3+$$
$$+ p_0q_0 - p_0q_1\mathbf{i} - p_0q_2\mathbf{j} - p_0q_3\mathbf{k} + p_1q_0\mathbf{i} + p_1q_1 - p_1q_2\mathbf{k} + p_1q_3\mathbf{j}+$$
$$+ p_2q_0\mathbf{j} + p_2q_1\mathbf{k} + p_2q_2 - p_2q_3\mathbf{i} + p_3q_0\mathbf{k} + p_3q_1\mathbf{j} + p_3q_2\mathbf{i} + p_3q_3\Big) =$$
$$= q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3,$$
$$\Re(q\overline{p}) = \Re((q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(p_0 - p_1\mathbf{i} - p_2\mathbf{j} - p_3\mathbf{k})) = q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3,$$
$$\Re(p\overline{q}) = \Re((p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k})) = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3;$$

(vi)

$$pq = (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) =$$
$$= p_0q_0 + p_0q_1\mathbf{i} + p_0q_2\mathbf{j} + p_0q_3\mathbf{k} + p_1q_0\mathbf{i} + p_1q_1\mathbf{i}^2 + p_1q_2\mathbf{ij} + p_1q_3\mathbf{ik}+$$

$$+ p_2q_0\mathbf{j} + p_2q_1\mathbf{ji} + p_2q_2\mathbf{j}^2 + p_2q_3\mathbf{jk} + p_3q_0\mathbf{k} + p_3q_1\mathbf{ki} + p_3q_2\mathbf{kj} + p_3q_3\mathbf{k}^2 =$$

$$= p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\mathbf{i}+$$

$$+ (p_0q_2 + p_1q_3 + p_2q_0 - p_3q_1)\mathbf{j} + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)\mathbf{k},$$

$$\langle q, pq\rangle = q_0^2p_0 - q_0q_1p_1 - q_0q_2p_2 - q_0q_3p_3 + q_1^2p_0 + q_0q_1p_1 + q_1q_3p_2 - q_1q_2p_3+$$

$$+ q_2^2p_0 + q_2q_3p_1 + q_0q_2p_2 - q_1q_2p_3 + q_3^2p_0 + q_2q_3p_1 - q_1q_3p_2 + q_0q_3p_3 =$$

$$= \left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right)p_0 = \|q\|^2\Re p.$$

5. Linear difference equations. Let us consider (1). The associated linear map is

$$Ax = px + xq. \tag{10}$$

If $p, q, x_n, x_{n+1} \in \mathbb{C},$ then (1) is reducing to

$$x_{n+1} = (p+q)x_n, \quad n \in \mathbb{N}_0. \tag{11}$$

It is known, that (11) is stable if and only if

$$|p+q| \leq 1$$

and asymptotically stable if and only if

$$|p+q| < 1.$$

Here we have an interest to study problem (1), where $p,\ q,$ and $x\ \in \mathbb{H}.$
Let us consider q and p defined by (2) and (4) respectively and

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

Then the right part of the Eq. (1) is

$$(p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})\times$$

$$\times(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) =$$

$$= p_0x_0 + p_0x_1\mathbf{i} + p_0x_2\mathbf{j} + p_0x_3\mathbf{k} + x_0q_0 + x_1q_0\mathbf{i} + x_2q_0\mathbf{j} + x_3q_0\mathbf{k}+$$

$$+ p_1x_0\mathbf{i} + p_1x_1\mathbf{i^2} + p_1x_2\mathbf{ij} + p_1x_3\mathbf{ik} + x_0q_1\mathbf{i} + x_1q_1\mathbf{i^2} + x_2q_1\mathbf{ji} + x_3q_1\mathbf{ki}+$$

$$+ p_2x_0\mathbf{j} + p_2x_1\mathbf{ji} + p_2x_2\mathbf{j^2} + p_2x_3\mathbf{jk} + x_0q_2\mathbf{j} + x_1q_2\mathbf{ij} + x_2q_2\mathbf{j^2} + x_3q_2\mathbf{kj}+$$

$$+ p_3x_0\mathbf{k} + p_3x_1\mathbf{ki} + p_3x_2\mathbf{kj} + p_3x_3\mathbf{k^2} + x_0q_3\mathbf{k} + x_1q_k\mathbf{ik} + x_2q_3\mathbf{jk} + x_3q_k\mathbf{k^2} =$$

$$= (p_0 + q_0)x_0 - (p_1 + q_1)x_1 - (p_2 + q_2)x_2 - (p_3 + q_3)x_3+$$

$$+ (p_1x_0 + p_0x_1 - p_3x_2 + p_2x_2 + x_0q_1 + x_1q_0 + x_2q_3 - x_3q_2)\mathbf{i}+$$

$$+ (p_2x_0 + p_3x_1 + p_0x_2 - p_1x_3 + x_0q_2 - x_1q_3 + x_2q_0 + x_3q_1)\mathbf{j}+$$

$$+ (p_3x_0 - p_2x_1 + p_1x_2 + p_0x_3 + x_0q_3 + x_1q_2 - x_2q_1 + x_3q_0)\mathbf{k} =$$
$$= (p_0+q_0)x_0 - (p_1+q_1)x_1 - (p_2+q_2)x_2 - (p_3+q_3)x_3 +$$
$$+ ((p_0+q_0)x_0 - (p_1+q_1)x_1 + (-p_3+q_3)x_2 + (p_2-q_2)x_3)\mathbf{i}+$$
$$+ ((p_2+q_2)x_0 + (p_3-q_0)x_1 + (p_0+q_0)x_2 + (-p_1+q_1)x_3)\mathbf{j}+$$
$$+ ((p_3+q_3)x_0 + (-p_2+q_2)x_1 + (p_1-q_1)x_2 + (p_0+q_0)x_3)\mathbf{k}.$$

The map A from (10) has the form

$$A = \begin{pmatrix} p_0+q_0 & -p_1-q_1 & -p_2-q_2 & -p_3-q_3 \\ p_1+q_1 & p_0+q_0 & q_3-p_3 & p_2-q_2 \\ p_2+q_2 & p_3-q_3 & p_0+q_0 & q_1-p_1 \\ p_3+q_3 & q_2-p_2 & p_1-q_1 & p_0+q_0 \end{pmatrix}.$$

Then, A have eigenvalues

$$\lambda_1 = p_0+q_0 - \sqrt{-2\sqrt{\left(p_1^2+p_2^2+p_3^2\right)\left(q_1^2+q_2^2+q_3^2\right)} - p_1^2-p_2^2-p_3^2-q_1^2-q_2^2-q_3^2},$$

$$\lambda_2 = p_0+q_0 + \sqrt{-2\sqrt{\left(p_1^2+p_2^2+p_3^2\right)\left(q_1^2+q_2^2+q_3^2\right)} - p_1^2-p_2^2-p_3^2-q_1^2-q_2^2-q_3^2},$$

$$\lambda_3 = p_0+q_0 - \sqrt{2\sqrt{\left(p_1^2+p_2^2+p_3^2\right)\left(q_1^2+q_2^2+q_3^2\right)} - p_1^2-p_2^2-p_3^2-q_1^2-q_2^2-q_3^2},$$

$$\lambda_4 = p_0+q_0 + \sqrt{2\sqrt{\left(p_1^2+p_2^2+p_3^2\right)\left(q_1^2+q_2^2+q_3^2\right)} - p_1^2-p_2^2-p_3^2-q_1^2-q_2^2-q_3^2}.$$

Clearly λ_1 and λ_2 are complex conjugated numbers. By using

$$4\left(p_1^2+p_2^2+p_3^2\right)\left(q_1^2+q_2^2+q_3^2\right) - \left(-p_1^2-p_2^2-p_3^2-q_1^2-q_2^2-{q_3}^2\right)^2 =$$
$$= -\left(p_1^2+p_2^2+p_3^2-q_1^2-q_2^2-q_3^2\right)^2,$$

we see that λ_3 and λ_4 are complex conjugated numbers for

$$p_1^2+p_2^2+p_4^2-q_1^2-q_2^2-q_3^2 \neq 0,$$

and they are real $\lambda_3 = \lambda_4$ for

$$p_1^2+p_2^2+p_4^2-q_1^2-q_2^2-q_3^2 = 0.$$

Thus,

$$|\lambda_1|^2 = |\lambda_2|^2 = (p_0+q_0)^2 +$$
$$+ 2\sqrt{\left(p_1^2+p_2^2+p_3^2\right)\left(q_1^2+q_2^2+q_3^2\right)} + p_1^2+p_2^2+p_3^2+q_1^2+q_2^2+q_3^2, \qquad (12)$$

$$|\lambda_3|^2 = |\lambda_4|^2 = (p_0 + q_0)^2 - \\ - 2\sqrt{\left(p_1^2 + p_2^2 + p_3^2\right)\left(q_1^2 + q_2^2 + q_3^2\right)} + p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2, \qquad (13)$$

and this gives

$$\rho(A) = \sqrt{(p_0 + q_0)^2 + 2\sqrt{\left(p_1^2 + p_2^2 + p_3^2\right)\left(q_1^2 + q_2^2 + q_3^2\right)} + p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2}. \qquad (14)$$

Returning back to the complex case for $p_2 = p_3 = q_2 = q_3 = 0,$ we have

$$\rho(A) = \sqrt{(p_0 + q_0)^2 + 2\sqrt{p_1^2 q_1^2} + p_1^2 + q_1^2} = \sqrt{(p_0 + q_0)^2 + (|p_1| + |q_1|)^2} \geq |p + q|.$$

Theorem 1. *If $\rho(A) < 1$ in (14), then solution of the Eq. (1) is asymptotically stable.*

Theorem 2. *If $\rho(A) > 1$ in (14), then solution of the Eq. (1) is unstable.*

Theorem 3. *If $|\lambda_i| \neq 1$ in (12) and (13) then the Eq. (1) is hyperbolic (see Definition 3).*

Theorem 4. *If p and q $\in \mathbb{C}$ satisfy*

$$\sqrt{(p_0 + q_0)^2 + (|p_1| + |q_1|)^2} > 1 > |p + q|,$$

then solution of the Eq. (1) is asymptotically stable in the space $\mathbb{C}$, where $A\colon \mathbb{C} \to \mathbb{C}$ and unstable in $\mathbb{H}$.

Remark 1. If some of $|\lambda_i| = 1$ and p and q depend from some parameters and $|\lambda_i| = 1,$ then we may have bifurcation in the equation (1).

Remark 2. If $|\lambda_1| = |\lambda_2| = 1$ and simultaneously $|\lambda_3| = |\lambda_4| \neq 1$ then we may have Neimark – Sacker bifurcation (see Definition 4) for certain nonlinear parametric perturbations.

Remark 3. If $p = -q \in \mathbb{C}$, then (11) reduce to equation $x_n = 0,\ n \in \mathbb{N}$, while $A \neq 0$ with $\rho(A) = 2|p_2|$.

Remark 4. It clear that the solution of (1) is given by

$$x_n = \sum_{i=0}^{n} \binom{n}{i} p^i x_0 q^{n-i}, \quad n \in \mathbb{N}_0.$$

Note when $p, q, x_0 \in \mathbb{C}$, then

$$x_n = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} x_0 = (p + q)^n x_0,$$

which, of course, gives the solution of (11).

6. Linear differential equations. Now we consider a linear differential equation

$$x'(t) = px(t) + x(t)q, \quad t \in \mathbb{R}_0 = [0, \infty), \qquad (15)$$

on $\mathbb{H}$. By using the spectrum results of A in Section 5, we have the following results.

Theorem 5. *If $p_0 + q_0 < 0$, then solution of the Eq. (15) is asymptotically stable and 0 is a sink.*

Theorem 6. *If $p_0 + q_0 > 0$, then solution of the Eq. (15) is unstable and 0 is a source/repeller.*

Theorem 7. *If $p_0 + q_0 = 0$ then solution of the Eq. (15) is stable and 0 is a center.*

We see that the dynamics of (1) is different than of (15). The Eq. (15) is never saddle hyperbolic, opposite to (1).

7. Linear fractional equations. Let $\alpha \in (0,1)$, $\mathbb{N}_{1-\alpha} = \{1-\alpha, 2-\alpha, 3-\alpha, \ldots\}$, $0 < \alpha \leq 1$. A linear fractional difference equation on $\mathbb{H}$ is given by

$$\triangle^\alpha u(k) = pu(k-1+\alpha) + u(k-1+\alpha)q, \quad k \in \mathbb{N}_{1-\alpha}, \tag{16}$$

where $\triangle^\alpha v(k)$ is the Caputo delta fractional difference [17]. By using [18], we have the following observation.

Theorem 8. *The Eq. (16) is asymptotically stable if and only if*

$$|\lambda| < \left(2\cos\frac{|\arg\lambda| - \pi}{2-\alpha}\right)^\alpha, \qquad |\arg\lambda| > \frac{\alpha\pi}{2} \quad \forall\lambda \in \sigma(A). \tag{17}$$

The conditions (17) are difficult to express for our concrete forms of λ, so we do not go in details.

Finally, we consider a linear fractional differential equation

$${}^cD_0^\alpha x(t) = px(t) + x(t)q, \quad t \in \mathbb{R}_0 = [0,\infty), \tag{18}$$

on $\mathbb{H}$, where ${}^cD_0^\alpha x(t)$ is the Caputo fractional derivative with a lower limit at 0 [19]. Applying [20], we obtain the following result.

Theorem 9. *The Eq. (18) is asymptotically stable if and only if*

$$|\arg\lambda| > \frac{\alpha\pi}{2} \quad \forall\lambda \in \sigma(A). \tag{19}$$

Again, the condition (19) is difficult to express for our concrete forms of λ, so we do not go in details.

8. Example. To illustrate the theory, we consider the following simple example for

$$p = 1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \quad q = 5 + 6\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}. \tag{20}$$

Then A from (10) has a form

$$A = \begin{pmatrix} 6 & -8 & -10 & -12 \\ 8 & 6 & 4 & -4 \\ 10 & -4 & 6 & 4 \\ 12 & 4 & -4 & 6 \end{pmatrix}.$$

We have

$$\sigma(A) = \left\{6 \pm \sqrt{2\left(\sqrt{4321} + 89\right)}\imath, 6 \pm \sqrt{2\left(89 - \sqrt{4321}\right)}\imath\right\} \simeq$$

$$\simeq \{6 \pm 17.5917\imath, 6 \pm 6.82139\imath\},$$

$$\rho(A) = \max\left\{\sqrt{2\left(107 + \sqrt{4321}\right)}, \sqrt{2\left(107 - \sqrt{4321}\right)}\right\} =$$

$$= \sqrt{2\left(107 + \sqrt{4321}\right)} \simeq 18.5868,$$

$$\min|\sigma(A)| = \sqrt{2\left(107-\sqrt{4321}\right)} \simeq 9.08468,$$

$$\arg\sigma(A) = \left\{\pm\arctan\left(\frac{1}{3}\sqrt{\frac{1}{2}\left(\sqrt{4321}+89\right)}\right),\ \pm\arctan\left(\frac{1}{3}\sqrt{\frac{1}{2}\left(89-\sqrt{4321}\right)}\right)\right\} \simeq$$

$$\simeq \{\pm 1.2421, \pm 0.849375\}.$$

By Theorem 2, (1) with (20) is unstable, and now 0 is a source/repeller. By Theorem 6, (15) with (20) is unstable and 0 is a source/repeller. Since

$$\min|\sigma(A)| \simeq 9.08468 > 2 \geq \left(2\cos\frac{|\arg\lambda|-\pi}{2-\alpha}\right)^{\alpha},$$

condition (17) is not satisfied, by Theorem 8, (16) with (20) is not stable. By Theorem 9, (18) with (20) is asymptotically stable if and only if

$$\alpha < \frac{2\arctan\left(\frac{1}{3}\sqrt{\frac{1}{2}(89-\sqrt{4321})}\right)}{\pi} \simeq 0.540729.$$

We see that as $\alpha \in (0,1]$ varies, the stability of (18) with (20) is changing: 0 is asymptotically stable for

$$\alpha \in \left(0, \frac{2\arctan\left(\frac{1}{3}\sqrt{\frac{1}{2}(89-\sqrt{4321})}\right)}{\pi}\right),$$

and 0 is unstable for

$$\alpha \in \left(\frac{2\arctan\left(\frac{1}{3}\sqrt{\frac{1}{2}(89-\sqrt{4321})}\right)}{\pi},\ 1\right].$$

9. Conclusions. The next possible direction could be the study the nonhomogenous case

$$x_{n+1} = px_n + x_nq + r_n, \quad n \in \mathbb{N}_0,$$

for $p, q, x_n, x_{n+1},\ r_n \in \mathbb{H}$. Then a general, higher dimensional case on $\mathbb{H}^{m\times m}$ given by

$$x_{n+1} = Px_n + \left(x_n^TQ\right)^T + r_n, \quad n \in \mathbb{N}_0, \tag{21}$$

for $P, Q \in \mathbb{H}^{m\times m}$ and $x_n, r_n \in \mathbb{H}^m$. So we have

$$Ax = Px + \left(x^TQ\right)^T.$$

Note it does not hold $\left(x^TQ\right)^T = Q^Tx$ since $\mathbb{H}$ is noncommutative. We see that (21) is a general (real) quaternion linear difference equation. Extensions of the above hints can be studied to linear differential and fractional equations mentioned above.

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