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ON ASYMPTOTIC STABILITY AND INSTABILITY OF A FUNCTIONAL-DIFFERENTIAL EQUATION WITH PERIODIC RIGHT-HAND SIDE

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We give modifications of theorems on asymptotic stability and instability using Lyapunov functionals.

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Let $\mathbb{R}^p$ be a real linear space of p-dimensional vectors with norm $|x|, \mathbb{R} =]-\infty, +\infty[$ the real axis, $h > 0$ a given real number, C a Banach space of continuous functions $\varphi : [-h, 0] \to \mathbb{R}^p$ with norm $\|\varphi\| = \max(|\varphi(s)|, -h \le s \le 0)$. For any continuous function $x : [\alpha - h, \beta[\to \mathbb{R}^p$, $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, and every $t \in [\alpha, \beta[$, define a function $x_t \in C$ by $x_t(s) = x(t+s), -h \le s \le 0$. By $\dot{x}(t)$, we understand the right-hand derivative.

Consider a time lag type functional-differential equation,

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where $f : \mathbb{R}^+ \times C_H \to \mathbb{R}^p$ is a completely continuous mapping, periodic in $t \in \mathbb{R}^+$, i.e., $f(t + T, \varphi) = f(t, \varphi)$ for all $(t, \varphi) \in \mathbb{R}^+ \times C_H$, satisfying the Lipschitz condition in φ,

$$|f(t, \varphi_2) - f(t, \varphi_1)| \le L\|\varphi_2 - \varphi_1\|, \qquad L = L(t), \tag{2}$$

for all $(t, \varphi_2),\ (t, \varphi_1) \in \mathbb{R}^+ \times K$, and every compact set $K \subset C_H$.

With these assumptions, by [1], a solution $x = x(t, \alpha, \varphi)$ of equation (1) is unique and satisfies the initial condition $x_\alpha = \varphi$; it is also continuous with respect to the initial conditions (α, φ). If the solution $x = x(t, \alpha, \varphi)$ can not be extended outside the interval $[\alpha - h, \beta[$, then $\|x_t(\alpha, \varphi)\| \to H$ for $t \to \beta$.

To describe the limiting behavior of the solution $x = x(t, \alpha, \varphi)$, as $t \to +\infty$, we make the following definition [1].

Definition 1. *The orbit $\gamma^+(\alpha, \varphi)$ of a solution $x = x(t, \alpha, \varphi)$ of equation (1) is the set*

$$\gamma^+(\alpha, \varphi) = \{\, x_t(\alpha, \varphi),\ t \ge \alpha \,\}.$$

Definition 2. *Let a solution $x = x(t, \alpha, \varphi)$ of equation (1) be defined for all $t \geq \alpha - h$. A point $p \in C_H$ is called a positive limit point of this solution if there exists a sequence $t_n \to +\infty$ such that*

$$\lim_{n\to\infty} x_{t_n}(\alpha, \varphi) = p.$$

The set of all limit points of a solution $x = x(t, \alpha, \varphi)$ makes a positive limit set, $w^+(\alpha, \varphi)$.

According to the definition,

$$w^+(\alpha, \varphi) = \bigcap_{t\geq\alpha} d\Big(\bigcup_{\tau\geq t} x_\tau(\alpha, \varphi)\Big),$$

where $d\,A$ denotes the closure of the set A.

One can deduce the following properties of the sets $\gamma^+(\alpha, \varphi)$ and $w^+(\alpha, \varphi)$.

Lemma 1. *Let $x = x(t, \alpha, \varphi)$ be a solution of equation (1), defined for all $t \geq \alpha$. If $\gamma^+(\alpha, \varphi)$ is precompact, then $w^+(\alpha, \varphi)$ is nonempty, compact, and connected. We also have that $x_t(\alpha, \varphi) \to w^+(\alpha, \varphi)$ for $t \to +\infty$.*

Lemma 2. *If a solution $x = x(t, \alpha, \varphi)$ is bounded, $|x(t, \alpha, \varphi)| \leq H_1 < H$ for all $t \geq \alpha - h$, and the right-hand side of equation (1) is bounded on this solution,*

$$|f(t, x_t(\alpha, \varphi))| \leq M \tag{3}$$

for all $t \geq \alpha$, then the orbit of this solution, $\gamma^+(\alpha, \varphi)$, is precompact.

Because the function $f = f(t, \varphi)$ is completely continuous independently of t or periodic in t, it follows that it is bounded on the set $\mathbb{R}^+ \times \overline{C}_{H_1}$ for any $H_1 < H$. This observation and Lemma 2 give the following.

Corollary 1. *If a solution $x = x(t, \alpha, \varphi)$ of periodic equation (1) is bounded, $|x(t, \alpha, \varphi)| \leq H_1 < H$ for all $t \geq \alpha - h$, then its orbit, $\gamma^+(\alpha, \varphi)$, is precompact.*

The most important property of the positive limit set $w^+(\alpha, \varphi)$ of a solution of an autonomous or periodic equation is that it is invariant. A detailed treatment of this problem is given in [1]. In what follows, we give a modification of the invariance property in the case of a periodic equation.

Let the right-hand side of equation (1) be a periodic function of t with some $T > 0$, i.e., we have $f(t+T, \varphi) = f(t, \varphi)$ for all $(t, \varphi) \in \mathbb{R}^+ \times C_H$. It is easy to see that a solution $x = x(t, \alpha, \varphi)$ of such an equation satisfies

$$x(t + nT, \alpha + nT, \varphi) = x(t, \alpha, \varphi), \qquad t \geq \alpha, \tag{4}$$

for every point $(\alpha, \varphi) \in \mathbb{R}^+ \times C_H$ and every number $n \in \mathbb{N}$. Hence, solutions $x(t, \alpha, \varphi)$, determined by initial conditions $(\alpha, \varphi) \in [0, T[\times C_H$, define a totality of all solutions of (1).

Definition 3. *A set $M \subset C_H$ is called invariant if for any point $\psi \in M$ there exists time $\alpha \in [0, T[$ such that the solution $x(t, \alpha, \psi)$ is defined and contained in M, $x_t(\alpha, \psi) \in M$ for all $t \in \mathbb{R}$.*

Theorem 1. *Let a solution $x = x(t, \alpha, \varphi)$ of equation (1) with a periodic right-hand side be bounded, $|x(t, \alpha, \varphi)| \leq H_1 < H$ for all $t \geq \alpha$.*

Then the positive limit set $w^+(\alpha, \varphi)$ of this solution is connected, compact, and invariant.

Proof. Lemmas 1 and 2 imply that the set $w^+(\alpha, \varphi)$ is connected and compact. Let us show that it is invariant.

Let $\psi \in w^+(\alpha, \varphi)$ so that there exists a sequence $t_n \to +\infty$ such that $\varphi_n = x_{t_n}(\alpha, \varphi) \to \psi$ for $n \to \infty$. Choose a subsequence $t_{n_j} \to +\infty$ (denote it in the sequel by $t_j \to +\infty$) such that, for some sequence of natural numbers $n_j \to +\infty$, we have $t_j - n_j T \to \alpha^*$, $0 \leq \alpha^* < T$, for $j \to \infty$. By the definition of solution and using (4) we get

$$x(t + t_j, \alpha, \varphi) = x(t + t_j, t_j, \varphi_j) = x(t + t_j - n_j T, t_j - n_j T, \varphi_j). \tag{5}$$

The sequence of solutions $x_j(t) = x(t + t_j - n_j T, t_j - n_j T, \varphi_j)$ converges to the solution $x = x(\alpha^* + t, \alpha^*, \psi)$, for $j \to \infty$, uniformly in $t \in [-h, \gamma]$, where $\gamma = \mathrm{const}$ is an arbitrary constant. Hence, for each $t \in \mathbb{R}^+$, we have

$$\lim_{j \to \infty} x_{t_j + t}(\alpha, \varphi) = x_{\alpha^* + t}(\alpha^*, \psi).$$

This implies that $x_t(\alpha^*, \psi) \in \omega^+(\alpha, \varphi)$ for all $t \geq \alpha^*$.

Relations (5) hold for all t, $t \geq \alpha - t_j$, where $t_j \to +\infty$ for $j \to \infty$. Hence, the solutions $x_j(t)$ can be extended to $t \leq 0$ and the solution $x = x(t, \alpha^*, \psi)$ can be extended to all $t < \alpha^*$. For $t \leq 0$, in a similar way, we also have $x_t(\alpha^*, \psi) \subset w^+(\alpha, \varphi)$. The theorem is proved.

Let $f(t, 0) \equiv 0$ and equation (1) have the zero solution, $x = 0$.

As in the case of ordinary differential equations [2], there is the following relation between stability properties.

Theorem 2. *Suppose the function $f(t, \varphi)$ in equation (1) does not explicitly depend on t or it is periodic in t, i.e., there exists $T > 0$ such that $f(t + T, \varphi) = f(t, \varphi)$ for all $(t, \varphi) \in \mathbb{R}^+ \times C_H$.*

Then stability and asymptotic stability of the solution $x = 0$ of equation (1) are uniform.

The direct Lyapunov method, applied to study stability of functional-differential equations, is based on the use of Lyapunov functionals and functions.

Let $V : \mathbb{R}^+ \times C_H \to \mathbb{R}$ be a continuous Lyapunov functional and $x = x(t, \alpha, \varphi)$ a solution of equation (1). The function $V(t) = V(t, x_t(\alpha, \varphi))$ is a continuous function of $t \geq \alpha$.

We call

$$\dot{V}^+(t, x_t(\alpha, \varphi)) = \lim_{\Delta t \to 0^+} \sup \frac{1}{\Delta t} \Big(V(t + \Delta t, x_{t+\Delta t}(\alpha, \varphi)) - V(t, x_t(\alpha, \varphi)) \Big)$$

the upper right-hand derivative of V along the solution $x(t, \alpha, \varphi)$ [1, 3, 4].

Define the set $\{\dot{V}^+(T, \varphi) = 0\}$ to be a set of points $\varphi \in C_H$ in which the upper right-hand derivative of the functional $V(t, \varphi)$ equals zero at time $t \in \mathbb{R}^+$.

Definition 4. *A set $M \subseteq \{\dot{V}^+(T,\varphi) = 0\}$ is an invariant subset of the set $\{\dot{V}^+(t,\varphi) = 0\}$ if it follows from the condition $\varphi \in \{\dot{V}^+(\alpha,\varphi) = 0\}$ that the solution $x = x(t,\alpha,\varphi)$ is such that $x_t(\alpha,\varphi) \in M$ on the whole interval where the solution is defined.*

Theorem 3. *Let, for equation (1) with a periodic right-hand side, there exist a Lyapunov functional $V = V(t,\varphi)$, periodic in t with period T, $V(t+T,\varphi) = V(t,\varphi)$ for all $(t,\varphi) \in \mathbb{R}^+ \times C_H$, such that its derivative $\dot{V}^+(t,\varphi) \leq 0$.*

Then, for each bounded solution $x = x(t,\alpha,\varphi)$, $|x(t,\alpha,\varphi)| \leq H_1 < H$, the set of limit points $w^+(\alpha,\varphi) \subset M$ for all $t \geq \alpha$, where M is a maximal invariant subset of the set $\{\dot{V}^+(\alpha,\varphi) = 0\}$.

Proof. Suppose $\psi \in w^+(\alpha,\phi)$ and there exists $t_n \to +\infty$ such that $x_{t_n}(\alpha,\varphi) \to \psi$ for $n \to \infty$. As in Theorem 1, define a sequence $n_j \to \infty$ and a number α^* such that $t_{n_j} - n_jT \to \alpha^*$. The sequence $x = x(t_{n_j}+t,\alpha,\varphi)$ converges to the solution $x = x(\alpha^*+t,\alpha^*,\psi)$ uniformly in $t \in [-\gamma,\gamma]$ for each number $\gamma > 0$.

The function $V(t) = V(t,x_t(\alpha,\varphi))$ is monotone decreasing, as a function of t, since $\dot{V}^+(t,x_t) \leq 0$. Because the functional $V(t,\varphi)$ is periodic in t, it is bounded in the region $\overline{C}_{H_1} = \{\varphi : \|\varphi\| \leq H_1\}$. Thus, the function $V(t)$ is lower bounded and, hence, the limit

$$\lim_{t\to+\infty} V(t,x_t(\alpha,\varphi)) = c_0 \tag{6}$$

exists. Since $V(t,\varphi)$ is periodic, it follows that for each $t \in [-\gamma,\gamma]$ and sufficiently large n_j,

$$V(\tau,x_\tau(\alpha,\varphi))|_{\tau=t_{n_j}+t} = V(t_{n_j}+t-n_jT,x_\tau(\alpha,\varphi))|_{\tau=t_{n_j}+t}.$$

By passing to the limit for $n_j \to \infty$, since $V(t,\varphi)$ is continuous and $x(t_{n_j}+t,\alpha,\varphi)$ converges to $x(t,\alpha^*,\varphi^*)$, we get from (6) that

$$V(\tau,x_\tau(\alpha^*,\psi))|_{\tau=\alpha^*+t} = c_0 = \mathrm{const}$$

for all $t \in \mathbb{R}$. This implies that the derivative of $V(t,\varphi)$ along a solution $x = x(t,\alpha^*,\psi)$ such that $x_t(\alpha^*,\psi) \in \omega^+(\alpha,\varphi)$ for all $t \in \mathbb{R}$ equals zero, $\dot{V}^+(t,x_t(\alpha,\psi)) \equiv 0$ for all $t \in \mathbb{R}$. The theorem is proved.

This theorem is an extension of the theorem due to J. Hale [1, 4] and theorem of J. P. La Salle proved for ordinary differential equations with periodic right-hand sides. An essential difference of the latter is that it gives invariance of $\omega^+(\alpha,\varphi)$. Namely, it is established without passing to discrete dynamical systems. From the proved theorem, it is easy to deduce corresponding theorems of N. N. Krasovskii on asymptotic stability and instability by using a Lyapunov functional with the derivative of constant sign [3].

Suppose that the function $f(t,\varphi)$, which is periodic in t, satisfies the condition $f(t,0) = 0$, hence equation (1) has a zero solution.

Let us consider an application of a Lyapunov functional of constant sign to the problem on asymptotic stability for an equation which is autonomous or periodic in time. To this end, let us make the following definition.

Definition 5. *Let $M \subset C_H$ be an open set containing a point $\varphi = 0$. The solution $x = 0$ is stable with respect to M if for any $\varepsilon > 0$ and each $\alpha \in \mathbb{R}^+$ there exists $\delta = \delta(\varepsilon,\alpha) > 0$ such that for all $\varphi \in \{\|\varphi\| < \delta\} \cap M$ and all $t \geq \alpha$, we have $|x(t,\alpha,\varphi)| < \varepsilon$.*

Definition 6. *The solution $x = 0$ is uniformly stable with respect to a set M if the number δ in Definition 5 depends only on ε, i.e., $\delta = \delta(\varepsilon)$.*

Definition 7. *The solution $x = 0$ is asymptotically stable with respect to a set M if it is stable with respect to M and for each $\alpha \in \mathbb{R}^+$ there exists $\eta = \eta(\alpha) > 0$ such that for all $\varphi \in \{\|\varphi\| < \eta\} \cap M$, we have $\lim\limits_{t \to +\infty} x(t, \alpha, \varphi) = 0$.*

Definition 8. *The solution $x = 0$ is uniformly asymptotically stable with respect to a set M if it is uniformly stable with respect to M and there exists $\eta > 0$ such that for any small $\varepsilon > 0$, one can find $\sigma = \sigma(\varepsilon) > 0$ such that for all $\alpha \in \mathbb{R}^+$, all $\varphi \in \{\|\varphi\| < \eta\} \cap M$, and all $t \geq \alpha + \sigma$, we have $|x(t, \alpha, \varphi)| < \varepsilon$.*

Remark. Following the proof of Theorem 2 it is easy to show that if equation (1) is periodic in t, then stability and asymptotic stability of $x = 0$ with respect to M is uniform.

Theorem 4. *Suppose that*

1) the right-hand side of equation (1), $f = f(t, \varphi)$, is periodic in t, $f(t + T, \varphi) = f(t, \varphi)$, $T > 0$;

2) there exists a Lyapunov functional, $V = V(t, \varphi) \geq 0$, periodic in t, $V(t+T, \varphi) = V(t, \varphi)$, with the derivative $\dot{V}^+(t, \varphi) \leq 0$;

3) the solution $x = 0$ is asymptotically stable with respect to the set $M = \{V(t, \varphi) = 0\}$. Then the solution $x = 0$ is uniformly stable.

Proof. By Theorem 2, it is sufficient to prove that $x = 0$ is stable. Assume the converse, i.e., $x = 0$ is not stable so that for some α, $0 \leq \alpha < T$, there exist $\varepsilon_0 > 0$, a sequence $\varphi_n \to 0$, as $n \to +\infty$, and a sequence $\{\beta_n\}$ such that $|x(\alpha + \beta_n, \alpha, \varphi_n)| = \varepsilon_0$.

Let $\eta > 0$ be the number defined in Theorem 3. Set $l = \min(\eta, \varepsilon_0)/2$. For the solution $x = x(t, \alpha, \varphi_n)$ there exists a sequence of times, $\{\gamma_n \leq \beta_n\}$, such that

$$|x(t, \alpha, \varphi_n)| < l, \qquad |x(\alpha + \gamma_n, \alpha, \varphi_n)| = l \tag{7}$$

for $\alpha \leq t < \alpha + \gamma_n$. Since the solution $x = 0$ is continuous, it follows that $\gamma_n \to +\infty$. However, since $V = V(t, \varphi)$ is continuous in the point $\varphi = 0$, we have $V(\alpha, \varphi_n) \to 0$ for $n \to \infty$. The function $V = V(t, x(t, \alpha, \varphi_n))$ is monotone in t, hence,

$$\lim_{n \to \infty} V(t, x_t(\alpha, \varphi_n)) = 0 \tag{8}$$

for each fixed $t \geq \alpha$.

Let $\sigma = \sigma(l)$ be the number defined by Condition 3 of the theorem according to Definitions 7 and 8. Set $\theta_n = \alpha + \gamma_n - \sigma$ and consider the sequence $\{\psi_n = x(\theta_n, \alpha, \varphi_n)\}$. Let $\psi_{n_k} \to \psi^*$ for $n_k \to \infty$ be a convergent subsequence. Since $x(\theta_{n_k} + t, \alpha, \varphi_{n_k}) = x(\theta_{n_k} + t, \theta_{n_k}, \psi_{n_k})$, we find from relations (7) and (8) that $\|\psi^*\| \leq l$ and also

$$\begin{gathered} |x(\theta_{n_k} + \sigma, \theta_{n_k}, \psi_{n_k}| = l, \\ \lim_{n \to \infty} V(\theta_{n_k} + t, x_{\tau+t}(\theta_{n_k}, \psi_{n_k})|_{\tau = \theta_{n_k}}) = 0 \end{gathered} \tag{9}$$

for each fixed $t \in \mathbb{R}$.

Let us now define a subsequence $n_{k_l} \to \infty$ such that for some sequence of natural numbers, $j_l \to \infty$, the sequence $\theta_{n_{k_l}} - j_l T \to \alpha^*$, $0 \le \alpha^* < T$. Then, since solutions of an equation that is periodic in t are continuous, $x(\theta_{n_{k_l}} + t, \theta_{n_{k_l}}, \psi_{n_{k_l}}) = x(\theta_{n_{k_l}} - j_l T + t, \theta_{n_{k_l}} - j_l T, \psi_{n_{k_l}}) \to x^*(\alpha^* + t, \alpha^*, \psi^*)$ for $j_l \to \infty$, where $x^*(t, \alpha^*, \psi^*)$ is a solution of equation (1). As it was noted above, $\|\psi^*\| \le l$.

Passing to limit as $n_{k_l} \to +\infty$ we get, from the first relation of (9), that $|x^*(\alpha^* + \sigma, \alpha^*, \psi)| = l$. The second relation, since the functional $V(t, \varphi)$ is periodic in t, i.e.,

$$V(\theta_{n_{k_l}} + t, x_{\tau+t}(\theta_{n_{k_l}}, \psi_{n_{k_l}})|_{\tau=\theta_{n_{k_l}}}) = V(\theta_{n_{k_l}} - j_l T + t, x_{\tau+t}(\tau, \psi_{n_{k_l}})|_{\tau=\theta_{n_{k_l}} - j_l T}),$$

and continuous, gives, as $l \to \infty$, that

$$V(\alpha^* + t, x^*_{\alpha^*+t}(\alpha^*, \psi^*)) = 0$$

for all $t \in \mathbb{R}^+$, and this contradicts Condition 3 of the theorem. The theorem is proved.

Theorem 5. *Suppose that conditions of Theorem 3 hold and*
4) the set $\{V(t, \varphi) > 0\}$ does not contain solutions along which $\dot{V}^+(t, \varphi) = 0$.
Then the solution $x = 0$ of equation (1) is uniformly asymptotically stable.

Proof. Let $\eta > 0$ and $\sigma > 0$ be the numbers defined by Condition 3 of the theorem, $\Gamma_0 = \{\|\varphi\| < \eta_0 > 0\}$ be the region from where solutions are bounded so that if $\varphi \in \Gamma_0$, then $|x(t, \alpha, \varphi)| < \eta/2$ for all $t \ge \alpha - h$. Such a region exists, since $x = 0$ is uniformly bounded by Theorem 3.

Let us show that the region Γ_0 is a region of attraction to the zero solution of equation (1).

Assume the converse, i.e., for some number $\varepsilon_0 > 0$ and an arbitrary sequence $\sigma_n \to +\infty$ there exists a sequence of initial conditions $\{\varphi_n : \|\varphi_n\| \le \eta_0\}$ such that, for the solution $x(t, \alpha, \varphi_n)$ of (1), we have

$$|x(\alpha + \sigma_n, \alpha, \varphi_n)| = \varepsilon_0. \tag{10}$$

The function $V(t, x_t(\alpha, \varphi_n))$ is decreasing in t along the solution $x(t, \alpha, \varphi_n)$ by Theorem 2. It follows from Theorem 3 and Condition 4 of the theorem that $\lim\limits_{t \to +\infty} V(t, x_t(\alpha, \varphi_n)) = 0$. Whence we find that

$$\lim_{n \to \infty} V(\tau_n, \psi_n) = 0 \tag{11}$$

for the sequence of times $\tau_n = \alpha + \sigma_n - \sigma$ and the sequence of points $\psi_n = x_{\tau_n}(\alpha, \varphi_n)$.

By choosing a subsequence if necessary, assume that $\psi_n \to \psi^*$ for $n \to \infty$. Consider the sequence of solutions $x = x(t, \tau_n, \psi_n) = x(t, \alpha, \varphi_n)$, $t \ge \tau_n$. It follows from (10) that these solutions satisfy

$$|x(\tau_n + \sigma, \tau_n, \psi_n)| = \varepsilon_0. \tag{12}$$

Find a sequence of natural numbers $j_n \to \infty$ such that, for $n \to \infty$,

$$\tau_n - j_n T \to \alpha^*, \qquad 0 \le \alpha^* < T.$$

Then, since V is periodic and continuous, it follows from (11) that

$$V(\alpha^*, \psi^*) = \lim_{n\to\infty} V(\tau_n - j_n T, \psi_n) = \lim_{n\to\infty} V(\tau_n, \psi_n) = 0$$

in the point (α^*, ψ^*). Because $f = f(t, \varphi)$ is periodic in φ and solutions depend on the initial conditions continuously, we get that the sequence of solutions $x = x(t, \tau_n, \psi_n) = x(t - j_n T, \tau_n - j_n T, \psi_n)$ converges to the solution $x(\alpha^* + t, \alpha^*, \psi^*)$ for $n \to \infty$. By (12), we have $|x(\alpha^* + \sigma, \alpha^*, \psi^*)| = \varepsilon_0 > 0$. But this contradicts Condition 3, which proves the theorem.

Theorems 4 and 5 extend and generalize, for functional-differential equations, the results of [5, 6] obtained for ordinary differential equations.

Example. Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= -a_1(t)x_1(t) + a_2(t)x_1(t-h), \\ \dot{x}_2(t) &= -a_3(t)x_2(t) + a_4(t)x_1(t-h), \end{aligned} \tag{13}$$

where $a_i(t)$, $i = \overline{1, 4}$, are functions periodic in t with period T and satisfying the inequalities

$$a_1(t) \geq a_0 + \varepsilon > 0, \qquad |a_2(t)| \leq a_0, \qquad a_3(t) \geq a_0 > 0, \qquad \varepsilon > 0.$$

For the Lyapunov functional,

$$V = \frac{\varphi_1^2(0)}{2} + \frac{a_0}{2} \int_{-h}^{0} \varphi_1^2(\tau)\, d\tau,$$

we find that its derivative

$$\begin{aligned} \dot{V} &= -\frac{a_1(t)}{2}\, \varphi_1^2(0) + a_2(t)\varphi_1(0)\varphi_1(-h) + \frac{a_0}{2}\, (\varphi_1^2(0) - \varphi_1^2(-h)) \\ &\leq -\frac{\varepsilon}{2}\, |\varphi_1(0)|^2 \leq 0. \end{aligned}$$

The solution $x_1 = x_2 = 0$ is asymptotically stable with respect to the set $\{V = 0\} = \{\varphi_1 = 0\}$, since on this set, the system reduces to the equation $\dot{x}_2(t) = -a_3(t)x_2^3(t)$ with the coefficient $a_3(t) \geq a_0 > 0$. By using Theorem 5, we see that the zero solution of (11) is uniformly asymptotically stable.

REFERENCES

1. *Hale J. K.* Theory of Functional Differential Equations, Mir, Moscow (1984).
2. *Rouche N., Habets P., and Laloy M.* Stability Theory by Liapunov's Direct Method, Ser. Appl. Math. Sci., Springer-Verlag, New York etc. (1977).

3. *Krasovskii N. N.* Some Problems on the Theory of Stability of Motion [in Russian], Fizmatgiz, Moscow (1959).

4. *Kim A. V.* A Direct Lyapunov Method in the Theory of Stability of Systems with an Aftereffect [in Russian], Izdatel'stvo Ural'skogo Universiteta, Ekaterinburg (1992).

5. *Samoilenko A. M.* "A study of dynamical systems with the use of functions of constant sign," Ukr. Mat. Zh., **24**, No 3, 374 – 386 (1972).

6. *Bulgakov N. G.* Functions of Constant Sign in Stability Theory [in Russian], Izdatel'stvo "Universitetskoe," Minsk (1984).

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