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OSCILLATION AND NONOSCILLATION RESULTS FOR CAPUTO FRACTIONAL q-DIFFERENCE EQUATIONS AND INCLUSIONS

РЕЗУЛЬТАТИ ЩОДО ОСЦИЛЯЦІЇ ТА НЕОСЦИЛЯЦІЇ ДЛЯ РІВНЯНЬ ТА ВКЛЮЧЕНЬ ДРОБОВОГО ПОРЯДКУ З q-РІЗНИЦЕВОЮ ПОХІДНОЮ КАПУТО

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This paper deals with existence, oscillation, and nonoscillation of solutions to some classes of Caputo fractional q-difference equations and inclusions. The technique of proof employs set-valued analysis, fixed point theory, and the method of upper and lower solutions.

Для деяких класів рівнянь та включень дробового порядку з q-різницевою похідною Капуто досліджено існування, осциляцію та неосциляцію розв'язків. При доведенні використано багатозначний аналіз, теореми про нерухому точку та метод верхнього й нижнього розв'язків.

1. Introduction. Fractional differential equations and inclusions have been applied in various areas of engineering, mathematics, physics, and other applied sciences (see [1 – 8], and the references therein). Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with Caputo fractional derivatives; for example, see [2, 9]. The method of upper and lower solutions has been successfully applied to study the existence of solutions to a variety of differential equations and inclusions; see, for example, [10 – 15] and the references therein.

The study of fractional q-difference equations was initiated early in the 20-th century [16, 17] and has received significant attention in recent years [18, 19]. Some interesting details concerning initial and boundary value problems for q-difference and fractional q-difference equations can be found in [19 – 23] and the included references.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of different types of dynamic equations and inclusions. We refer the reader to the papers [24 – 26] and the references cited therein. In this paper, we discuss the existence and the oscillatory and nonoscillatory behavior of solutions to the fractional q-difference equation

$$\left({}^cD_q^\alpha u\right)(t) = f(t, u(t)), \quad t \in I := [0, T], \tag{1}$$

with the initial condition

$$u(0) = u_0 \in \mathbb{R}, \tag{2}$$

where $q \in (0,1),\ \alpha \in (0,1],\ T > 0,\ f : I \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α (as defined below).

We also investigate the existence and the oscillatory and nonoscillatory behavior of solutions to the fractional q-difference inclusion

$$\left({}^cD_q^\alpha u\right)(t) \in F(t, u(t)), \quad t \in I, \tag{3}$$

with the initial condition (2) , where $F \colon I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

This paper initiates the study of the oscillation and nonoscillation of solutions to Caputo q-fractional difference equations and inclusions.

2. Preliminaries. Consider the Banach space $C(I) := C(I, \mathbb{R})$ of continuous functions from I into $\mathbb{R}$ equipped with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t \in I} |u(t)|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|v\|_1 = \int\limits_0^T |v(t)|\, dt.$$

We now recall some definitions and properties from the fractional q-calculus. For $a \in \mathbb{R}$ set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q analogue of the power $(a - b)^n$ is

$$(a - b)^{(0)} = 1, \qquad (a - b)^{(n)} = \Pi_{k=0}^{n-1}(a - bq^k), \quad a, b \in \mathbb{R}, \quad n \in \mathbb{N}.$$

In general,

$$(a - b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}}\right), \quad a, b, \alpha \in \mathbb{R}.$$

Definition 2.1 [27]**.** *The q-gamma function is defined by*

$$\Gamma_q(\xi) = \frac{(1 - q)^{(\xi - 1)}}{(1 - q)^{\xi - 1}} \quad \textit{for} \quad \xi \in \mathbb{R} - \{0, -1, -2, \ldots\}.$$

Notice that the q-gamma function satisfies $\Gamma_q(1+\xi) = [\xi]_q \Gamma_q(\xi)$.

Next, we give definitions of different types of q-derivatives and q-integrals and indicate some of their properties.

Definition 2.2 [27]**.** *The q-derivative of order $n \in \mathbb{N}$ of a function $u\colon I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$,*

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1-q)t}, \quad t \neq 0, \qquad (D_q u)(0) = \lim_{t\to 0}(D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t), \qquad t \in I, \quad n \in \{1, 2, \ldots\}.$$

We set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3 [27]**.** *The q-integral of a function $u\colon I_t \to \mathbb{R}$ is defined by*

$$(I_q u)(t) = \int\limits_0^t u(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 2.4 [28]**.** *The Riemann–Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $u\colon I \to \mathbb{R}$ is defined by $(I_q^0 u)(t) = u(t)$, and*

$$(I_q^\alpha u)(t) = \int\limits_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s)\, d_q s, \quad t \in I.$$

Lemma 2.1 [29]**.** *For $\alpha \in \mathbb{R}_+ := [0,\infty)$ and $\lambda \in (-1, \infty)$, we have*

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(t-a)^{(\lambda+\alpha)}, \quad 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 2.5 [30]**.** *The Riemann–Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u\colon I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$, and*

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t), \quad t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 2.6 [30]. *The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u: I \to \mathbb{R}$ is defined by $\left({}^C D_q^0 u\right)(t) = u(t)$ and*

$$\left({}^C D_q^{\alpha} u\right)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t), \quad t \in I.$$

As a simple illustration, we have the following example.

Example 2.1. Let $\alpha > 0$ and $\beta > [\alpha]$. Then for each $t > 0$, we have

$$\begin{cases} \left({}^C D_q^{\alpha}\right)(t^{\beta-1}) = t^{\beta-\alpha-1} \dfrac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)}, & \beta \notin \mathbb{N}, \\ \left({}^C D_q^{\alpha}\right)(t^{\beta-1}) = 0, & \beta \in \{1, 2, \ldots, [\alpha]-1\}. \end{cases}$$

Lemma 2.2 [30]. *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$\left(I_q^{\alpha}\, {}^C D_q^{\alpha} u\right)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$\left(I_q^{\alpha}\, {}^C D_q^{\alpha} u\right)(t) = u(t) - u(0).$$

For a given Banach space $(X, \|\cdot\|)$, we define the following subsets of $\mathcal{P}(X)$:
$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$,
$P_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex}\}$,
$P_{cp,cv}(X) = P_{cp}(X) \Cap P_{cv}(X)$.

The following properties of multivalued maps will be needed.

Definition 2.7. *A multivalued map $G : X \to \mathcal{P}(X)$ is said to be convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}$ exists).*

Definition 2.8. *A multivalued map $G : X \to \mathcal{P}(X)$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set $N \subset X$ containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$.*

Definition 2.9. *The multivalued map $G : X \to \mathcal{P}(X)$ is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.*

Definition 2.10. *Let $G : X \to \mathcal{P}(X)$ be completely continuous with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$).*

Definition 2.11. *A multivalued map $G : X \to \mathcal{P}(X)$ has a fixed point if there is $x \in X$ such that $x \in G(x)$.*

We denote by $FixG$ the set of fixed points of the multivalued operator G.

Definition 2.12. *A multivalued map* $G\colon J \to P_{cl}(\mathbb{R})$ *is said to be measurable if for every* $y \in \mathbb{R}$, *the function*

$$t \to d(y, G(t)) = \inf\big\{|y - z| : z \in G(t)\big\}$$

is measurable.

The following relationship between upper semi-continuous maps and closed graphs is well known.

Lemma 2.3 [31]. *Let G be a completely continuous multivalued map with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph.*

Definition 2.13. *A multivalued map* $F\colon I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is said to be Carathéodory if:*

(i) $t \to F(t, u)$ *is measurable for each* $u \in \mathbb{R}$;

(ii) $u \to F(t, u)$ *is upper semicontinuous for almost all* $t \in I$.

Moreover, F *is said to be* L^1*-Carathéodory if (1), (2), and the following condition hold:*

(iii) For each $q > 0$, *there exists* $\varphi_q \in L^1(I, \mathbb{R}_+)$ *such that*

$$\|F(t, u)\|_{\mathcal{P}} = \sup\big\{|v| : v \in F(t, u)\big\} \le \varphi_q \ \textit{for all} \ |u| \le q \ \textit{and for a.e.} \ t \in I.$$

For each $u \in C(I, \mathbb{R})$, we define the set of selections of F by

$$S_{F \circ u} = \big\{v \in L^1(I, \mathbb{R}) : v(t) \in F(t, u(t)) \ \text{ a.e. } \ t \in I\big\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. The function $H_d\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(A, b)\right\}$$

is known as the Hausdorff–Pompeiu metric. For more details on multivalued maps, see the monograph of Hu and Papageorgiou [31].

3. Caputo fractional q-difference equations. We begin by defining what we mean by a solution, an upper solution, and a lower solution to the problem (1), (2).

Definition 3.1. *A function* $u \in C(I)$ *is said to be a solution of problem (1), (2), if* $u(0) = u_0$ *and* ${}^C D_q^{\alpha} u(t) = f(t, u(t))$ *on* I.

Definition 3.2. *A function* $w \in C(I)$ *is said to be an upper solution of (1), (2) if* $w(0) \ge u_0$ *and* ${}^C D_q^{\alpha} w(t) \ge f(t, w(t))$ *on* I. *Similarly, a function* $v \in C(I)$ *is said to be a lower solution of (1), (2) if* $v(0) \le u_0$, *and* ${}^C D^{\alpha} v(t) \le f(t, v(t))$ *on* I.

In the sequel, we will need the following fixed point theorem.

Theorem 3.1 (Schauder's fixed point theorem [32]). *Let B be a closed, convex, and nonempty subset of a Banach space X. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of X. Then N has at least one fixed point in B.*

3.1. Existence of solutions. We now present an existence result for the problem (1), (2).

Theorem 3.2. *Assume that:*

(H) There exist v and $w \in C$, lower and upper solutions for the problem (1), (2) respectively, such that $v \le w$.

Then the problem (1), (2) has at least one solution u *such that*

$$v(t) \leq u(t) \leq w(t) \quad \textit{for all} \quad t \in I.$$

Proof. Consider the modified problem

$$\begin{cases} \left({}^cD_q^\alpha u\right)(t) = g(t, u(t)), \quad t \in I := [0, T], \\ u(0) = u_0, \end{cases} \tag{4}$$

where

$$g(t, u(t)) = f(t, h(t, u(t))) \qquad \text{and} \qquad h(t, u(t)) = \max\left\{v(t), \min\{u(t), w(t)\}\right\}$$

for each $t \in I$. A solution of problem (4) is a fixed point of the operator $N : C \to C$ defined by

$$(Nu)(t) = u_0 + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s))\, d_q s.$$

Notice that the functions $f,\ v,$ and w are continuous, and from the definition of the function g, we have

$$|g(t, u(t))| \leq \sup_{t \in I} \left| f\left(t, \max\left\{v(t), \min\{u(t), w(t)\}\right\}\right)\right| := M.$$

Set

$$\eta = |u_0| + \frac{MT^{(\alpha)}}{\Gamma_q(1 + \alpha)},$$

and

$$D = \left\{u \in C : \|u\|_C \leq \eta\right\}.$$

Clearly, D is a closed, bounded convex subset of C and that N maps D into itself. We shall show that N satisfies the assumptions of Theorem 3.1. The proof will be given in several steps.

Step 1. *N is continuous and $N(D)$ is bounded.* It is clear that $N(D)$ is bounded since $N(D) \subset D$ and D is bounded.

Next, let $\{u_n\}$ be a sequence such that $u_n \to u$ in D. Then

$$|(Nu_n)(t) - (Nu)(t)| \leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|g(s, u_n(s)) - g(s, u(s))\right| d_q s \leq$$

$$\leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \sup_{s \in I} \left|g(s, u_n(s)) - g(s, u(s))\right| d_q s.$$

For each $t \in I,$ set $(g \circ u)(t) := g(t, u(t)).$ Thus,

$$|(Nu_n)(t) - (Nu)(t)| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \sup_{s\in I} \big|(g\circ u_n)(s) - (g\circ u)(s)\big|\, d_qs \leq$$

$$\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|g\circ u_n - g\circ u\|_C\, d_qs \leq$$

$$\leq \frac{T^{(\alpha)}}{\Gamma_q(1+\alpha)} \|g\circ u_n - g\circ u\|_C.$$

From Lebesgue's dominated convergence theorem and the continuity of the function $g,$ we see that

$$\big|(Nu_n)(t) - (Nu)(t)\big| \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. *$N(D)$ is equicontinuous.* Let $t_1, t_2 \in I$ with $t_1 < t_2,$ and let $u \in D.$ Then,

$$\|(Nu)(t_2) - (Nu)(t_1)\| \leq \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s,u(s))|\, d_qs +$$

$$+ \int_0^{t_1} \left| \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| |g(s,u(s))|\, d_qs \leq$$

$$\leq M \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\, d_qs +$$

$$+ M \int_0^{t_1} \left| \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_qs.$$

As $t_1 \to t_2,$ the right-hand side of the above inequality tends to zero.

As a consequence of the above two steps and the Arzelà-Ascoli theorem, we can conclude that N is a continuous and completely continuous operator. An application of Theorem 3.1 yields that N has a fixed point u that in turn is a solution of problem (4).

Step 3. *The solution u of (4) satisfies*

$$v(t) \leq u(t) \leq w(t) \quad \textit{for all} \quad t \in I.$$

Let u be the above solution to (4). We wish to show that

$$u(t) \leq w(t) \quad \text{for all} \quad t \in I.$$

Assume that $u - w$ attains a positive maximum on I at $\bar{t} \in I,$ that is,

$$(u-w)(\bar{t}) = \max\{u(t) - w(t) : t \in I\} > 0.$$

We distinguish the following cases.

Case 1. If $\bar{t} \in (0,T)$ then, there exists $t^* \in [0,\bar{t})$ such that

$$0 < u(t) - w(t) \leq u(\bar{t}) - w(\bar{t}) \quad \text{for all} \quad t \in [t^*, \bar{t}]. \tag{5}$$

From the definition of h,

$$\left({}^cD_q^{\alpha} u\right)(t) = g(t, u(t)), \tag{6}$$

for all $t \in [t^*, \bar{t}]$, where

$$g(t, u(t)) = f(t, w(t)), \quad t \in [t^*, \bar{t}].$$

An integration of (6) yields

$$u(t) = u_0 + \int\limits_{t^*}^{t} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, w(s))\, d_q s.$$

Using the fact that w is an upper solution to (1), (2) we get

$$u(t) - u(t^*) \leq w(t) - w(t^*) - \int\limits_{t^*}^{t} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, w(s))\, d_q s < w(t) - w(t^*). \tag{7}$$

Thus from (5) and (7), we obtain the contradiction

$$u(\bar{t}) - w(\bar{t}) < u(t^*) - w(t^*) \quad \text{for all} \quad t \in [t^*, \bar{t}\,].$$

Case 2. If $\bar{t} = 0$, then

$$w(0) < u(0) \leq w(0)$$

which is a contradiction. Thus,

$$u(t) \leq w(t) \quad \text{for all} \quad t \in I.$$

Analogously, we can prove that

$$u(t) \geq v(t) \quad \text{for all} \quad t \in I.$$

This shows that the problem (4) has a solution u satisfying $v \leq u \leq w$ that is also a solution of problem (1), (2).

3.2. Nonoscillation and oscillation of solutions. Our next theorem gives sufficient conditions to ensure the nonoscillation of solutions of problem (1), (2). We begin with the definition of an oscillatory solution.

Definition 3.3. *A solution $u \in C$ of problem (1), (2) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, u is called nonoscillatory.*

Theorem 3.3. *In addition to condition (H), assume that:*

(H′) v is an eventually positive nondecreasing lower solution, or w is an eventually negative nonincreasing upper solution of (1), (2).

Then every solution u of (1), (2) such that $u \in [v, w]$ is nonoscillatory.

Proof. Assume that v is an eventually positive. Thus there exists $T_v > 0,$ such that

$$v(t) > 0 \quad \text{for all} \quad t > T_v.$$

Hence, $u(t) > 0$ for all $t > T_v$. This means that y is nonoscillatory.

Analogously, if w is an eventually negative, then there exists $T_w > 0,$ such that $u(t) < 0$; for all $t > T_w,$ which means that u is nonoscillatory.

The following theorem concerns the oscillatory behavior of the solutions of problem (1), (2).

Theorem 3.4. *In addition to condition (H) assume that:*

(H″) v and w are oscillatory lower and upper solutions, respectively, of (1), (2).

Then every solution u of (1), (2) such that $u \in [v, w]$ is oscillatory.

Proof. Assume that problem (1), (2) has a nonoscillatory solution u on I. Then there exists $T_u > 0$ such that $u(t) > 0$ or $u(t) < 0$ for all $t > T_u$. In the case where $u(t) > 0$ for all $t > T_u,$ we have $w(t) > 0$ for all $t > T_u$, which is a contradiction since w is an oscillatory upper solution. Analogously, if $u(t) < 0$ for all $t > T_u,$ we have $v(t) < 0$ for all $t > T_u,$ which again is a contradiction since v is an oscillatory lower solution.

4. Caputo fractional q-difference inclusions. We begin by defining what we mean by a solution of the fractional inclusion problem (2), (3).

Definition 4.1. *A function $u \in AC(I)$ is said to be a solution of problem (2), (3) if $u(0) = u_0$, and there exists a function $f \in S_{F\circ u}$ such that ${}^C D_q^\alpha u(t) = f(t)$ a.e. $t \in I$.*

Definition 4.2. *A function $w \in AC(I)$ is said to be an upper solution of (2), (3) if $w(0) \geq u_0$, and there exists a function $v_1 \in S_{F\circ w}$ such that ${}^C D_q^\alpha w(t) \geq v_1(t)$ a.e. $t \in I$. Similarly, a function $v \in AC(I)$ is said to be a lower solution of (2), (3) if $v(0) \geq u_0$ and there exists a function $v_2 \in S_{F\circ v}$ such that ${}^C D^\alpha v(t) \leq v_2(t)$ a.e. $t \in I$.*

We will need the following fixed point theorem.

Theorem 4.1 (Martelli's fixed point theorem [33])**.** *Let X be a Banach space and $N : X \to \mathcal{P}_{cl,cv}(X)$ be an upper semicontinuous and condensing multivalued operator. If the set $\Omega := \{u \in X : \lambda u \in N(u)$ for some $\lambda > 1\}$ is bounded, then N has a fixed point.*

4.1. Existence of solutions.

Theorem 4.2. *Assume that the following conditions hold:*

(H1) $F : I \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is Carathéodory;

(H2) There exist $v, w \in AC(I)$ which are lower and upper solutions, respectively, of problem (2), (3) such that $v \leq w$;

(H3) There exists $l \in L^1(I, \mathbb{R}^+)$ such that

$$H_d(F(t, u), F(t, \bar{u})) \leq l(t)|u - \bar{u}| \quad \textit{for all} \quad u, \bar{u} \in \mathbb{R},$$

and

$$d(0, F(t, 0)) \leq l(t) \quad \textit{a.e.} \quad t \in I.$$

Then the problem (2), (3) has at least one solution u defined on I such that

$$v \leq u \leq w.$$

Proof. Consider the multivalued operator $N: C(I) \to \mathcal{P}(C(I))$ defined by

$$N(u) = \left\{ h \in C(I) \colon h(t) = u_0 + \int\limits_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s)\, d_q s, \;\; v \in S_{F\circ u} \right\}.$$

Clearly, the fixed points of N are solutions of the problem (2), (3).

Consider the following modified problem:

$$^C D_q^\alpha u(t) \in F\big(t, \tau(u(t))\big), \quad \text{for a.e.} \quad t \in I, \tag{8}$$

$$u(0) = u_0, \tag{9}$$

where

$$\tau(u(t)) = \max\big\{v(t), \min\{u(t), w(t)\}\big\},$$

and

$$\overline{u}(t) = \tau(u(t)).$$

A solution to (8), (9) is a fixed point of the operator $N: C(I) \to \mathcal{P}(C(I))$ defined by

$$N(u) = \big\{h \in C(I) \colon h(t) = u(0) + (I_q^\alpha \nu)(t)\big\},$$

where

$$\nu \in \Big\{x \in \widetilde{S}^1_{F\circ\tau(u)} \colon x(t) \geq v_1(t) \;\text{ on }\; A_1 \;\text{ and }\; x(t) \leq v_2(t) \;\text{ on }\; A_2\Big\},$$

$$\widetilde{S}^1_{F\circ\tau(y)} = \big\{x \in L^1(I) \colon x(t) \in F(t, (\tau u)(t)) \;\text{ a.e. }\; t \in I\big\},$$

$$A_1 = \big\{t \in I \colon u(t) < v(t) \leq w(t)\big\}, \qquad A_2 = \big\{t \in I \colon v(t) \leq w(t) < u(t)\big\}.$$

Remark 4.1. (i) For each $u \in C(I)$, the set $\widetilde{S}^1_{F\circ\tau(u)}$ is nonempty. In fact, (H1) implies that there exists $v_3 \in S^1_{F\circ\tau(u)}$ so we set

$$v = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3},$$

where

$$A_3 = \big\{t \in I \colon v(t) \leq u(t) \leq w(t)\big\}.$$

Then by decomposability, $x \in \widetilde{S}^1_{F\circ\tau(u)}$.

(ii) From the definition of τ, it is clear that $F(\cdot, \tau u(\cdot))$ is an L^1-Carathéodory multi-valued map with compact convex values and there exists $\phi_1 \in C(I, \mathbb{R}^+)$ such that

$$\big\|F(t, \tau u(t))\big\|_{\mathcal{P}} \leq \phi_1(t) \quad \text{for each} \quad u \in \mathbb{R}.$$

Now set

$$R := |u_0| + \frac{\|\phi_1\|_\infty T^{(\alpha)}}{\Gamma_q(1+\alpha)},$$

and consider the closed and convex subset of $C(I)$ given by

$$B = \{u \in C(I) \colon \|u\|_\infty \leq R\}.$$

We shall show that the operator $N \colon B \to \mathcal{P}_{cl,cv}(B)$ satisfies all the assumptions of Theorem 4.1. The proof will be given in steps.

Step 1. *$N(u)$ is convex for each $u \in B$.* Let $h_1,\ h_2$ belong to $N(u)$; then there exist $\nu_1, \nu_2 \in \widetilde{S}^1_{F\circ\tau(u)}$ such that, for each $t \in I$ and $i = 1, 2,$ we have

$$h_i(t) = u(0) + (I_q^\alpha \nu_i)(t).$$

Let $0 \leq d \leq 1$. Then, for each $t \in I,$ we have

$$\big(dh_1 + (1-d)h_2\big)(t) = u(0) + \int\limits_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big[d\nu_1(s) + (1-d)\nu_2(s)\big]\, d_qs.$$

Since $S_{F\circ\tau(u)}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(u).$$

Step 2. *N maps bounded sets into bounded sets in B.* For each $h \in N(u),$ there exists $\nu \in \widetilde{S}^1_{F\circ\tau(u)}$ such that

$$h(t) = u(0) + \int\limits_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\nu(s)\, d_qs.$$

From conditions (H1) – (H3), for each $t \in I,$ we have

$$|h(t)| \leq |u(0)| + \left|\int\limits_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|\nu(s)|\, d_qs\right| \leq$$

$$\leq |u_0| + \frac{\|\phi_1\|_\infty T^{(\alpha)}}{\Gamma_q(1+\alpha)}.$$

Thus,

$$\|h\|_\infty \leq R.$$

Step 3. *N maps bounded sets into equicontinuous sets of B.* Let $t_1, t_2 \in I$ with $t_1 < t_2,$ and let $u \in B$ and $h \in N(u)$. Then

$$\big|h(t_2) - h(t_1)\big| = \left|\int\limits_0^{t_1} \frac{\big|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\big|}{\Gamma_q(\alpha)}\nu(s)\, d_qs + \right.$$

$$+\int\limits_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\nu(s)\,d_qs\Bigg|\leq$$

$$\leq\int\limits_0^{t_1}\frac{\left|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right|}{\Gamma_q(\alpha)}|\nu(s)|\,d_qs+$$

$$+\int\limits_{t_1}^{t_2}\frac{\left|(t_2-qs)^{(\alpha-1)}\right|}{\Gamma_q(\alpha)}|\nu(s)|\,d_qs\leq$$

$$\leq\|\phi_1\|_\infty\int\limits_0^{t_1}\frac{\left|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right|}{\Gamma_q(\alpha)}\,d_qs+$$

$$+\|\phi_1\|_\infty\int\limits_{t_1}^{t_2}\frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}\,d_qs\to 0\quad\text{as}\quad t_1\to t_2.$$

As a consequence of the three steps above, we can conclude from the Arzelà – Ascoli theorem that $N: C(I)\to\mathcal{P}(C(I))$ is continuous and completely continuous.

Step 4. *N has a closed graph.* Let $u_n\to u_*,\ h_n\in N(u_n)$, and $h_n\to h_*$. We need to show that $h_*\in N(u_*)$. Now $h_n\in N(u_n)$ implies there exists $\nu_n\in\widetilde{S}^1_{F\circ\tau(u_n)}$ such that, for each $t\in I$,

$$h_n(t)=u(0)+\int\limits_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\nu_n(s)\,d_qs.$$

We must show that there exists $\nu_*\in\widetilde{S}^1_{F\circ\tau(u_*)}$ such that, for each $t\in I$,

$$h_*(t)=u(0)+\int\limits_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\nu_*(s)\,d_qs.$$

Since $F(t,\cdot)$ is upper semi-continuous, for every $\epsilon>0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n\geq n_0(\epsilon)$, we have

$$\nu_n(t)\in F(t,\tau u_n(t))\subset F(t,u_*(t))+\epsilon B(0,1)\quad\text{a.e.}\quad t\in I.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $\nu_{n_m}(\cdot)$ such that

$$\nu_{n_m}(\cdot)\to\nu_*(\cdot)\quad\text{as}\quad m\to\infty,$$

and

$$\nu_*(t)\in F(t,\tau u_*(t))\quad\text{a.e.}\quad t\in I.$$

For every $w\in F(t,\tau u_*(t))$, we have

$$\left|\nu_{n_m}(t)-\nu_*(t)\right|\leq|\nu_{n_m}(t)-w|+|w-\nu_*(t)|.$$

Hence,

$$\big|\nu_{n_m}(t)-\nu_*(t)\big| \le d\big(\nu_{n_m}(t), F(t,\tau u_*(t))\big).$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_* to obtain

$$\big|\nu_{n_m}(t)-\nu_*(t)\big| \le H_d\big(F(t,\tau u_{n_m}(t)), F(t,\tau u_*(t))\big) \le l(t)\|u_{n_m}-u_*\|_\infty.$$

Thus,

$$\big|h_{n_m}(t)-h_*(t)\big| \le \int\limits_0^t \frac{\big|(t-qs)^{(\alpha-1)}\big|}{\Gamma_q(\alpha)}\big|\nu_{n_m}(s)-\nu_*(s)\big|\,d_qs \le$$

$$\le \|u_{n_m}-u_*\|_\infty \int\limits_0^t \frac{\big|(t-qs)^{(\alpha-1)}\big|}{\Gamma_q(\alpha)} l(s)\,d_qs.$$

Therefore,

$$\|h_{n_m}-h_*\|_\infty \le \|u_{n_m}-u_*\|_\infty \int\limits_0^t \frac{\big|(t_1-qs)^{(\alpha-1)}\big|}{\Gamma_q(\alpha)} l(s)\,d_qs \to 0 \quad \text{as} \quad m\to\infty,$$

so Lemma 2.3 implies that N is upper semicontinuous.

Step 5. *The set* $\Omega=\{u\in C\colon \lambda u\in N(u)$ *for some* $\lambda>1\}$ *is bounded.* Let $u\in\Omega$. Then, there exists $f\in\lambda(\tilde S_{F\circ g(u)})$ such that

$$\lambda u(t) = |u_0| + \int\limits_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s)|\,d_qs.$$

As in Step 2, this implies that for each $t\in I$, we have

$$\|u\|_C \le \frac{R}{\lambda} < \ell.$$

This shows that Ω is bounded. As a consequence of Theorem 4.1, N has a fixed point that in turn is a solution of (2), (3) on I.

Step 6. *Every solution* u *of (8), (9) satisfies* $v(t)\le u(t)\le w(t)$ *for all* $t\in I$. Let u be a solution of (8), (9). To prove that $v(t)\le u(t)$ for all $t\in I$, suppose this is not the case. Then there exist $t_1,\ t_2$, with $t_1<t_2$, such that $v(t_1)=u(t_1)$ and $v(t)>u(t)$ for all $t\in(t_1,\,t_2)$. In view of the definition of τ,

$${}^C D_q^\alpha u(t) \in F(t, v(t)) \quad \text{for all} \quad t\in(t_1,\,t_2).$$

Thus, there exists $y\in S_{F\circ\tau(v)}$ with $y(t)\ge v_1(t)$ a.e. on $(t_1,\,t_2)$ such that

$${}^C D_q^\alpha u(t) = y(t) \quad \text{for all} \quad t\in(t_1,\,t_2).$$

An integration on $(t_1,\,t]$, with $t\in(t_1,\,t_2)$, yields

$$u(t) - y(t_1) = \int\limits_{t_1}^{t} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \nu(s)\, d_q.$$

Since v is a lower solution of (2), (3),

$$v(t) - v(t_1) \leq \int\limits_{t_1}^{t} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_1(s) d_q, \quad t \in (t_1, t_2).$$

From the facts that $u(t_0) = v(t_0)$ and $\nu(t) \geq v_1(t)$, it follows that

$$v(t) \leq u(t) \quad \text{for all} \quad t \in (t_1, t_2).$$

This is a contradiction, since $v(t) > u(t)$ for all $t \in (t_1, t_2)$. Consequently,

$$v(t) \leq u(t) \quad \text{for all} \quad t \in I.$$

Similarly, we can prove that

$$u(t) \leq w(t) \quad \text{for all} \quad t \in I.$$

This shows that

$$v(t) \leq u(t) \leq w(t) \quad \text{for all} \quad t \in I.$$

Therefore, the problem (8), (9) has a solution u that is also a solution of (2), (3) and satisfies $v \leq u \leq w$.

4.2. Nonoscillation and oscillation of solutions. As in Theorems 3.3 and 3.4, the following results ensure the nonoscillation and oscillation of solutions of problem (2), (3).

Theorem 4.3. *In addition to conditions (H1) – (H3), assume that:*

(H4) v is an eventually positive nondecreasing lower solution, or w is an eventually negative nonincreasing upper solution of (2), (3).

Then every solution u of (2), (3) such that $u \in [v, w]$ is nonoscillatory.

Theorem 4.4. *In addition to conditions (H1) – (H3), assume that:*

(H5) v and w are oscillatory lower and upper solutions, respectively, of (2), (3).

Then every solution u of (2), (3) such that $u \in [v, w]$ is oscillatory.

References

1. S. Abbas, M. Benchohra, J. R. Graef, J. Henderson, *Implicit fractional differential and integral equations: Existence and stability*, De Gruyter, Berlin (2018).
2. S. Abbas, M. Benchohra, G. M. N'Guérékata, *Topics in fractional differential equations*, Springer, New York (2012).
3. S. Abbas, M. Benchohra, G. M. N'Guérékata, *Advanced fractional differential and integral equations*, Nova Sci. Publ., New York (2015).
4. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Stud., **204**, Elsevier Sci., Amsterdam (2006).
5. J. A. Tenreiro Machado, V. Kiryakova, *The chronicles of fractional calculus*, Fract. Calc. Appl. Anal., **20**, 307–336 (2017).

6. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach, Amsterdam (1987).
7. V. E. Tarasov, *Fractional dynamics: application of fractional calculus to dynamics of particles, fields and media*, Higher Education Press, Beijing, & Springer, Heidelberg etc. (2010).
8. Y. Zhou, *Basic theory of fractional differential equations*, World Sci., Singapore (2014).
9. A. A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc., **38**, 1191–1204 (2001).
10. S. Abbas and M. Benchohra, *Upper and lower solutions method for Darboux problem for fractional order implicit impulsive partial hyperbolic differential equations*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., **51**, 5–18 (2012).
11. S. Abbas and M. Benchohra, *The method of upper and lower solutions for partial hyperbolic fractional order differential inclusions with impulses*, Discuss. Math. Differ. Incl. Control Optim., **30**, 141–161 (2010).
12. S. Abbas, M. Benchohra, S. Hamani, J. Henderson, *Upper and lower solutions method for Caputo – Hadamard fractional differential inclusions*, Math. Morav., **23**, 107–118 (2019).
13. S. Abbas, M. Benchohra, A. Hammoudi, *Upper, lower solutions method and extremal solutions for impulsive discontinuous partial fractional differential inclusions*, PanAmer. Math. J., **24**, 31–52 (2014).
14. S. Abbas, M. Benchohra, J. J. Trujillo, *Upper and lower solutions method for partial fractional differential inclusions with not instantaneous impulses*, Progr. Fract. Differentiation Appl., **1**, No 1,+ 11–22 (2015).
15. M. Benchohra, S. K. Ntouyas, *The lower and upper solutions method for first order differential inclusions with nonlinear boundary conditions*, J. Inequal. Pure Appl. Math., **3**, Issue 1, 1–8 (2002).
16. C. R. Adams, *On the linear ordinary q-difference equation*, Ann. of Math. (2), **30**, 195–205 (1928).
17. R. D. Carmichael, *The general theory of linear q-difference equations*, Amer. J. Math., **34**, 147–168 (1912).
18. M. H. Annaby, Z. S. Mansour, *q-fractional calculus and equations*, Lect. Notes Math., **2056**, Springer, Heidelberg (2012).
19. T. Ernst, *A comprehensive treatment of q-calculus*, Birkhäuser, Basel (2012).
20. B. Ahmad, *Boundary value problem for nonlinear third order q-difference equations*, Electron. J. Differential Equations, **2011**, № 94, 1–7 (2011).
21. B. Ahmad, S. K. Ntouyas, L. K. Purnaras, *Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations*, Adv. Difference Equ., **2012**, 140 (2012).
22. M. El-Shahed, H. A. Hassan, *Positive solutions of q-difference equation*, Proc. Amer. Math. Soc., **138**, 1733–1738 (2010).
23. S. Etemad, S. K. Ntouyas, B. Ahmad, *Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders*, Mathematics, **7**, 659, 1–15 (2019).
24. M. Benchohra, S. Hamani, Y. Zhou, *Oscillation and nonoscillation for Caputo – Hadamard impulsive fractional differential inclusions*, Adv. Difference Equ., **2019**, 74 (2019)
25. J. R. Graef, J. Karsai, *Oscillation and nonoscillation in nonlinear impulsive systems with increasing energy*, Proc. Int. Conf. Dyn. Syst. Differ. Equ., **7**, 161–173 (2000).
26. S. Harikrishnan, P. Prakash, J. J. Nieto, *Forced oscillation of solutions of a nonlinear fractional partial differential equation*, Appl. Math. Comput., **254**, 14–19 (2015).
27. V. Kac, P. Cheung, *Quantum Calculus*, Springer, New York (2002).
28. R. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Cambridge Philos. Soc., **66**, 365–370 (1969).
29. P. M. Rajkovic, S. D. Marinkovic, M. S. Stankovic, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math., **1**, 311–323 (2007).
30. P. M. Rajkovic, S. D. Marinkovic, M. S. Stankovic, *On q-analogues of Caputo derivative and Mittag – Leffler function*, Fract. Calc. Appl. Anal., **10**, 359–373 (2007).
31. Sh. Hu, N. Papageorgiou, *Handbook of multivalued analysis, Vol. I: Theory*, Kluwer, Dordrecht etc. (1997).
32. A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York (2003).
33. M. Martelli, *A Rothe's type theorem for noncompact acyclic-valued map*, Boll. Unione Mat. Ital., **11**, 70–76 (1975).

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