UDC 517.9

ATTAINABILITY ISSUE FOR OPTIMAL CONTROL PROBLEM IN COEFFICIENTS FOR DEGENERATE PARABOLIC VARIATIONAL INEQUALITY

ПРОБЛЕМА ДОСЯЖНОСТІ ДЛЯ ЗАДАЧІ ОПТИМАЛЬНОГО КЕРУВАННЯ КОЕФІЦІЄНТАМИ ВИРОДЖЕНОЇ ПАРАБОЛІЧНОЇ ВАРІАЦІЙНОЇ НЕЛІНІЙНОСТІ

N. V. Kasimova

Kyiv Nat. Taras Shevchenko Univ.
Acad. Hlushkov AVE, 4, Kyiv, 03127, Ukraine
e-mail: zadoianchuk.nv@gmail.com

We investigate the optimal control problem with respect to coefficients of the degenerate parabolic variational inequality. Since problems of this type can have the Lavrent'ev effect, we consider the optimal control problem in a class of so-called H-admissible solutions. We substantiate the attainability of H-optimal pairs via optimal solutions of some nondegenerate perturbed optimal control problems under the condition of solvability of the original degenerate problem.

Досліджено задачу оптимального керування в коефіцієнтах для виродженої параболічної варіаційної нерівності. Оскільки в задачах такого типу може виникати ефект Лаврентьєва, ми розглядаємо задачу в класі так званих H-допустимих розв'язків. Обґрунтовано доступність H-оптимальних пар оптимальними розв'язками деяких невироджених збурених задач оптимального керування за умови розв'язності вихідної виродженої задачі.

1. Introduction. The purpose of this paper is to investigate optimal control problem associated with a degenerate parabolic inequality. The control is a matrix of coefficients in the main part of elliptic operator. It is well known that degenerate control problems of this type may admit non-uniqueness of admissible solution classes, which implies non-uniqueness of optimal solutions of particular kind and the optimal control problem in the coefficients can be stated in different forms depending on the choice of the class of admissible solutions (for example W- or H-solutions if we consider the weight Sobolev space W or its subspace H as the phase space, correspondingly) (see [1 – 5]).

Note, that optimal control problems in coefficients for PDE are not new in the literature, and as F. Murat shows in [6], in general, such problems have no solution even if the original elliptic equation is non-degenerate. But such problems are widely studied by many authors since this topic includes optimal shape design problems, optimization of certain evolution systems, some problems originating in mechanics and others. We could mention Butazzo and Dal Maso [7], Lions [8], Murat [9] and others.

Taking into account a wide spectrum of application of the optimal control theory, in particular, we deal with possibilities of some types of approximation of original problems by those that are better researched and converge to the original problems by suitable way. For example, we refer to [10] and references there, where the authors justify the application of the averaging method to optimal control problems for systems of differential equations on the half-line, for optimal control problems for systems of differential equations linear in the control the authors prove the existence of optimal controls for the exact and averaged problems and show that an optimal control in the averaged problem is ε-optimal in the exact problem. It is known that rather popular in such class of problems is the problem of approximation of controls. In [11] for a problem of optimal control for a parabolic equation, in the case of bounded control, the authors construct and justify an approximate averaged control in the form of feedback. In [12] the authors construct approximations of optimal bounded controls for optimal control synthesis in a parabolic problem with fast oscillatory coefficients and prove their convergence to the exact values. As for problem studied in the given paper, in application a degenerate weight ρ occurs as the limit of a sequence of non-degenerate weights ρ_ε for which the corresponding "approximate" optimal control problem is solvable. Thus, naturally, it arises the question: if limit points of the family of admissible solutions to the perturbed problems appear to be admissible solutions to the original problem, whether all optimal solutions are attainable in this sense? Note that for some optimal control problems the attainability and approximability questions remain in the focus of attention. In particular, similar questions were raised in [1, 3] for the degenerate boundary value problems without controls. In [13] the author studies the attainability issue for optimal control problem in coefficients for degenerate variational inequality of monotone type in the class of H-admissible solutions. In [5] the authors prove the existence of W-solutions to the optimal control problem and provide way for their approximation.

Here we concentrate on the optimal control problem in coefficients in the so-called class of H-admissible solutions. Moreover, we are interested about attainability of H-optimal solutions to degenerate problems via optimal solutions of non-degenerate problems. The paper is organized as follows. In Section 2 we give the collection of preliminary results. In Section 3 we state the problem of optimal control in coefficients and prescribe the solvability of degenerate variational inequality that gives us the regularity of the original problem. Section 4 is devoted to the attainability of H-optimal solutions via the optimal solutions to the special perturbed problems for non-degenerate variational inequalities.

2. Notations and Preliminaries. ***2.1. Weighted Sobolev Spaces.*** Let Ω be a bounded subset of $\mathbb{R}^N$ $(N \geq 1)$ with a Lipschitz boundary. Let χ_E be the characteristic function of a subset $E \subseteq \Omega$, i.e., $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$. The space $W_0^{1,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the classical Sobolev space $W^{1,1}(\Omega)$. For any subset $E \subset \Omega$ we denote by $|E|$ its N-dimensional Lebesgue measure $\mathcal{L}^N(E)$.

Hereinafter by a weight we mean a locally integrable function ρ on $\mathbb{R}^N$ such that $\rho(x) > 0$ for a.e. $x \in \mathbb{R}^N$. As a matter of fact every weight ρ gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will also be denoted by ρ. Thus $\rho(E) = \int_E \rho\, dx$ for measurable sets $E \subset \mathbb{R}^N$. We will use the standard notation $L^2(\Omega, \rho\, dx)$ for the set of measurable functions f on Ω such that

$$\|f\|_{L^2(\Omega,\rho\, dx)} = \left(\int_\Omega f^2 \rho\, dx \right)^{1/2} < +\infty.$$

Definition 1. *We say that a weight function $\rho:\mathbb{R}^N\to\mathbb{R}_+$ is degenerate on Ω if*

$$\rho+\rho^{-1}\in L^1_{\mathrm{loc}}(\mathbb{R}^N) \tag{1}$$

and the sum $\rho+\rho^{-1}$ does not belong to $L^\infty(\Omega)$.

With each of the degenerate weight functions ρ we will associate two weighted Sobolev spaces $W(\Omega,\rho\,dx)$ and $H(\Omega,\rho\,dx)$, where $W(\Omega,\rho\,dx)$ is the set of functions $y\in W^{1,1}_0(\Omega)$ for which the norm

$$\|y\|_\rho=\left(\int\limits_\Omega\left(y^2+\rho|\nabla y|^2\right)\,dx\right)^{1/2} \tag{2}$$

is finite, and $H(\Omega,\rho\,dx)$ is the closure of $C^\infty_0(\Omega)$ in $W(\Omega,\rho\,dx)$-norm. Note that due to the compact embedding $W^{1,1}_0(\Omega)\hookrightarrow L^1(\Omega)$ and estimates

$$\int\limits_\Omega|y|\,dx\le|\Omega|^{1/2}\left(\int\limits_\Omega|y|^2\,dx\right)^{1/2}\le\sqrt{|\Omega|}\|y\|_\rho,$$

$$\int\limits_\Omega|\nabla y|\,dx\le\left(\int\limits_\Omega|\nabla y|^2\rho\,dx\right)^{1/2}\left(\int\limits_\Omega\rho^{-1}dx\right)^{1/2}\le C\|y\|_\rho,$$

we come to the following result (we refer to [3, 14] for the details):

Theorem 1. *Let $\rho:\mathbb{R}^N\to\mathbb{R}_+$ be a degenerate weight on Ω. Then*

(i) *the spaces $H(\Omega,\rho\,dx)$ and $W(\Omega,\rho\,dx)$ are complete with respect to the norm $\|\cdot\|_\rho$;*

(ii) *$H(\Omega,\rho\,dx)\subseteq W(\Omega,\rho\,dx)$, and $W(\Omega,\rho\,dx)$, $H(\Omega,\rho\,dx)$ are Hilbert spaces;*

(iii) *$H(\Omega,\rho\,dx)\subset W^{1,1}_0(\Omega)$, $W(\Omega,\rho\,dx)\subset W^{1,1}_0(\Omega)$, and the estimate*

$$\|v\|_{W^{1,1}_0(\Omega)}\le\left(\sqrt{|\Omega|}+\left(\int\limits_\Omega\rho^{-1}dx\right)^{1/2}\right)\|v\|_\rho$$

is valid for every element $v\in H(\Omega,\rho\,dx)\cup W(\Omega,\rho\,dx)$;

(iv) *the embeddings $H(\Omega,\rho\,dx)\hookrightarrow L^1(\Omega)$ and $W(\Omega,\rho\,dx)\hookrightarrow L^1(\Omega)$ are compact.*

If ρ is non-degenerate weight function, that is, ρ is bounded between two positive constants, then it is easy to verify that $W(\Omega,\rho\,dx)=H(\Omega,\rho\,dx)$. However, for a "typical" degenerate weight ρ the space of smooth functions $C^\infty_0(\Omega)$ is not dense in $W(\Omega,\rho\,dx)$. Hence the identity $W(\Omega,\rho\,dx)=H(\Omega,\rho\,dx)$ is not always valid (for the corresponding examples we refer to [15, 16].

We recall that by Riesz Representation Theorem the dual space $\big(H(\Omega,\rho\,dx)\big)^*$ of weighted Sobolev space $H(\Omega,\rho\,dx)$ can be characterized as follows: if $g\in\big(H(\Omega,\rho\,dx)\big)^*$ then there exist functions $g_0\in L^2(\Omega)$ and $\vec{g}_1\in L^2(\Omega,\rho\,dx)^N$ such that

$$\langle g,y\rangle=\int\limits_\Omega g_0y\,dx+\int\limits_\Omega(\vec{g_1},\nabla y)_{\mathbb{R}^N}\rho\,dx\quad\forall y\in H(\Omega,\rho\,dx), \tag{3}$$

where by $\langle \cdot, \cdot \rangle$ we denote the duality between elements of $\big(H(\Omega, \rho\, dx)\big)^*$ and $H(\Omega, \rho\, dx)$, respectively. Furthermore,

$$\|g\|_{(H(\Omega,\rho\, dx))^*} = \inf \left\{ \left(\int\limits_{\Omega} |g_0|^2 dx + \int\limits_{\Omega} \|\vec{g_1}\|^2_{\mathbb{R}^N} \rho\, dx \right)^{1/2} : \ g \ \text{ satisfies } (3) \right\}.$$

Remark 1. Note that under some additional suppositions Theorem 1 can be specified as follows: assume that there exists $v \in (N/2, +\infty)$ such that $\rho^{-v} \in L^1(\Omega)$. Then

$$\big\||y|\big\| = \left(\int\limits_{\Omega} \rho |\nabla y|^2 dx \right)^2$$

is a norm defined on $H(\Omega, \rho\, dx)$ and it's equivalent to (2) and that, the embedding $H(\Omega, \rho\, dx) \hookrightarrow$ $\hookrightarrow L^2(\Omega)$ is compact [17, p. 46].

To conclude this section we recall some results concerning variational triplets. Let $V_- =$ $= H(\Omega, \rho\, dx), V = L^2(\Omega)$ and let $V_-^* = \big(H(\Omega, \rho\, dx)\big)^*$. Let $\mathcal{X} = L^2(0, T; V_-)$. Then the dual space of $\mathcal{X}$ is $\mathcal{X}^* = L^2(0, T; V_-^*)$. For any $y \in \mathcal{X}$, let y' denotes the generalized derivative of $y(t) = y(t, \cdot),$, i.e.,

$$\int\limits_0^T y'(t)\varphi(t)\, dt = -\int\limits_0^T y(t)\varphi'(t)\, dt \quad \forall \varphi \in C_0^\infty([0, T]).$$

Then we have the following result (see [18]):

Lemma 1. *Assume that there exists $v \in (N/2, +\infty)$ such that $\rho^{-v} \in L^1(\Omega)$. Then $V_- \subseteq$ $\subseteq V \subseteq V_-^*$ is an evolution triple, i.e., the embeddings $V_- \hookrightarrow V \hookrightarrow V_-^*$ are continuous, and the embedding $V_- \hookrightarrow V$ is compact. Moreover, $\mathcal{W} = \{y \in \mathcal{X}, y' \in \mathcal{X}^*\}$ equipped with the norm*

$$\|y\|_{\mathcal{W}} = \|y\|_{\mathcal{X}} + \|y'\|_{\mathcal{X}^*} := \|y\|_{L^2(0,T;H(\Omega,\rho dx))} + \|y'\|_{L^2(0,T;(H(\Omega,\rho\, dx))^*)}$$

is a Banach space such that:

(i) *the embedding $\mathcal{W} \hookrightarrow C\big([0, T]; L^2(\Omega)\big)$ is continuos;*

(ii) *the embedding $\mathcal{W} \hookrightarrow L^2\big(0, T; L^2(\Omega)\big)$ is compact.*

2.2. Conditions for operator A. Let V and H will be real Hilbert spaces, V is a dense subspace of H and

$$V \subset H \subset V^*$$

algebraically and topologically and let K be some closed convex subset of V. We shall denote by $|\cdot|$ and $\|\cdot\|$ the norms in H and V, respectively, and by $(\cdot, \cdot)$ the scalar product in H and the pairing between V and its dual V^*. The norm of V^* will be denoted $\|\cdot\|_*$.

We are given a linear continuous and symmetric operator from V to V^* satisfying for some $\omega > 0$, and real α, the coercivity condition

$$(Ay, y) + \alpha|y|^2 \geq \omega\|y\|^2 \quad \text{for all} \quad y \in V. \tag{4}$$

Assume in addition that for some $\omega_1 > 0$

$$(Av, v) \geq \omega_1 \|v\|^2 \quad \forall v \in V \tag{5}$$

Referring to [19] we make use the following assumption.

Hypothesis A. There exists $h \in H$ such that

$$(I + \varepsilon A_H)^{-1}(v + \varepsilon h) \in K \quad \text{for all } \varepsilon > 0 \quad \text{and all } v \in K, \quad A_H y = Ay \cap H.$$

2.3. Smoothing. Throughout the paper ε denotes a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. When we write $\varepsilon > 0,$ we consider only the elements of this sequence, while writing $\varepsilon \geq 0,$ we also consider its limit $\varepsilon = 0.$

Definition 2. *We say that a weight function ρ with properties (1) is approximated by non-degenerate weight functions $\{\rho^\varepsilon\}_{\varepsilon>0}$ on Ω if:*

$$\rho^\varepsilon(x) > 0 \quad \textit{a.e. in} \quad \Omega, \quad \rho^\varepsilon + (\rho^\varepsilon)^{-1} \in L^\infty(\Omega) \quad \forall \varepsilon > 0, \tag{6}$$

$$\rho^\varepsilon \to \rho, \quad (\rho^\varepsilon)^{-1} \to \rho^{-1} \quad \textit{in} \quad L^1(\Omega) \quad \textit{as} \quad \varepsilon \to 0. \tag{7}$$

Remark 2. The family $\{\rho^\varepsilon\}_{\varepsilon>0}$ satisfying properties (6), (7) is called the non-degenerate perturbation of the weight function ρ.

Examples of such perturbations can be constructed using the classical smoothing. For instance, let Q be some positive compactly supported function such that $Q \in L^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} Q(x)\, dx = 1,$ and $Q(x) = Q(-x)$. Then, for a given weight function $\rho \in L^1_{\text{loc}}(\mathbb{R}^N)$, we can take $\rho^\varepsilon = (\rho)_\varepsilon,$ where

$$(\rho)_\varepsilon(x) = \frac{1}{\varepsilon^N} \int\limits_{\mathbb{R}^N} Q\left(\frac{x-z}{\varepsilon}\right) \rho(z)\, dz = \int\limits_{\mathbb{R}^N} Q(z)\rho(x + \varepsilon z)\, dz.$$

In this case, we say that the perturbation $\left\{\rho^\varepsilon = (\rho)_\varepsilon\right\}_{\varepsilon>0}$ of the original degenerate weight function ρ is constructed by the “direct” smoothing scheme.

Lemma 2 [1]**.** *If $\rho, \rho^{-1} \in L^1_{\text{loc}}(\mathbb{R}^N)$, then the “direct” smoothing $\left\{\rho^\varepsilon = (\rho)_\varepsilon\right\}_{\varepsilon>0}$ possesses properties (6), (7).*

2.4. Weak Compactness Criterion in $L^1(\Omega)$. Throughout the paper we will often use the concepts of the weak and strong convergence in $L^1(\Omega)$. Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $L^1(\Omega)$. We recall that $\{a_\varepsilon\}_{\varepsilon>0}$ is called equi-integrable if for any $\delta > 0$ there exists $\tau = \tau(\delta)$ such that $\int_S |a_\varepsilon|\, dx < \delta$ for every $\varepsilon > 0$ and every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$. Then the following assertions are equivalent:

(i) a sequence $\{a_\varepsilon\}_{\varepsilon>0}$ is weakly compact in $L^1(\Omega)$;

(ii) the sequence $\{a_\varepsilon\}_{\varepsilon>0}$ is equi-integrable;

(iii) given $\delta > 0$ there exists $\lambda = \lambda(\delta)$ such that $\sup_{\varepsilon>0} \int_{\{|a_\varepsilon|>\lambda\}} |a_\varepsilon|\, dx < \delta.$

Theorem 2 (Lebesgue’s Theorem)**.** *If a bounded sequence $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ is equi-integrable and $a_\varepsilon \to a$ almost everywhere on Ω, then $a_\varepsilon \to a$ in $L^1(\Omega)$.*

2.5. Radon measures and convergence in variable spaces. By a nonnegative Radon measure on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω. The space of all nonnegative Radon measures on Ω will be denoted by $\mathcal{M}_+(\Omega)$. If μ is nonnegative Radon measure on Ω, we will use $L^r(\Omega, d\mu),\ 1 \le r \le \infty$, to denote the usual Lebesgue space with respect to the measure μ with the corresponding norm $\|f\|_{L^r(\Omega,d\mu)} = \left(\int_\Omega |f(x)|^r\, d\mu\right)^{1/r}$.

Let $\{\mu_\varepsilon\}_{\varepsilon>0}, \mu$ be Radon measures such that $\mu_\varepsilon \xrightarrow{*} \mu$ in $\mathcal{M}_+(\Omega)$; that is,

$$\lim_{\varepsilon\to 0} \int\limits_\Omega \varphi\, d\mu_\varepsilon = \int\limits_\Omega \varphi\, d\mu \quad \forall \varphi \in C_0(\mathbb{R}^N),$$

where $C_0(\mathbb{R}^N)$ is the space of all compactly, supported continuous functions. A typical example of such measures is $d\mu_\varepsilon = \rho^\varepsilon(x)\, dx,\ d\mu = \rho(x)\, dx$, where $0 \le \rho^\varepsilon \rightharpoonup \rho$ in $L^1(\Omega)$. Let us recall the definition and main properties of convergence in the variable L^2-space [2].

1. A sequence $\left\{v_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)\right\}$ is called bounded if

$$\limsup_{\varepsilon\to 0} \int\limits_\Omega |v_\varepsilon|^2\, d\mu_\varepsilon < +\infty.$$

2. A bounded sequence $\left\{v_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)\right\}$ converges weakly to $v \in L^2(\Omega, d\mu)$ if

$$\lim_{\varepsilon\to 0} \int\limits_\Omega v_\epsilon \varphi\, d\mu_\varepsilon = \int\limits_\Omega v\varphi\, d\mu$$

for any $\varphi \in C_0^\infty(\Omega)$ and we write $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon)$.

3. The strong convergence $v_\varepsilon \to v$ in $L^2(\Omega, d\mu_\varepsilon)$ means that $v \in L^2(\Omega, d\mu)$ and

$$\lim_{\varepsilon\to 0} \int\limits_\Omega v_\varepsilon z_\varepsilon\, d\mu_\varepsilon = \int\limits_\Omega vz\, d\mu \quad \text{as} \quad z_\varepsilon \rightharpoonup z \quad \text{in} \quad L^2(\Omega, d\mu_\varepsilon). \tag{8}$$

The following convergence properties in variable spaces hold:

(a) Compactness criterium: if a sequence is bounded in $L^2(\Omega, d\mu_\varepsilon)$, then this sequence is compact with respect to the weak convergence.

(b) Property of lower semicontinuity: if $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)$, then

$$\liminf_{\varepsilon\to 0} \int\limits_\Omega |v_\varepsilon|^2\, d\mu_\varepsilon \ge \int\limits_\Omega v^2\, d\mu.$$

(c) Criterium of strong convergence: $v_\varepsilon \to v$ if and only if $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \int\limits_\Omega |v_\varepsilon|^2\, d\mu_\varepsilon = \int\limits_\Omega v^2\, d\mu.$$

Concluding this section, we recall some well-known results concerning the convergence in the variable space $L^2(\Omega, \rho^\varepsilon\, dx)$:

Lemma 3 [1, 2]. *If $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a non-degenerate perturbation of the weight function $\rho(x) \geq 0$, then:*

(A$_1$) $(\rho^\varepsilon)^{-1} \to \rho^{-1}$ *in* $L^2(\Omega, \rho^\varepsilon\, dx)$*;*

(A$_2$) $\big[v_\varepsilon \to v \text{ in } L^2(\Omega, \rho^\varepsilon\, dx)\big] \Longrightarrow \big[v_\varepsilon \to v \text{ in } L^1(\Omega)\big]$*;*

(A$_3$) *If a sequence $\big\{v_\varepsilon \in L^2(\Omega, \rho^\varepsilon\, dx)\big\}_{\varepsilon>0}$ is bounded, then the weak convergence $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, \rho^\varepsilon\, dx)$ is equivalent to the weak convergence $\rho^\varepsilon v_\varepsilon \rightharpoonup pv$ in $L^1(\Omega)$;*

(A$_4$) *If $a \in L^\infty(\Omega)$ and $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, \rho^\varepsilon\, dx)$, then $av_\varepsilon \rightharpoonup av$ in $L^2(\Omega, \rho^\varepsilon dx)$.*

2.6. Variable Sobolev spaces. Let $\rho(x)$ be a degenerate weight function and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of the function ρ in the sense of Definition 1. We denote by $H(\Omega, \rho^\varepsilon dx)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\rho^\varepsilon}$. Since for every ε the function ρ^ε is non-degenerate, the space $H(\Omega, \rho^\varepsilon dx)$ coincides with the classical Sobolev space $W_0^{1,p}(\Omega)$.

Definition 3. *We say that a sequence $\big\{y_\varepsilon \in H(\Omega, \rho^\varepsilon dx)\big\}_{\varepsilon>0}$ converges weakly to an element $y \in W(\Omega, \rho\, dx)$ as $\varepsilon \to 0$, if the following hold:*

(i) *this sequence is bounded;*

(ii) $y_\varepsilon \rightharpoonup y$ *in* $L^2(\Omega)$*;*

(iii) $\nabla y_\varepsilon \rightharpoonup \nabla y$ *in* $L^2(\Omega, \rho^\varepsilon dx)^N$.

2.7. Compensated compactness lemma in variable Lebesgue and Sobolev spaces. Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function ρ. We associate to every ρ^ε the space

$$X_{\rho^\varepsilon} = \Big\{\vec{f} \in L^2\big(0,T; L^2(\Omega, \rho^\varepsilon dx)^N\big) \;\big|\; \operatorname{div}(\rho^\varepsilon \vec{f}) \in L^2\big(0,T; L^2(\Omega)\big)\Big\} \quad \forall \varepsilon > 0$$

and endow it with the norm

$$\|\vec{f}\|_{X_{\rho^\varepsilon}} = \Big(\|\vec{f}\|^2_{L^2(0,T;L^2(\Omega,\rho^\varepsilon dx)^N)} + \|\operatorname{div}(\rho^\varepsilon \vec{f})\|^2_{L^2(0,T;L^2(\Omega))}\Big)^{1/2}.$$

We call a sequence $\big\{\vec{f}_\varepsilon \in X_{\rho^\varepsilon}\big\}_{\varepsilon>0}$ bounded if

$$\varlimsup_{\varepsilon\to 0} \|\vec{f}_\varepsilon\|_{X_{\rho^\varepsilon}} < +\infty.$$

Also let us consider the space $\mathcal{H}_\varepsilon = \big\{y \in H(\Omega, \rho^\varepsilon dx) \;\big|\; y' \in L^2\big(0,T; L^2(\Omega)\big)\big\}$. Composing suggestions of [20] (Lemma 4) and [21] (Theorem 2) we obtain the next result, which is rather useful for investigating of the attainability.

Lemma 4. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function $\rho(x) > 0$. Let $\big\{\vec{f}_\varepsilon \in L^2\big(0,T; L^2(\Omega, \rho^\varepsilon dx)^N\big)\big\}_{\varepsilon>0}$ and $\{g_\varepsilon \in \mathcal{H}_\varepsilon\}_{\varepsilon>0}$ be such that $\{\vec{f}_\varepsilon\}_{\varepsilon>0}$ is bounded in the variable space X_{ρ_ε}, $\vec{f}_\varepsilon \rightharpoonup \vec{f}$ in $L^2\big(0,T; L^2(\Omega, \rho^\varepsilon dx)^N\big)$ as $\varepsilon \to 0$, $\{g_\varepsilon\}_{\varepsilon>0}$ is bounded in the variable space $\mathcal{H}_\varepsilon$, $g_\varepsilon \rightharpoonup g$ in $L^2\big(0,T; L^2(\Omega)\big)$, and $\nabla g_\varepsilon \rightharpoonup \nabla g$ in $L^2\big(0,T; L^2(\Omega, \rho^\varepsilon dx)^N\big)$, $g'_\varepsilon \rightharpoonup g'$ in $L^2\big(0,T; L^2(\Omega)\big)$ as $\varepsilon \to 0$. Then*

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \varphi\big(\vec{f}_\varepsilon, \nabla g_\varepsilon\big)_{\mathbb{R}^N} \rho^\varepsilon \psi\, dx\, dt = \int_0^T \int_\Omega \varphi(\vec{f}, \nabla g)_{\mathbb{R}^N} \rho \psi\, dx\, dt,$$

$$\forall \varphi \in C_0^\infty(\Omega), \quad \psi \in C_0^\infty(0,T).$$

3. Setting of the Optimal Control Problem (OCP). Let ρ be given element of $L^1(\Omega)$ satisfying the conditions

$$0 < \rho(x) \quad \text{a.e. in} \quad \Omega, \qquad \rho^{-\nu} \in L^1(\Omega) \quad \text{for some} \quad \nu \in (N/2, +\infty).$$

Then in view of the estimate

$$\int\limits_\Omega \rho^{-1}\, dx \le \left(\int\limits_\Omega \rho^{-\nu}\, dx\right)^{1/\nu} \left(\int\limits_\Omega dx\right)^{1/\nu^*} = \|\rho^{-\nu}\|^{1/\nu}_{L^1(\Omega)} |\Omega|^{1/\nu^*},$$

where $\nu^* = \nu/(1-\nu)$ is the conjugate of ν, we have: $\rho^{-1} \in L^1(\Omega)$, i.e.,, ρ is a degenerate weight in the sense of Definition 1.

Let $\mathcal{K}$ be a non-empty convex closed subset of the space $L^2\big(0,T; H(\Omega, \rho\, dx)\big)$ such that $0 \in \mathcal{K}$, $y_{ad}, f \in L^2\big(0,T; L^2(\Omega)\big)$ be given elements. Consider the next OCP in coefficients for degenerate variation parabolic inequality:

$$I(U,y) = \int\limits_0^T \int\limits_\Omega |y - y_{ad}|^2\, dx\, dt \to \inf, \tag{9}$$

$$\int\limits_0^T \int\limits_\Omega y'(v-y)\, dx\, dt + \int\limits_0^T \left(\sum_{i,j=1}^N \int\limits_\Omega \left(a_{i,j}(x) \frac{\partial y}{\partial x_j}\right) \frac{\partial (v-y)}{\partial x_i} \rho\, dx + \int\limits_\Omega y(v-y)\, dx \right) dt \ge$$

$$\ge \int\limits_0^T \int\limits_\Omega f(v-y)\, dx\, dt, \qquad \forall v \in \mathcal{K}, \tag{10}$$

$$U \in \mathcal{U}_{ad}, \quad y \in \mathcal{K}, \quad y' \in L^2\big(0,T; L^2(\Omega)\big), \tag{11}$$

$$y(0,x) = 0, \quad x \in \Omega. \tag{12}$$

Here

$$\mathcal{U}_{ad} = \Big\{ U = [\vec{a_1}, \ldots, \vec{a_N}] \in M_2^{\alpha,\beta}(\Omega) \;\Big|\; \big|\operatorname{div}(\rho \vec{a_i})\big| \le \gamma_i, \quad \text{a.e. in} \quad \Omega \quad \forall i = 1, \ldots, N \Big\},$$

where $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ is a strictly positive vector, $M_2^{\alpha,\beta}(\Omega)$ $(0 < \alpha \le \beta < +\infty)$ is a set of all symmetric matrices $U(x) = \{a_{i,j}(x)\}_{1 \le i,j \le N}$ in $L^\infty(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ such that the following conditions are fulfilled:

$$|a_{i,j}(x)| \le \beta \quad \text{a.e. in} \quad \Omega \quad \forall i,j \in \{1, \ldots N\}, \tag{13}$$

$$(U(x)(\xi - \eta), \xi - \eta)_{\mathbb{R}^N} \ge 0 \quad \text{a.e. in} \quad \Omega \quad \forall \xi, \eta \in \mathbb{R}^N, \tag{14}$$

$$(U(x)\xi, \xi)_{\mathbb{R}^N} = \sum_{i,j=1}^N a_{i,j}(x) \xi_i \xi_j \ge \alpha |\xi|^2 \quad \text{a.e. in} \quad \Omega. \tag{15}$$

Definition 4. *We say that $y \in \mathcal{K}$, for which the inequality (10)–(12) takes place is called an H-solution.*

For every fixed control $U \in M_2^{\alpha,\beta}(\Omega)$ let us consider the linear operator $A : H(\Omega, \rho\, dx) \to \to \big(H(\Omega, \rho\, dx)\big)^*$ defined as

$$\langle A(y), v\rangle = \sum_{i,j=1}^{N} \int\limits_{\Omega} \left(a_{i,j}(x)\frac{\partial y}{\partial x_j}\right)\frac{\partial v}{\partial x_i}\rho\, dx + \int\limits_{\Omega} yv\, dx \quad \text{for} \quad v \in H(\Omega, \rho\, dx). \tag{16}$$

It can be shown that taking into account (13)–(15) we obtain (4), and as a corollary (5).

Hereinafter we shall suggest that the Hypothesis A is fulfilled for $V = H(\Omega, \rho\, dx),\ H = = L^2(\Omega),\ A : H(\Omega, \rho\, dx) \to \big(H(\Omega, \rho\, dx)\big)^*$, defined by (16).

Note, that the set of optimal solutions for the problem (9)–(12) is nonempty (see for details [22] (Theorem 5)).

4. Attainability of *H*-optimal solutions. In this section we show that H-optimal solutions of (9)–(12) can be attained by optimal solutions of perturbed problems considering an appropriate non-degenerate perturbation for the original OCP.

Let ρ be a degenerate weight function with properties (1), and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a direct smoothing of a degenerate weight function $\rho(x) \geq 0$.

Definition 5. *We say that a bounded sequence*

$$\{(U_\varepsilon, y_\varepsilon)\}_{\varepsilon>0} \in \mathbb{Y}_\varepsilon = L^\infty(\Omega; \mathbb{R}^{N\times N}) \times \mathcal{H}_\varepsilon$$

w-converges to

$$(U, y) \in L^\infty(\Omega; \mathbb{R}^{N\times N}) \times \mathcal{H} = \big\{y \in L^2(0, T; H(\Omega, \rho\, dx)) \mid y' \in L^2\big(0, T; L^2(\Omega)\big)\big\}$$

in the variable space $\mathbb{Y}_\varepsilon$ as $\varepsilon \to 0$, if

$$U_\varepsilon \to U \quad \textit{weakly-}* \ \textit{in} \quad L^\infty(\Omega; \mathbb{R}^{N\times N}), \qquad y_\varepsilon \rightharpoonup y \quad \textit{in} \quad L^2\big(0, T; L^2(\Omega)\big),$$
$$\nabla y_\varepsilon \rightharpoonup \nabla y \in L^2(0, T; (L^2(\Omega, \rho^\varepsilon dx))^N), \qquad (y_\varepsilon)' \rightharpoonup y' \quad \textit{in} \quad L^2\big(0, T; L^2(\Omega)\big).$$

Similarly to [13] (Definition 8) and [23] (Definition 5.13) we consider the next concept.

Definition 6. *We say that a minimization problem*

$$\left\langle \inf_{(U,y)\in\Xi_H} I(U, y)\right\rangle \tag{17}$$

is a weak variational limit (or variational w-limit) of the sequence

$$\left\{\left\langle \inf_{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(U_\varepsilon, y_\varepsilon)\right\rangle;\ \ \Xi_\varepsilon \in \mathbb{Y}_\varepsilon,\ \ \varepsilon > 0\right\}, \tag{18}$$

with respect to w-convergence in the variable space $\mathbb{Y}_\varepsilon$, if the following conditions are satisfied:

(i) *if $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, and a sequence $\{(U_k, y_k) \in \in \Xi_{\varepsilon_k}\}_{\varepsilon>0}$ w-converges to a pair (U, y), then*

$$(U, y) \in \Xi_H : \ I(U, y) \leq \varliminf_{k\to\infty} I_{\varepsilon_k}(U_k, y_k); \tag{19}$$

(ii) *for every pair* $(U, y) \in \Xi_H$ *and any value* $\delta > 0$ *there exists a realizing sequence* $\big\{(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}_\varepsilon\big\}_{\varepsilon>0}$ *such that*

$$(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon \quad \forall \varepsilon > 0, \qquad (\hat{U}_\varepsilon, \hat{y}_\varepsilon) \xrightarrow{w} (\hat{U}, \hat{y}),$$

$$\|U - \hat{U}\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \|y - \hat{y}\|_{L^2(0,T;H(\Omega,\rho dx))} + \|y' - \hat{y}'\|_{L^2(0,T;L^2(\Omega))} \le \delta,$$

$$I(U, y) \ge \varlimsup_{\varepsilon\to 0} I_\varepsilon(\hat{U}_\varepsilon, \hat{y}_\varepsilon) - \delta.$$

Applying similar suggestions to [13] (Theorem 4) and [23] (Theorem 5.4) we obtain the next result.

Theorem 3. *Assume that (17) is a weak variational limit of the sequence (18), and the constrained minimization problem (17) has a solution. Suppose* $\big\{(U^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\big\}_{\varepsilon>0}$ *is a sequence of optimal pairs to (18). Then there exists a pair* $(U^0, y^0) \in \Xi_H$ *such that* $(U^0_\varepsilon, y^0_\varepsilon) \xrightarrow{w} (U^0, y^0)$ *and*

$$\inf_{(U,y)\in\Xi_H} I(U, y) = I(U^0, y^0) = \lim_{\varepsilon\to 0} \inf_{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(U_\varepsilon, y_\varepsilon).$$

Let us consider the sequence $\{\mathcal{K}_\varepsilon\}_{\varepsilon>0}$ of non-empty closed and convex subsets, which sequentially converges to the set $\mathcal{K}$ in the sense of Kuratovski as $\varepsilon \to 0$ with respect to the weak topology of the space $L^2\big(0, T; H(\Omega, \rho^\varepsilon dx)\big)$. Taking into account Theorem 3, we consider the following collection of perturbed OCPs in coefficients for non-degenerate parabolic variational inequalities:

$$\text{Minimize } \left\{ I_\varepsilon(U, y) = \int_0^T \int_\Omega (y(x, t) - y_{ad}(x, t))^2 \, dx \, dt \right\}, \tag{20}$$

$$\begin{aligned} &\langle y', v - y\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} + \\ &\qquad + \langle -\operatorname{div}(\rho^\varepsilon U\nabla y) + y, v - y\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} \ge \\ &\ge \langle f, v - y\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} \quad \forall v \in \mathcal{K}_\varepsilon, \end{aligned} \tag{21}$$

$$U \in U^\varepsilon_{ad}, \quad y \in \mathcal{K}_\varepsilon, \quad y' \in L^2\big(0, T; L^2(\Omega)\big), \tag{22}$$

$$y(0, x) = 0, \; x \in \Omega, \tag{23}$$

$$U^\varepsilon_{ad} = \Big\{ U = [\vec{a}_1, \ldots, \vec{a}_N] \in M^{\alpha,\beta}_2(\Omega) \;\Big|\; |\operatorname{div}(\rho^\varepsilon \vec{a}_i)| \le \gamma_i, \text{ a.e. in } \Omega \;\; \forall i = 1, \ldots, N \Big\}, \tag{24}$$

where the elements $y_{ad},\ f \in L^2\big(0, T; L^2(\Omega)\big)$ and $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ are the same as for the original problem (9)–(12). For every $\varepsilon > 0$ we define Ξ_ε as a set of all admissible pairs to the problem (20)–(24), namely $(U, y) \in \Xi_\varepsilon$ if and only if the pair (U, y) satisfies (20)–(24).

Note that each of perturbed OCPs (20)–(24) is solvable provided $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a non-degenerate perturbation of $\rho > 0$, in particular, for "direct" smoothing of $\rho > 0$ (see for details [19] (Proposition 5.1) taking into account properties of variable Sobolev Spaces).

Remark 3. Let us recall that sequential K-upper and K-lower limits of a sequence of sets $\{E_k\}_{k\in\mathbb{N}}$ are defined as follows, respectively:

$$K_s - \overline{\lim}\, E_k = \big\{y \in X:\ \exists\sigma(k) \to \infty,\ \exists y_k \to y\ \forall k \in \mathbb{N}:\ y_k \in E_{\sigma(k)}\big\},$$
$$K_s - \underline{\lim}\, E_k = \big\{y \in X:\ \exists y_k \to y,\ \exists k \geq k_0 \in \mathbb{N}:\ y_k \in E_k\big\}.$$

The sequence $\{E_k\}_{k\in\mathbb{N}}$ sequentially converges in the sense of Kuratovski to the set E (shortly, K_s-converges), if $E = K_s - \overline{\lim}\, E_k = K_s - \underline{\lim}\, E_k$.

We are now in position to show that each optimal solution to the problem (9)–(12) can be attained by admissible solutions to perturbed problems (20)–(24), however there exists at least one optimal solution $(U_0, y_0) \in \Xi_H$ which can be attained by optimal solutions to perturbed problems (20)–(24). Namely, the next results take place.

Lemma 5. *Let $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho(x) > 0$. Let $\big\{(U_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\big\}_{\varepsilon>0}$ be a sequence of admissible pairs to the problem (20)–(24). Then there exists a pair (U^*, y^*) and a subsequence $\big\{(U_{\varepsilon_k}, y_{\varepsilon_k})\big\}_{k\in\mathbb{N}}$ of $\big\{(U_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\big\}_{\varepsilon>0}$ such that $(U_{\varepsilon_k}, y_{\varepsilon_k}) \xrightarrow{w} (U^*, y^*)$ as $k \to \infty$ and $(U^*, y^*) \in \Xi_H$.*

Proof. Let us consider the relation:

$$\begin{aligned}\langle y_\varepsilon', v_\varepsilon - y_\varepsilon\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} &+ \\ &+ \langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon \nabla y_\varepsilon) + y_\varepsilon, v_\varepsilon - y_\varepsilon\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} \geq \\ &\geq \langle f, v_\varepsilon - y_\varepsilon\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} \quad \forall v_\varepsilon \in \mathcal{K}_\varepsilon.\end{aligned} \tag{25}$$

As follows from (24) the sequence $\{U_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega;\mathbb{R}^{N\times N})$. Let us suppose that the sequence $\{y_\varepsilon'\}_{\varepsilon>0}$ is bounded in $L^2\big(0,T;L^2(\Omega)\big)$ and prove the boundedness of $\{y_\varepsilon\}_{\varepsilon>0}$ in $L^2(0,T;H(\Omega;\rho^\varepsilon dx))$. By contradiction, suppose that $\|y_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} \to \infty,\ \varepsilon \to 0$. Then on the one hand

$$\begin{aligned}\big\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon \nabla y_\varepsilon) + y_\varepsilon, y_\varepsilon - v_\varepsilon\big\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} &\leq \\ \leq \langle -y_\varepsilon' + f, y_\varepsilon - v_\varepsilon\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx)} &\leq \\ \leq \Big(\|y_\varepsilon'\|_{L^2([0,T]:L^2(\Omega))} + \|f\|_{L^2(0,T;L^2(\Omega))}\Big)&\|y_\varepsilon - v_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx)} \quad \forall v_\varepsilon \in \mathcal{K}_\varepsilon \quad \forall \varepsilon > 0.\end{aligned} \tag{26}$$

On the other hand, for arbitrary fixed elements $v \in \mathcal{K}$ let us consider the sequence $\{v_\varepsilon \in \mathcal{K}_\varepsilon\}_{\varepsilon>0}$ such that $v_\varepsilon \rightharpoonup v$ in $L^2(0,T;L^2(\Omega;\rho^\varepsilon dx))$ (such sequence always exists provided $\mathcal{K} = K_s - \lim \mathcal{K}_\varepsilon$), and then taking into account properties for operator $A: H(\Omega,\rho\, dx) \to \big(H(\Omega,\rho\, dx)\big)^*$ and definition of norm in $L^2\big(0,T;H(\Omega,\rho\, dx)\big)$, we can consider the next estimation for A: $L^2\big(0,T;H(\Omega,\rho\, dx)\big) \to L^2\big(0,T;\big(H(\Omega,\rho\, dx)\big)^*\big)$: for every fixed $U \in M_2^{\alpha,\beta}(\Omega)$

$$\begin{aligned}\langle A(U,y), y - v\rangle \geq \min\{\alpha,1\}&\|y\|^2_{L^2(0,T;H(\Omega,\rho dx))} - \\ &- \max\{\beta,1\}\|v\|_{L^2(0,T;H(\Omega,\rho dx))}\|y\|_{L^2(0,T;H(\Omega,\rho dx))},\end{aligned}$$

$$v \in L^2\big(0,T; H(\Omega, \rho\, dx)\big).$$

Thus, we obtain the following relations:

$$\frac{\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon \nabla y_\varepsilon) + y_\varepsilon, y_\varepsilon - v_\varepsilon\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon - v_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}} \geq$$

$$\geq \frac{\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon \nabla y_\varepsilon) + y_\varepsilon, y_\varepsilon - v_\varepsilon\rangle_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} + \|v_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}} \geq$$

$$\geq \frac{\min\{\alpha,1\}\|y\|^2_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} - \max\{\beta,1\}\|v\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}\|y\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} + \|v_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}} =$$

$$= \|y_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))} \frac{\left(\min\{\alpha,1\} - \dfrac{\max\{\beta,1\}\|v_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}}\right)}{\left(1 + \dfrac{\|v_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^2(0,T;H(\Omega,\rho^\varepsilon dx))}}\right)} \to \infty, \quad \varepsilon \to 0,$$

since the sequence $\{v_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2\big(0,T; H(\Omega, \rho^\varepsilon dx)\big)$. The obtained contradiction with (26) implies that $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2\big(0,T; H(\Omega, \rho^\varepsilon dx)\big)$.

Hence, there exists a subsequence $\{\varepsilon_k\}$ of the sequence $\{\varepsilon\}$, converging to 0 and elements $U^* \in M_2^{\alpha,\beta}(\Omega)$, $y^* \in L^2\big(0,T; L^2(\Omega)\big)$, $\vec{v} \in L^2\big(0,T; (L^2(\Omega, \rho\, dx))^N\big)$ and $\vec{\xi} \in L^2\big(0,T; (L^2(\Omega, \rho^\varepsilon dx))^N\big)$ such that: $U_{\varepsilon_k} \to U^*$ weakly-$*$ in $L^\infty(\Omega; \mathbb{R}^{N\times N})$, $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^2\big(0,T; L^2(\Omega)\big)$, $\nabla y_{\varepsilon_k} \rightharpoonup \vec{v}$ in $L^2\big(0,T; (L^2(\Omega, \rho^\varepsilon dx))^N\big)$, $y'_{\varepsilon_k} \rightharpoonup (y^*)'$ in $L^2\big(0,T; L^2(\Omega)\big)$, $U_{\varepsilon_k}\nabla y_{\varepsilon_k} =: \vec{\xi_k} \rightharpoonup \vec{\xi}$ in $L^2\big(0,T; (L^2(\Omega, \rho^\varepsilon dx))^N\big)$ as $k \to \infty$.

By [13] (Theorem 3), taking into account properties of the Bochner integral and Definition of equivalent functions (see [24] (Definition 1.6) and [24] (Definition 1.8)), we have that $y^* \in L^2\big(0,T; H(\Omega, \rho\, dx)\big)$, and $v = \nabla y^*$, and, moreover, we have $y^* \in \mathcal{K}$, $(y^*)' \in L^2\big(0,T; L^2(\Omega)\big)$.

Following arguments of the proof of [13] (Lemma 11), we obtain that $U^* \in U_{ad}$.

In what follows, we consider the relation (25) for $(U_{\varepsilon_k}, y_{\varepsilon_k})$ and pass to the limit in it as $k \to \infty$.

Let us prove that

$$\big\langle -\operatorname{div}(\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k}), y_{\varepsilon_k}\big\rangle_{L^2(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} \to \big\langle -\operatorname{div}(\rho\vec{\xi}), y^*\big\rangle. \tag{27}$$

Taking into account (5) and Hypothesis A, in view of [22] (Proposition 1), we have $-\operatorname{div}\big(\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k}\big) + y_{\varepsilon_k} \in L^2\big(0,T; L^2(\Omega)\big)$ $\forall k \in \mathbb{N}$ and, obviously, $\operatorname{div}\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k} \in L^2\big(0,T; L^2(\Omega)\big)$ $\forall k \in \mathbb{N}$.

The following relation

$$\int_0^T \int_\Omega \varphi \operatorname{div}(\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k})\psi\, dx\, dt =$$

$$= -\int_0^T \int_\Omega (\vec{\xi}_{\varepsilon_k}, \nabla\varphi)_{\mathbb{R}^N}\psi\rho^{\varepsilon_k}\, dx \to -\int_0^T \int_\Omega (\vec{\xi}, \nabla\varphi)_{\mathbb{R}^N}\rho\psi\, dx\, dt$$

$$\forall\varphi\in C_0^\infty(\Omega),\quad \psi\in C_0^\infty(0,T),\quad k\to\infty,$$

means that $\operatorname{div}(\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k}) \to \operatorname{div}(\rho\vec{\xi})$ weakly in $L^2\big(0,T;L^2(\Omega)\big)$. Therefore, the sequence $\big\{\operatorname{div}(\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k})\big\}_{k\in\mathbb{N}}$ is bounded in $L^2\big(0,T;L^2(\Omega)\big)$ and $\{\vec{\xi}_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is bounded in $X_{\rho^{\varepsilon_k}}$. Taking into account Lemma 4 we obtain (27).

Let us now prove that

$$\int_0^T\int_\Omega y'_{\varepsilon_k}(v_{\varepsilon_k}-y_{\varepsilon_k})\,dx\,dt\to\int_0^T\int_\Omega (y^*)'(v-y^*)\,dx\,dt,\quad k\to\infty.$$

Having put $z_{\varepsilon_k}:=v_{\varepsilon_k}-y_{\varepsilon_k},\ z_{\varepsilon_k}\in L^2\big(0,T;H(\Omega,\rho^{\varepsilon_k}dx)\big)$ we rewrite this relation in the form:

$$\langle y'_{\varepsilon_k},z_{\varepsilon_k}\rangle_{L^2(0,T;H(\Omega,\rho^{\varepsilon_k}dx))}\to\langle (y^*)',z\rangle,\quad k\to\infty,\tag{28}$$

where $v\in L^2\big(0,T;H(\Omega,\rho\,dx)\big)$ is the weak limit of the sequence $\{v_{\varepsilon_k}\}_{k\in\mathbb{N}}$ in $L^2(0,T;H(\Omega,\rho^\varepsilon dx))$ as $k\to\infty$ and $z=v-y^*,\ z\in L^2\big(0,T;H(\Omega,\rho\,dx)\big)$.

Let us consider the left hand side of (28):

$$\int_0^T\int_\Omega y'_{\varepsilon_k}z_{\varepsilon_k}\,dx\,dt\pm\int_0^T\int_\Omega y'_{\varepsilon_k}z\,dx\,dt=$$

$$=\int_0^T\int_\Omega y'_{\varepsilon_k}(z_{\varepsilon_k}-z)\,dx\,dt+\int_0^T\int_\Omega y'_{\varepsilon_k}z\,dx\,dt=I_1+I_2.$$

It is obvious, that

$$I_2\to\int_0^T\int_\Omega (y^*)'z\,dx\,dt,\quad k\to\infty.$$

Let us consider now I_1. Since the sequence $\big\{y'_{\varepsilon_k}\in L^2\big(0,T;L^2(\Omega)\big)\big\}$ is bounded and $y'_{\varepsilon_k}\rightharpoonup (y^*)'$ in $L^2\big(0,T;L^2(\Omega)\big)$ as $k\to\infty$, we have that $y'_{\varepsilon_k}\rightharpoonup (y^*)'$ in $L^1\big([0,T];L^1(\Omega)\big)$. Hence the family $\{y'_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is equi-integrable on $([0,T])\times\Omega$. Let us show that an element $z\in L^2\big(0,T;L^2(\Omega)\big)$ can be interpreted as the strong limit of the sequence $\big\{z_{\varepsilon_k}\in L^2(0,T;H(\Omega,\rho^{\varepsilon_k}dx))\big\}_{k\in\mathbb{N}}$ in $L^1\big([0,T];L^1(\Omega)\big)$. Indeed, as follows from initial assumptions and estimates

$$\int_0^T\int_\Omega |z_{\varepsilon_k}|\,dx\,dt\le\int_0^T\left(\int_\Omega |z_{\varepsilon_k}|^2\,dx\right)^{1/2}|\Omega|^{1/2}dt\le\int_0^T C|\Omega|^{1/2}dt=TC|\Omega|^{1/2},$$

$$\int_0^T\int_\Omega |\nabla z_{\varepsilon_k}|_2\,dx\,dt\le\int_0^T\left(\int_\Omega |\nabla z_{\varepsilon_k}|_2^2\rho^{\varepsilon_k}dx\right)^{1/2}\left(\int_\Omega(\rho^{\varepsilon_k})^{-1}dx\right)^{1/2}dt\le$$

$$\leq \int_0^T C \left(\int_\Omega (\rho^{\varepsilon_k})^{-1} dx \right)^{1/2} dt = CT \left(\int_\Omega (\rho^{\varepsilon_k})^{-1} dx \right)^{1/2},$$

the family $\{z_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is equi-integrable on $(0,T)\times\Omega$ and bounded in $L^1\big([0,T];W^{1,1}(\Omega)\big)$. Hence by compact embedding $L^1\big([0,T];W^{1,1}(\Omega)\big) \hookrightarrow L^1\big([0,T];L^1(\Omega)\big)$ we can assume that there exists an element $z^* \in L^1\big([0,T];L^1(\Omega)\big)$ such that $z_{\varepsilon_k} \to z^*$ in $L^1\big([0,T];L^1(\Omega)\big)$. Taking into account that $z_{\varepsilon_k} \rightharpoonup z$ in $L^2\big(0,T;L^2(\Omega)\big)$ we obtain

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \varphi z_{\varepsilon_k} \psi \, dx\, dt = \int_0^T \int_\Omega \varphi z^* \psi \, dx\, dt$$

and

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \varphi z_{\varepsilon_k} \psi \, dx\, dt = \int_0^T \int_\Omega \varphi z \psi \, dx\, dt \quad \forall \varphi \in C_0^\infty(\Omega), \quad \psi \in C_0^\infty(0,T).$$

We get $z = z^*$ almost everywhere on $(0,T)\times\Omega$. It means that up to a subsequence $z_{\varepsilon_k} \to z$ a.e. in $(0,T)\times\Omega$.

Therefore, because of boundedness of $\{z_{\varepsilon_k} - z\}_{k\in\mathbb{N}}$ the sequence $\{(z_{\varepsilon_k} - z)y'_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is equi-integrable on $(0,T)\times\Omega$ as well. Using that fact that $z_{\varepsilon_k} \to z$ a.e. in $(0,T)\times\Omega$, the Lebesgue Theorem implies that $(z_{\varepsilon_k} - z)y'_{\varepsilon_k} \to 0$ in $L^1\big([0,T];L^1(\Omega)\big)$. Thus, we have (28).

Therefore, as a result of limit passage in (25), we obtain

$$\begin{aligned} &\big\langle (y^*)', v - y^* \big\rangle_{L^2(0,T;H(\Omega,\rho\, dx))} + \big\langle -\operatorname{div}(\rho\vec{\xi}), v - y^* \big\rangle_{L^2(0,T;H(\Omega,\rho\, dx))} + \\ &\quad + \langle y^*, v \rangle_{L^2(0,T;H(\Omega,\rho\, dx))} - \varlimsup_{k\to\infty} \langle y_{\varepsilon_k}, y_{\varepsilon_k} \rangle_{L^2(0,T;H(\Omega,\rho^{\varepsilon_k} dx))} \geq \\ &\geq \langle f, v - y^* \rangle_{L^2(0,T;H(\Omega,\rho\, dx))}, \quad \forall v \in \mathcal{K}. \end{aligned} \tag{29}$$

It is left to show that $\vec{\xi} = U^* \nabla y^*$. However it can be done in a similar manner as we did it proving [22] (Theorem 4).

Now, let us show that

$$\lim_{k\to\infty} \langle y_{\varepsilon_k}, y_{\varepsilon_k} \rangle_{L^2(0,T;H(\Omega,\rho^{\varepsilon_k} dx))} = \lim_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^2 \, dx\, dt = \int_0^T \int_\Omega |y^*|^2 \, dx\, dt.$$

On the one hand, in view of the property of lower semicontinuity, weak convergence $y_{\varepsilon_k} \to y^*$ in $L^2\big(0,T;L^2(\Omega)\big)$ as $k\to\infty$ implies that:

$$\int_0^T \int_\Omega |y^*|^2 \, dx\, dt \leq \varliminf_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^2 \, dx\, dt.$$

On the other hand, from (29), taking into account the representation of the vector-function ξ, we obtain:

$$\varlimsup_{k\to\infty}\int\limits_0^T\int\limits_\Omega |y_{\varepsilon_k}|^2\,dx\,dt \le$$

$$\le \left\langle (y^*)' - \operatorname{div}(U^*(x)\rho(x)\nabla y^*) - f, v - y^*\right\rangle_{L^2(0,T;H(\Omega,\rho\,dx))} +$$

$$+ \langle y^*, v\rangle_{L^2(0,T;H(\Omega,\rho\,dx))} \quad \forall v \in \mathcal{K}.$$

Having put in the last inequality $v = y^*$, we get

$$\varlimsup_{k\to\infty}\int\limits_0^T\int\limits_\Omega |y_{\varepsilon_k}|^2\,dx\,dt \le \int\limits_0^T\int\limits_\Omega |y^*|^2\,dx\,dt.$$

Hence, taking into account the chain of inequalities

$$\int\limits_0^T\int\limits_\Omega |y^*|^2\,dx\,dt \le \varliminf_{k\to\infty} |y_{\varepsilon_k}|^2\,dx\,dt \le \varlimsup_{k\to\infty} |y_{\varepsilon_k}|^2\,dx\,dt \le \int\limits_0^T\int\limits_\Omega |y^*|^2\,dx\,dt$$

we obtain that $y_{\varepsilon_k} \to y^*$ in $L^2\big(0,T;L^2(\Omega)\big)$ as $k\to\infty$.

Therefore, variational inequality (29) can be represented in the form

$$\left\langle (y^*)', v - y^*\right\rangle_{L^2(0,T;H(\Omega,\rho\,dx))} +$$

$$+ \left\langle -\operatorname{div}(\rho U^*\nabla y^*) + y^*, v - y^*\right\rangle_{L^2(0,T;H(\Omega,\rho\,dx))} \ge$$

$$\ge \langle f, v - y^*\rangle_{L^2(0,T;H(\Omega,\rho\,dx))} \quad \forall v\in\mathcal{K}.$$

Thus, w-limit pair (U^*, y^*) is admissible to the problem (9)–(12). Hence, $(U^*, y^*) \in \Xi_H$.

As an evident consequence of these suggestions and the lower semicontinuity property of the cost functional (20) with respect to w-convergence in variable space $\mathbb{Y}_\varepsilon$, we have the following conclusion: let $\{\varepsilon_k\}$ be a subsequence of indices $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k\to\infty$, and let $\big\{(U_k, y_k) \in \Xi_{\varepsilon_k}\big\}_{k\in\mathbb{N}}$ be a sequence of admissible solutions to corresponding perturbed problems (20)–(24) such that $(U_k, y_k) \xrightarrow{w} (U, y)$. Then properties (19) are valid.

Theorem 4. *Let $\big\{\rho^\varepsilon = (\rho)_\varepsilon\big\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho(x) > 0$. Then the minimization problem (9)–(12) is a weak variational limit of the sequence (20)–(24) as $\varepsilon\to 0$ with respect to the w-convergence in the variable space $\mathbb{Y}_\varepsilon$.*

Proof. Under preconditions of the theorem let $(U, y) \in \Xi_H$ be any admissible pair. Firstly, let us show that there exists a realizing sequence $\big\{(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}_\varepsilon\big\}_{\varepsilon>0}$ such that

$$(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon \quad \forall\varepsilon>0, \qquad \hat{U}_\varepsilon \to U \quad \text{weakly-}* \text{ in } \quad L^\infty(\Omega;\mathbb{R}^{N\times N});$$

$$\operatorname{div}(\rho^\varepsilon \vec{a}_{i_\varepsilon}) \rightharpoonup \operatorname{div}(\rho\vec{a}_i) \quad \text{in} \quad L^2\big(0,T;L^2(\Omega)\big) \quad \forall i\in\{1,\dots,N\};$$

$$\hat{y}_\varepsilon \to y \quad \text{in} \quad L^2\big(0,T;L^2(\Omega)\big); \qquad \nabla\hat{y}_\varepsilon \rightharpoonup \nabla y \quad \text{in} \quad L^2(0,T;(L^2(\Omega,\rho^\varepsilon dx))^N);$$
$$\hat{y}'_\varepsilon \rightharpoonup y' \quad \text{in} \quad L^2\big(0,T;L^2(\Omega)\big).$$

To begin with, we assume that a given control U is such that $U=[\vec{a}_1,\ldots,\vec{a}_N]\in M_2^{\alpha,\beta}(\Omega)$ and $|\operatorname{div}(\rho\vec{a}_i)|\le\gamma_i,$ a.e. in $\Omega,\ \forall i=1,\ldots,N.$ Further we construct the sequence $\big\{(\hat{U}_\varepsilon,\hat{y}_\varepsilon)\big\}_{\varepsilon>0}$ as follows:

$$\hat{U}_\varepsilon(x)=(U)_\varepsilon(x)=\big[(\vec{a}_1)_\varepsilon(x),\ldots,(\vec{a}_N)_\varepsilon(x)\big]=\int\limits_{\mathbb{R}^N}Q(z)U(x+\varepsilon z)\,dz,$$

$\hat{y}\in L^2\big(0,T;H(\Omega,\rho\,dx)\big)$ is an H-solution of (21) corresponding to $U=\hat{U}_\varepsilon$. Applying similar suggestions to [13] (Lemma 12) it can be shown that for every $\varepsilon>0$ the pair $(\hat{U}_\varepsilon,\hat{y}_\varepsilon)$ is admissible to the corresponding optimal control problem (20)–(24). As a result, following arguments of the proof of Lemma 5, we have that

$$\hat{y}_\varepsilon \to y \quad \text{in} \quad L^2\big(0,T;L^2(\Omega)\big), \qquad \nabla\hat{y}_\varepsilon \rightharpoonup \nabla y \quad \text{in} \quad L^2(0,T;(L^2(\Omega,\rho^\varepsilon dx))^N),$$
$$\hat{y}'_\varepsilon \rightharpoonup y' \quad \text{in} \quad L^2\big(0,T;L^2(\Omega)\big),$$

and

$$\hat{U}_\varepsilon\nabla\hat{y}_\varepsilon \rightharpoonup U\nabla y \quad \text{in} \quad L^2\big(0,T;(L^2(\Omega,\rho^\varepsilon dx))^N\big),$$

where $y=y(U)$, for any subsequence of $\{\hat{y}_\varepsilon\in\mathcal{H}_\varepsilon\}_{\varepsilon>0}$ and, hence, for the entire sequence. Here $(U,y)\in\Xi_H$ is a given H-admissible solution to problem (9) – (12).

Note that in view of 5 we have that $I(U,y)=\lim_{\varepsilon\to0}I_\varepsilon(\hat{U}_\varepsilon,\hat{y}_\varepsilon)$.

Taking into account Definition 6, Lemma 5 and previous suggestions we obtain the statement of the theorem.

References

1. S. E. Pastuhova, *Degenerate equations of monotone type: Lavrent'ev phenomenon and attainability problems*, Mathematics, **198**, Issue 10, 1465 – 1494 (2007).
2. V. V. Zhikov, *Weighted Sobolev spaces*, Mathematics, **189**, Issue 8, 27 – 58 (1998).
3. V. V. Zhikov, S. E. Pastukhova, *Homogenization of degenerate elliptic equations*, Sib. Math. J., **25**, Issue 1, 80 – 101 (2006).
4. P. I. Kogut, G. Leugering, *Optimal L^1-control in coefficients for Dirichlet elliptic problems: H-optimal solutions*, Z. Anal. Anwend., **31**, Issue 1, 31 – 53 (2012).
5. P. I. Kogut, G. Leugering, *Optimal L^1-control in coefficients for Dirichlet elliptic problems: W-optimal solutions*, J. Optim. Theory Appl., **150**, 205 – 232 (2011).
6. F. Murat, *Un contre-exemple pour le probléme de de contrôle dans les coefficients*, C.R.A.S. Paris, Sér. A, **273**, 708 – 711 (1971).
7. G. Butazzo, G. Dal Maso, A. Garroni, A. Malusa, *On the relaxed formulation of some shape optimization problems*, Adv. Math. Sci. Appl, **1**, Issue 7, 1 – 24 (1997).
8. J.-L. Lions, *Optimal control of systems governed by partial differential equations*, Springer Verlag, New York (1971).
9. F. Murat, *Compacité par compensation*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser., Issue 5, 489 – 507 (1978).

10. A. N. Stanzhitskii, T. V. Dobrodzii, *Study of optimal control problems on the half-line by the averaging method*, Differ. Equ., **47**, 264-–277 (2011).

11. A. V. Sukretna, O. A. Kapustyan, *Approximate averaged synthesis of the problem of optimal control for a parabolic equation*, Ukr. Mat. Zh., **56**, 1384 – 1394 (2004).

12. E. A. Kapustyan, A. G. Nakonechnyi, *Optimal bounded control synthesis for a parabolic boundary-value problem with fast oscillatory coefficients*, J. Autom. Inform. Sci., **31**, 33 – 44 (1999).

13. O. P. Kupenko, *Optimal control problems in coefficients for degenerate variational inequalities of monotone type. II. Attainability problem*, J. Comput. Appl. Math., **1**, № 107, 15 – 34 (2012).

14. A. A. Kovalevsky, Yu. S. Gorban, *Degenerate anisotropic variational inequalities with L^1-data*, C. R. Math. Acad. Sci. Paris, **345**, № 8, 441 – 444 (2007).

15. V. Chiadó Piat, F. Serra Cassano, *Cassano some remarks about the density of smooth functions in weighted Sobolev spaces*, J. Convex Anal., **1**, Issue 2, 135 – 142 (1994).

16. V. V. Zhikov, *On Lavrentiev phenomenon*, Russ. J. Math. Phys, **3**, Issue 2, 249 – 269 (1994).

17. F. Nicolosi, P. Drabek, A. Kufner, *Nonlinear elliptic equations, singular and degenerate cases*, Univ. West Bohemia, Pilsen (1996).

18. E. Zeider, *Nonlinear analysis and its applications*, IIA, IIB, Springer-Verlag, New York; Heidelberg (1990).

19. V. Barbu, *Optimal control of variational inequalities*, Pitman Adv. Publ. Program, London (1984).

20. O. P. Kupenko, *Optimal control problems in coefficients for degenerate variational inequalities of monotone type. I. Existence of optimal solutions*, J. Comput. Appl. Math., **3**, № 106, 88 – 104 (2011).

21. I. G. Balanenko, P. I. Kogut, *H-optimal control in coefficients for Dirichlet parabolic problems*, Bull. Dnipropetrovsk Univ.: Ser. Commun. Math. Model. Differ. Equ. Theory, **18**, Issue 8, 45 – 63 (2010).

22. N. V. Kasimova, *Solvability issue for optimal control problem in coefficients for degenerate parabolic variational inequality*, Underst. Complex Syst. (2020) (in print).

23. C. D'Apice, U. De Maio, P. I. Kogut, *Suboptimal boundary control for elliptic equations in critically perforated domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **25**, 1073 – 1101 (2008).

24. Ch. Gaevskii, K. Greger, K. Zacharias, *Nonlinear operator equations and operator differential equations* [in Russian], Mir, Moscow (1978).

Received 31.03.20,
after revision — 09.04.20