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EXPONENTIAL STABILITY OF INVARIANT MANIFOLD FOR NONLINEAR IMPULSIVE MULTIFREQUENCY SYSTEM*

ЕКСПОНЕНЦІАЛЬНА СТІЙКІСТЬ ІНВАРІАНТНОГО МНОГОВИДУ НЕЛІНІЙНОЇ ІМПУЛЬСНОЇ БАГАТОЧАСТОТНОЇ СИСТЕМИ

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We study the exponential stability of a trivial invariant manifold of nonlinear extension of the dynamical system on a torus with impulsive jumps at nonfixed moments of time. The derived sufficient conditions for the exponential stability of the trivial torus take advantage of the information on qualitative properties of the system dynamics on the invariant manifold and relax sufficient conditions available in the literature for a wide class of dynamical systems. New theorems set constraints in a nonwandering set of the dynamical system that guarantee the exponential stability of trivial manifold and are especially beneficial for the stability analysis of extensions of dynamical systems with a simple structure of limit sets and recurrent trajectories.

Досліджено експоненціальну стійкість тривіального інваріантного многовиду нелінійного розширення динамічної системи на торі з імпульсними збуреннями в нефіксовані моменти часу. Одержані достатні умови експоненціальної стійкості тривіального тора суттєвим чином враховують якісну поведінку траєкторій системи на інваріантному многовиді. Нові теореми про експоненціальну стійкість тривіального тороїдального многовиду встановлюють обмеження на множині неблукаючих точок динамічної системи та можуть бути застосовані, зокрема, для аналізу стійкості інваріантних торів розширень динамічних систем з простою структурою граничних множин і рекурентних траєкторій.

1. Introduction. Invariant toroidal manifold is the central object of investiga tions in the qualitative theory of multifrequency oscillations. The existence of invariant tori is a necessary condition for the existence of multifrequency oscillations, which are formed by quasiperiodic solutions to a dynamical system [1]. Fundamental results on the existence of invariant toroidal manifolds of linear systems in $\mathcal{T}_m \times \mathbb{R}^n$, perturbation theory of invariant manifolds for

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nonlinear systems, smoothness, and stability properties of invariant tori have been developed by A. M. Samoilenko and are summarized in [1]. In [2], the stability properties of invariant tori of systems defined in the direct product of m-dimensional torus $\mathcal{T}_m$ and n-dimensional Euclidean space $\mathbb{R}^n$ have been studied in terms of sign-definite quadratic forms. Sufficient conditions for the existence and stability of invariant tori of linear extensions of dynamical systems on torus that undergo impulsive perturbations [3 – 5] have been derived in [6, 7].

In this paper, we derive new sufficient conditions for the exponential stability of trivial torus of nonlinear extension of dynamical system on $\mathcal{T}_m$ that undergoes impulsive perturbations when the trajectory on torus intersects a predefined submanifold of $\mathcal{T}_m$. We consider the invariant manifold not just as a set of points, but rather as a set of trajectories of the dynamical system and account for the system's dynamics on the surface of the torus. This approach leads us to sufficient conditions for the exponential stability of trivial torus in terms of quadratic forms which are sign-definite not on the whole surface of $\mathcal{T}_m$, but only in non-wandering set of dynamical system.

A similar technique for the stability analysis of invariant tori of linear extensions of dynamical systems on $\mathcal{T}_m$ has been used in [8, 9] for the impulse-free case and in [10 – 13] for the systems with impulsive perturbations. Sufficient conditions for the exponential stability and instability of trivial torus of nonlinear extensions of dynamical systems on torus without impulses in terms of sign-indefinite on $\mathcal{T}_m$ quadratic forms have been derived in [14].

Notation. By $C(\mathcal{T}_m)$ we denote the space of continuous functions $F = F(\varphi)$ defined on torus $\mathcal{T}_m$ which are 2π-periodic with respect to each of the components $\varphi_v,\ v = 1, \ldots, m$. The subspace $C^1(\mathcal{T}_m) \subseteq C(\mathcal{T}_m)$ denotes the space of continuously differentiable functions $F \in C(\mathcal{T}_m)$. By $C(\mathcal{T}_m \times \mathbb{R}^n)$ we denote the space of continuous functions $F = F(\varphi, x)$ defined on $\mathcal{T}_m \times \mathbb{R}^n$ that are of class $C(\mathcal{T}_m)$ for every fixed $x \in \mathbb{R}^n$. Finally, $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ denotes the space of continuously differentiable functions $F = F(\varphi, x)$ defined on $\mathcal{T}_m \times \mathbb{R}^n$ that are of class $C^1(\mathcal{T}_m)$ for every fixed $x \in \mathbb{R}^n$. For any $r > 0$ the r-neighbourhood of a set Ω is denoted by $O_r(\Omega)$. For any matrix S the expressions $S > 0,\ S \geq 0,\ S < 0$, and $S \leq 0$ mean that S is positive definite, positive semidefinite, negative definite, and negative semidefinite, respectively.

2. Problem statement. We consider a system of impulsive differential equations defined in the direct product of m-dimensional torus $\mathcal{T}_m$ and n-dimensional Euclidean space $\mathbb{R}^n$:

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \qquad \triangle x|_{\varphi\in\Gamma} = I(\varphi, x)x, \tag{2.1}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)^\top \in \mathcal{T}_m,\ x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$, functions $P, I \in C(\mathcal{T}_m \times \mathbb{R}^n)$, $a \in C(\mathcal{T}_m)$. We assume that the following conditions hold true:

(A1) $\exists M > 0$ such that $\forall(\varphi, x) \in \mathcal{T}_m \times \mathbb{R}^n$

$$\|P(\varphi, x)\| \leq M; \tag{2.2}$$

(A2) $\forall r > 0\ \exists L = L(r) > 0$ such that $\forall x', x'',\ \|x'\| \leq r,\ \|x''\| \leq r\ \forall\varphi \in \mathcal{T}_m$

$$\left\|P(\varphi, x'') - P(\varphi, x')\right\| \leq L\left\|x'' - x'\right\|;$$

(A3) $\exists A > 0$ such that $\forall\varphi', \varphi'' \in \mathcal{T}_m$

$$\left\|a(\varphi'') - a(\varphi')\right\| \leq A\left\|\varphi'' - \varphi'\right\|. \tag{2.3}$$

Condition (2.3) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{2.4}$$

generates a dynamical system on $\mathcal{T}_m$, which we shall denote by $\varphi_t(\varphi)$. For any set $\Omega \subset \mathcal{T}_m$ let $\varphi_t(\Omega) = \bigcup_{\varphi \in \Omega} \varphi_t(\varphi)$.

The impulsive set Γ is defined by

$$\Gamma = \{\varphi \in \mathcal{T}_m \mid \Phi(\varphi) = 0\},$$

where $\Phi \in C(\mathcal{T}_m)$. We assume that for any $\varphi \in \mathcal{T}_m$ there exist $\{t_i(\varphi)\}_{i=1}^{\infty} \subset (0, +\infty)$ which are the roots of the gather $\Phi(\varphi_t(\varphi)) = 0$, and

$$\exists \quad \theta > 0 \quad \forall \varphi \in \mathcal{T}_m \quad \forall i \geq 1 \qquad t_{i+1}(\varphi) - t_i(\varphi) \geq \theta. \tag{2.5}$$

The latter condition excludes the occurrence of a so-called beating or Zeno phenomenon, which is characterized by infinitely many impulsive jumps over a finite period of time. The Zeno phenomenon leads to the loss of forward completeness of solutions to impulsive system and complicates their asymptotic characterization. An attempt to prolong solutions beyond Zeno time and to study the asymptotic properties of the prolonged solutions can be found in [15, 16].

Under the made assumptions (A1) – (A3) for any initial value $x^0 \in \mathbb{R}^n$ there exists a unique solution to the Cauchy problem

$$\frac{dx}{dt} = P(\varphi_t(\varphi), x)x, \quad t \neq t_i(\varphi),$$

$$\triangle x|_{t=t_i(\varphi)} = I(\varphi_t(\varphi), x)x,$$

$$x(0) = x^0,$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter. We denote this solution by $x(t, \varphi, x^0)$. System (2.1) possesses invariant toroidal manifold $x = 0$, $\varphi \in \mathcal{T}_m$ which is called trivial.

In this paper, we aim to derive less restrictive compared to [2, 12] sufficient conditions for the exponential stability of trivial invariant torus of system (2.1).

Definition 2.1 [1]. *Trivial invariant torus*

$$x = 0, \quad \varphi \in \mathcal{T}_m$$

of the system (2.1) is called exponentially stable if there exist constants $K > 0$, $\gamma > 0$, *and* $\delta > 0$ *such that for all* $\varphi \in \mathcal{T}_m$ *and for all* $x^0 \in \mathbb{R}^n$, $\|x^0\| \leq \delta$ *it holds that*

$$\forall t \geq 0 \quad \left\|x\left(t, \varphi, x^0\right)\right\| \leq K \left\|x^0\right\| e^{-\gamma t}. \tag{2.6}$$

In order to derive the main result of the paper we make use of the concept of a non-wandering set of a dynamical system.

Definition 2.2 [17]. *A point* $\varphi \in \mathcal{T}_m$ *is called a wandering point of the dynamical system (2.4) if there exist a neighbourhood* $O_r(\varphi)$, $r = r(\varphi) > 0$, *and a moment of time* $T = T(\varphi) > 0$ *such that*

$$O_r(\varphi) \cap \varphi_t(O_r(\varphi)) = \varnothing \quad \forall t \geq T.$$

Let $\mathcal{W}$ be the set of all wandering points of (2.4) and let $\Omega = \mathcal{T}_m \setminus \mathcal{W}$ be a set of all non-wandering points of dynamical system (2.4). Since $\mathcal{T}_m$ is a compact set, the set Ω is a nonempty, invariant, and compact subset of $\mathcal{T}_m$ [1]. Additionally, the following lemma holds true.

Lemma 2.1 [17]. *For any $r > 0$ there exist $T(r) > 0$ and $N(r) > 0$ such that for any $\varphi \notin \Omega$ the corresponding trajectory $\varphi_t(\varphi)$ spends only a finite time that is bounded by $T(r)$ outside the r-neighbourhood of the set Ω and leaves this set not more than $N(r)$ times.*

Lemma 2.1 suggests that the trajectories of the dynamical system $\varphi_t(\varphi)$ spend most of their life-time in a vicinity of non-wandering set Ω. This observation motivates us to establish the sufficient conditions for the exponential stability of trivial torus by imposing some restrictions on the system dynamics only in a vicinity of Ω and not on the whole surface of torus $\mathcal{T}_m$.

3. Main results. For any $\varphi \in \mathcal{T}_m,\ x \in \mathbb{R}^n$ let us denote

$$\hat{S}(\varphi, x) = \frac{\partial S(\varphi, x)}{\partial \varphi} a(\varphi) + \frac{\partial S(\varphi, x)}{\partial x} \left(P(\varphi, x)x\right) + S(\varphi, x)P(\varphi, x) + P^{\top}(\varphi, x)S(\varphi, x),$$

where $S = S(\varphi, x)$ is a symmetric matrix of class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$.

Theorem 3.1. *Let (A1) – (A3) hold true and there exist a symmetric matrix $S = S(\varphi, x)$ of class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that*

$$\forall \varphi \in \Omega \quad S(\varphi, 0) > 0, \quad \hat{S}(\varphi, 0) < 0. \tag{3.1}$$

If for some $r > 0$ the dwell-time condition

$$\frac{1}{\theta} \ln (K\alpha) - \frac{\gamma}{2C} < 0 \tag{3.2}$$

is fulfilled, where

$$\alpha = \max_{\varphi \in \Gamma} \|E + I(\varphi, 0)\|, \tag{3.3}$$

$$K = \left(\frac{C}{\gamma}\right)^{\frac{N(r)+1}{2}} e^{2\left(M + \frac{\gamma}{2C}\right)T(r)},$$

and the constants $C > 0,\ \gamma > 0$ defined by the inequalities

$$\forall \varphi \in O_r(\Omega) \quad \forall x \in \mathbb{R}^n, \quad \|x\| \leq r, \quad S(\varphi, x) - \gamma E \geq 0, \quad \hat{S}(\varphi, x) + \gamma E \leq 0, \tag{3.4}$$

$$\forall \varphi \in \mathcal{T}_m \quad \forall x \in \mathbb{R}^n, \quad \|x\| \leq r, \quad \|S(\varphi, x)\| + \left\|\hat{S}(\varphi, x)\right\| \leq C, \tag{3.5}$$

for $N > 0,\ T > 0$ from Lemma 2.1, then the trivial torus $x = 0,\ \varphi \in \mathcal{T}_m$ of system (2.1) is exponentially stable.

Proof. Let us fix $r > 0,\ \gamma = \gamma(r)$, and $C = C(r) > 0$ such that (3.4) and (3.5) hold. Note that due to (3.1) and continuous dependence of the polynomial's roots on its coefficients [18], there exists $r_S > 0$ such that (3.4) holds for $r \in [0, r_S]$ and some $\gamma = \gamma(r) > 0$.

The proof of the theorem is divided into two parts. In the first part, we study the evolution of solutions to (2.1) in the vicinity of invariant manifold under the assumption that no impulsive perturbations occur. In the second part, we investigate the influence of impulsive jumps on the solutions to (2.1).

We start considering system (2.1) under the assumption that no impulses occur. Following the arguments of [14], let us distinguish three qualitatively different types of behaviour of trajectories of the dynamical system $\dot{\varphi} = a(\varphi)$:

(A) a trajectory starts within the r-neighbourhood of the non-wandering set Ω and remains there for all times;

(B) a trajectory starts within the r-neighbourhood of Ω, but leaves it within a finite time;

(C) a trajectory starts outside the r-neighbourhood of Ω.

Case A. Let

$$\varphi \in O_r(\Omega) \quad \text{and} \quad \forall\, s \geq 0 \quad \varphi_s(\varphi) \in O_r(\Omega).$$

Then, there exists $T \in (0, +\infty]$ such that for the solution $x(t) = x\left(t, \varphi, x^0\right)$ to (2) with $\left\|x^0\right\| < r$ the estimate $\|x(t)\| < r$ holds for $t \in [0, T)$. Hence, denoting by

$$V(\varphi, x) = (S(\varphi, x)x, x),$$

from (3.4), (3.5) we obtain the following estimates:

$$\gamma\|x(t)\|^2 \leq V\left(\varphi_t(\varphi), x(t)\right) \leq C\|x(t)\|^2,$$

$$\frac{d}{dt} V(\varphi_t(\varphi), x(t)) \leq -\gamma\|x(t)\|^2,$$

hold for $t \in [0, T)$. From the latter inequalities we get that

$$V(\varphi_t(\varphi), x(t)) \leq V\left(\varphi, x^0\right) e^{-\frac{\gamma}{C}t}.$$

Hence, there exist constants $K_1 = \left(\dfrac{C}{\gamma}\right)^{\frac{1}{2}} \geq 1$ and $\gamma_1 = \dfrac{\gamma}{2C} > 0$ such that $\forall t \in [0, T)$

$$\|x(t)\| \leq K_1 \left\|x^0\right\| e^{-\gamma_1 t}. \tag{3.6}$$

For $\|x^0\| < \dfrac{r}{K_1}$ we obtain that $T = +\infty$ and the inequality (3.6) holds for all $t \geq 0$.

Case B. Now, let $\varphi \in O_r(\Omega)$, but there exists $t_1 > 0$ such that

$$\forall t \in [0, t_1) \quad \varphi_t(\varphi) \in O_r(\Omega), \quad \varphi_{t_1}(\varphi) \notin O_r(\Omega).$$

From Lemma 2.1, for any $r > 0$ there exist uniform w.r.t. φ upper bounds $T(r)$ and $N(r)$ for the total time $T(\varphi, r)$ spent by the trajectory $\varphi_t(\varphi)$ outside the set $O_r(\Omega)$ and for the number of times $N(\varphi, r)$ the trajectory leaves $O_r(\Omega)$. Associating the time periods $t_i(\varphi, t)$ and $\tau_i(\varphi, t)$, which denotes the time spent by the trajectory $\varphi_t(\varphi)$ in the set $O_r(\Omega)$ and outside $O_r(\Omega)$, with every index $i \in \{1, \ldots, N(\varphi, r)\}$ we obtain the sequences

$$\{\tau_i(\varphi, r)\}_{i=1}^{N(\varphi,r)}, \qquad \{t_i(\varphi, r)\}_{i=1}^{N(\varphi,r)}, \qquad N(\varphi, r) \leq N(r),$$

$$\sum_{i=1}^{N(\varphi,r)} \tau_i(\varphi, r) =: T(\varphi, r) \leq T(r)$$

such that

$$\varphi_t(\varphi) \in O_r(\Omega),$$

$$\forall t \in (0, t_1) \cup \bigcup_{k=1}^{N(\varphi,r)-1} \left(\sum_{i=1}^{k}(\tau_i + t_i), \sum_{i=1}^{k}(\tau_i + t_i) + t_{k+1}\right) \cup \left(\sum_{i=1}^{N(\varphi,r)} (\tau_i + t_i), +\infty\right). \tag{3.7}$$

Then, from (A), for $t \in [0, t_1]$

$$\|x(t)\| \le K_1 \|x^0\| e^{-\gamma_1 t} < r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1}.$$

For $t \in [t_1, t_1 + \tau_1],$ from (2.2) and Wazewski inequality [19],

$$\|x(t)\| \le \|x(t_1)\| e^{M(t-t_1)} \le K_1 \|x^0\| e^{(\gamma_1+M)\tau_1} e^{-\gamma_1 t} < r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1 e^{(\gamma_1+M)\tau_1}}.$$

For $t \in [t_1 + \tau_1, t_1 + \tau_1 + t_2],$

$$\|x(t)\| \le K_1^2 \|x^0\| e^{(\gamma_1+M)\tau_1} e^{-\gamma_1 t} < r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1^2 e^{(\gamma_1+M)\tau_1}}.$$

Continuing this process, due to (3.7), we finally conclude that

$$\forall t \ge 0 \quad \|x(t)\| \le K_2 \|x^0\| e^{-\gamma_1 t}$$

for

$$K_2 := K_1^{N(r)} e^{(\gamma+M)T(r)}, \qquad \delta_2 := \frac{r}{K_1^{N(r)} e^{(\gamma+M)T(r)}}, \quad \|x^0\| < \delta_2.$$

Case C. Now, let us consider the case of $\varphi \notin O_r(\Omega).$ In this case,

$$\exists \tau_0 \in (0, T(r)), \quad \varphi_{\tau_0}(\varphi) \in O_r(\Omega)$$

and

$$\forall t \in [0, \tau_0], \quad \|x(t)\| \le K_1 \|x^0\| e^{(M+\gamma_1)T(r)} e^{-\gamma_1 t}.$$

Then, from (B), for

$$K_3 = K_1^{N(r)+1} e^{2(\gamma+M)T(r)}, \qquad \delta_3 = \frac{r}{K_1^{N(r)+1} e^{2(\gamma+M)T(r)}}$$

we obtain the required estimate (2.6).

Summarizing the first phase of the proof, we have proven that in the case of absence of impulses, for some $r > 0$ there exist

$$K = K(r) = \max\{K_1(r), K_2(r), K_3(r)\} \ge 1 \qquad \text{and} \qquad \delta = \frac{r}{K(r)} > 0$$

such that the solutions to (2) with $\|x^0\| \le \delta$ satisfy

$$\forall t \ge 0 \quad \|x(t, \varphi, x^0)\| \le K \|x^0\| e^{-\frac{\gamma}{2C} t}. \tag{3.8}$$

Now, let us study the impact of the impulsive jumps on solutions to (2). Without loss of generality, we assume that $\alpha > 1$ (i.e., impulsive jumps play destabilizing role in the system dynamics). Let us fix sufficiently small $r > 0$ such that condition (3.2) holds and for some positive $\mu > 0$

$$\frac{1}{\theta} \ln (K\alpha(r)) - \frac{\gamma}{2C} \le -\mu, \tag{3.9}$$

where

$$\alpha(r) = \max_{\varphi\in\Gamma, \|x\|\le r} \|E + I(\varphi, x)\| . \tag{3.10}$$

The existence of positive μ follows from (3.3) and continuity property of the map $I(\varphi,\cdot)$ for every fixed $\varphi\in\Gamma$.

We pick arbitrary $x^0\in\mathbb{R}^n$ with $\|x^0\| \le \dfrac{r}{\alpha(r)K^2}$, $\varphi\in\mathcal{T}_m$ and estimate the norm of solution $x(t,\varphi,x^0)$ using (3.8) and (3.10):

$$\begin{gathered}
\forall t\in[0,t_1(\varphi)], \quad \left\|x\left(t,\varphi,x^0\right)\right\| \le K\frac{r}{\alpha K^2}\, e^{-\frac{\gamma}{2C}t} \le \frac{r}{\alpha(r)K} \le \frac{r}{K}, \\
\left\|x\big(t_1(\varphi)+0,\varphi,x^0\big)\right\| \le \alpha(r)\,\frac{r}{\alpha(r)K}\, e^{-\frac{\gamma}{2C}t_1(\varphi)} \le \frac{r}{K}.
\end{gathered} \tag{3.11}$$

Since $\left\|x\big(t_1(\varphi)+0,\varphi,x^0\big)\right\| \le \dfrac{r}{K}$ we may use inequality (3.8) to estimate the norm of solution in the interval $t\in[t_1(\varphi),t_2(\varphi)]$. Analogously to (3.11) we get

$$\begin{aligned}
\forall t\in[t_1(\varphi),t_2(\varphi)] \quad \left\|x\left(t,x^0,\varphi\right)\right\| &\le K\left\|x\big(t_1(\varphi)+0,\varphi,x^0\big)\right\| e^{-\frac{\gamma}{2C}(t-t_1(\varphi))} \le \\
&\le K\alpha(r)\left\|x\left(t_1(\varphi),\varphi,x^0\right)\right\| e^{-\frac{\gamma}{2C}(t-t_1(\varphi))} \le \\
&\le K^2\alpha(r)\left\|x^0\right\| e^{-\frac{\gamma}{2C}t_1(\varphi)}e^{-\frac{\gamma}{2C}(t-t_1(\varphi))} \le \\
&\le K^2\alpha(r)\left\|x^0\right\| e^{-\frac{\gamma}{2C}t} \le re^{-\frac{\gamma}{2C}t}.
\end{aligned}$$

From (3.9), (3.10) we derive the estimate for the solution's norm after the impulsive jump:

$$\begin{aligned}
\left\|x\big(t_2(\varphi)+0,\varphi,x^0\big)\right\| &\le \alpha(r)re^{-\frac{\gamma}{2C}t_2(\varphi)} \le \alpha(r)re^{-\frac{\gamma}{2C}\theta} \\
&= \frac{r}{K}e^{\left(\frac{1}{\theta}\ln(\alpha(r)K)-\frac{\gamma}{2C}\right)\theta} \le \frac{r}{K}e^{-\mu\theta} \le \frac{r}{K}.
\end{aligned}$$

Since $\left\|x\big(t_2(\varphi)+0,\varphi,x^0\big)\right\| \le \dfrac{r}{K}$, we can estimate the norm of the solution in the interval $[t_2(\varphi),t_3(\varphi)]$. From (3.8), (3.9), (3.10) we get that $\forall t\in[t_2(\varphi),t_3(\varphi)]$

$$\begin{aligned}
\left\|x\left(t,\varphi,x^0\right)\right\| &\le K\left\|x\big(t_2(\varphi)+0,\varphi,x^0\big)\right\| e^{-\frac{\gamma}{2C}(t-t_2(\varphi))} \le \\
&\le K\alpha(r)\left\|x\left(t_2(\varphi),\varphi,x^0\right)\right\| e^{-\frac{\gamma}{2C}(t-t_2(\varphi))} \le \\
&\le K^2\alpha(r)\left\|x\left(t_1(\varphi)+0,\varphi,x^0\right)\right\| e^{-\frac{\gamma}{2C}(t_2(\varphi)-t_1(\varphi))}e^{-\frac{\gamma}{2C}(t-t_2(\varphi))} \le \\
&\le K^2\alpha^2(r)\left\|x\left(t_1(\varphi),\varphi,x^0\right)\right\| e^{-\frac{\gamma}{2C}\,(t-t_1(\varphi))} \le \\
&\le K^3\alpha^2(r)\left\|x^0\right\| e^{-\frac{\gamma}{2C}t_1(\varphi)}e^{-\frac{\gamma}{2C}(t-t_1(\varphi))} \le \\
&\le K^3\alpha^2(r)\left\|x^0\right\| e^{-\frac{\gamma}{2C}t} = K^2\alpha(r)\left\|x^0\right\| e^{\frac{1}{\theta}\ln(\alpha(r)K)\theta}e^{-\frac{\gamma}{2C}t} \le
\end{aligned}$$

$$\leq K^2\alpha(r)\left\|x^0\right\| e^{\left(\frac{1}{\theta}\ \ln(\alpha(r)K)-\frac{\gamma}{2C}\right)t} \leq$$

$$\leq K^2\alpha(r)\left\|x^0\right\| e^{-\mu t}.$$

Continuing this process, after the n-th impulse we obtain

$$\left\|x\big(t_n(\varphi)+0,\varphi,x^0\big)\right\| \leq \frac{r}{K}\, e^{-(n-1)\mu\theta} \leq \frac{r}{K},$$

$$\forall t \in [t_n(\varphi), t_{n+1}(\varphi)] \quad \left\|x\left(t,\varphi,x^0\right)\right\| \leq K^2\alpha(r)\left\|x^0\right\| e^{-\mu t}$$

that imply the desired exponential stability of trivial torus $x = 0,\ \varphi \in \mathcal{T}_m$.

This completes the proof.

Theorem 3.1 requires the knowledge of the constants $N,\ T$ from Lemma 2.1 in order to check the dwell-time condition (3.2) and conclude exponential stability of the trivial torus. If the non-wandering set Ω and impulsive set Γ do not intersect, the following theorem, which does not require the fulfillment of the condition (3.2), holds true.

Theorem 3.2. *Let (A1) – (A3) hold and there exist a symmetric matrix $S = S(\varphi, x)$ of class $C^1\left(\mathcal{T}_m \times \mathbb{R}^n\right)$ such that*

$$\forall\varphi \in \Omega \quad S(\varphi, 0) > 0, \quad \hat{S}(\varphi, 0) < 0.$$

If

$$\Omega \cap \Gamma = \varnothing \tag{3.12}$$

and for some $\theta > 0$ the condition (2.5) holds true, then the trivial torus $x = 0,\ \varphi \in \mathcal{T}_m$ of system (2.1) is exponentially stable.

Proof. The proof is based on the parts (A), (B), and (C) from the proof of Theorem 3.1. From (3.12) it follows that there exists $r > 0$ such that trajectories that start in the r-neighbourhood of Ω and remain there for all times (case (A) from the proof of Theorem 3.1) do not undergo impulsive jumps. Then, there exist constants $K_1 \geq 1$ and $\gamma > 0$ such that $\forall t \geq 0$

$$\|x(t)\| \leq K_1\left\|x^0\right\| e^{-\gamma_1 t}$$

for $\|x^0\| < \dfrac{r}{K_1}$.

Since every trajectory $\varphi_t(\varphi)$ resides outside the $O_r(\Omega)$ only for a finite time $T(r)$, the condition (2.5) implies that the system may undergo only a finite number of impulses $N_{\max} \leq \leq \dfrac{T(r)}{\theta}$. Then, from (B) and (C) we derive the estimate (3.8) with

$$K = K(r) = \alpha(r)^{\frac{T(r)}{\theta}} \max\left\{K_1(r), K_2(r), K_3(r)\right\}.$$

This completes the proof.

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