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TWO FUNCTIONAL BOUNDARY-VALUE PROBLEMS WITH SINGULARITIES IN PHASE VARIABLES*

ДВІ ГРАНИЧНІ ФУНКЦІОНАЛЬНІ ЗАДАЧІ З ОСОБЛИВОСТЯМИ У ФАЗОВИХ ЗМІННИХ

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The differential equation $x'' = f(t, x, x')$ together with two functional boundary conditions is considered. Here $f(t, x, y)$ is local Carathéodory function which may be singular at the points $x = 0$ and $y = 0$ of the phase variables x and y. The main common feature for these two singular problems is the fact that any solution or the derivative of any solution “pass through” the singularities of f somewhere inside of $[0, T]$. Existence results are proved by the regularization and sequential techniques and using the Borsuk antipodal theorem, the Leray – Schauder degree and the Vitali's convergence theorem.

Розглядається диференціальне рівняння $x'' = f(t, x, x')$ з двома функціональними граничними умовами. Тут $f(t, x, y)$ локально є функцією Каратеодорі, що може мати особливість відносно фазових змінних x та y в точках $x = 0$ та $y = 0$. Основною спільною властивістю цих двох задач з особливостями є те, що будь-який розв'язок або похідна будь-якого розв'язку „проходить” через особливості f всередині $[0, T]$. Результати про існування доведено за допомогою регуляризації та послідовностей, а також з використанням антимодальної теореми Барсука, степеня Лере – Шаудера та теореми Віталі про збіжність.

1. Introduction. Let T be a positive constant, $J = [0, T]$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. Throughout the paper we denote by $\|x\| = \max\{|x(t)| : t \in J\}$, $\|x\|_L = \int_0^T |x(t)|\, dt$ and $\|x\|_\infty = \operatorname{ess\,max}\{|x(t)| : t \in J\}$ the norm in the space $C^0(J)$, $L_1(J)$ and $L_\infty(J)$, respectively. $AC^1(J)$ is the set of all functions having the first derivatives absolutely continuous on J. For any measurable set $\mathcal{M} \subset \mathbb{R}$, $\mu(\mathcal{M})$ denotes the Lebesgue measure of $\mathcal{M}$.

Let $\varepsilon \in [0, T)$ and $\mathcal{A}_\varepsilon$ be the set of all functionals $\alpha : C^0([\varepsilon, T]) \to \mathbb{R}$ which are

(a) continuous, $\alpha(0) = 0$, and

(b) increasing (i.e. $x, y \in C^0([\varepsilon, T])$, $x(t) < y(t)$ for $t \in [\varepsilon, T] \Rightarrow \alpha(x) < \alpha(y)$) (see [1 – 4]).

Example 1.1. Let $\varepsilon \in [0, T)$, $k \in C^0(\mathbb{R})$, be an increasing function, $k(0) = 0$, $r \in L_1([\varepsilon, T])$, $r > 0$ a.e. on $[\varepsilon, T]$, $\varepsilon \le t_1 < t_2 \le T$, $\varepsilon \le \xi_1 < \xi_2 < \ldots \xi_n \le T$ and $a_j > 0$, $j = 1, 2, \ldots, n$, be constants. Then the functionals

$$\max\{x(t) : t_1 \le t \le t_2\}, \quad \min\{x(t) : t_1 \le t \le t_2\},$$

$$\int_{t_1}^{t_2} r(t)k(x(t))\, dt, \quad \int_{t_1}^{t_2}\int_{t_1}^{t} r(s)k(x(s))\, ds\, dt, \quad \sum_{j=1}^{n} a_j k(x(\xi_j))$$

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and all their linear combinations with positive coefficients belong to $\mathcal{A}_\varepsilon$ (see [1–4]).

In the paper we consider the second order differential equation

$$x''(t) = f(t, x(t), x'(t)), \tag{1.1}$$

together with the functional boundary conditions either

$$\alpha(x) = 0, \quad x'(0) = 0, \qquad \alpha \in \mathcal{A}_\varepsilon \text{ with } \varepsilon \in [0, T) \tag{1.2}$$

or

$$x(0) = 0, \quad \alpha(x) = 0, \qquad \alpha \in \mathcal{A}_\varepsilon \text{ with } \varepsilon \in (0, T). \tag{1.3}$$

Here f satisfies the local Carathéodory conditions on $J \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}_0^2$ ($f \in \mathrm{Car}\,(J \times \mathcal{D})$) and $f(t, x, y)$ may be singular at the points $x = 0$ and $y = 0$ of the phase variables x and y.

Definition 1.1. *We say that a function $x \in AC^1(J)$ is a solution of BVP (1.1), (1.2) if x satisfies the boundary conditions (1.2)*[1] *and (1.1) holds for a.e. $t \in J$.*

Definition 1.2. *By a solution of BVP* (1.1), (1.3) *we understand a function $x \in AC^1(J)$ that satisfies the boundary conditions* (1.3) *and, for a.e. $t \in J$, fulfils* (1.1).

The aim of this paper is to give conditions on the function f which guarantee the solvability of BVPs (1.1), (1.2) and (1.1), (1.3). We note (see Lemma 2.1) that the condition $\alpha(x) = 0$ in (1.2) and (1.3) implies $x(\xi) = 0$ for some $\xi \in [\varepsilon, T]$. Hence BVPs (1.1), (1.2) and (1.1), (1.3) are singular with f having singularities in both phase variables. Indeed, if with some $\xi \in (0, T)$ $x(\xi) = 0$ for a solution x of BVP (1.1), (1.2), which always occurs if $\varepsilon \in (0, T)$, then the singularities of f "appear" at the fixed point $t = 0$, where x' vanishes, and the inner point ξ of J, where x' "passes through" a singularity of f. For BVP (1.1), (1.3) the functional boundary conditions (1.3) imply that the derivative of any solution to this problem vanishes at an inner point of J. Hence the singularities of f "appear" now in any solution x of BVP (1.1), (1.3) at the fixed point $t = 0$, where x vanishes, and an inner point of J, where x' vanishes. So, the main common feature of BVPs (1.1), (1.2) and (1.1), (1.3) is the fact that any solution or the derivative of any solution "passes through" singularities of f somewhere inside J. As we know, these problems for equation (1.1) with a one-parameter family of the functional boundary conditions (1.2) or (1.3) with the parameter ε has not been considered, yet.

In the special case where $\alpha(x) = x(T)$ in (1.3), BVP (1.1), (1.3) is the Dirichlet boundary-value problem. We recall that this problem was considered with f having singularities in phase variables in many papers (see, e.g., [5–22] and the references therein) with solutions in the class $C^0(J) \cap C^2((0, T))$ or $C^1(J) \cap C^2((0, T))$ or $C^0(J) \cap AC^1_{\mathrm{loc}}((0, T))$. Here $AC^1_{\mathrm{loc}}((0, T))$ denotes the set of functions whose first derivatives are absolutely continuous on each $[a, b] \subset (0, T)$. The nonlinearities of equations are usually nonpositive ([5, 6, 10, 11, 14–20, 22]) but in [7–9], [12, 13, 21] this assumption is overcome.

We note that functional boundary-value conditions for regular (in phase variables) differential equations and functional differential equations are used in many papers, for instance in [1–4, 23–31] and the references therein.

[1] If $x \in C^0(J)$ and $\alpha \in \mathcal{A}_\varepsilon$ with $\varepsilon \in [0, T)$, then throughout the paper $\alpha(x)$ means $\alpha(x|_{[\varepsilon, T]})$ where $x|_{[\varepsilon, T]}$ denotes the restriction of x to the interval $[\varepsilon, T]$.

In the paper the following two assumptions on the function f in (1.1) are used for BVP (1.1), (1.2):

(H_1) $f \in \mathrm{Car}\,(J \times \mathbb{R}_0 \times \mathbb{R}_+)$ and there exist $\psi \in L_1(J)$ and positive constants $a,\ \gamma$ such that

$$0 < \psi(t) \le f(t,x,y) \quad \text{for a.e. } t \in J \text{ and each } (x,y) \in \mathbb{R}_0 \times \mathbb{R}_+$$

and

$$\int\limits_0^t \psi(s)\,ds \ge at^\gamma \quad \text{for } t \in J; \tag{1.4}$$

(H_2) for a.e. $t \in J$ and each $(x,y) \in \mathbb{R}_0 \times \mathbb{R}_+$,

$$f(t,x,y) \le \phi(t) + q_0(t)\omega_0(|x|) + q_1(t)\omega_1(y) + h_0(t)|x| + h_1(t)y,$$

where $\phi,\ h_i \in L_1(J)$, $q_i \in L_\infty(J)$ are nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing, $i = 0,1$,

$$\int\limits_0^T \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt < \infty, \quad \int\limits_0^T \omega_1(at^\gamma)\,dt < \infty, \tag{1.5}$$

and

$$T\|h_0\|_L + \|h_1\|_L < 1, \tag{1.6}$$

and for BVP (1.1), (1.3):

(H_3) $f \in \mathrm{Car}\,(J \times \mathbb{R}_0^2)$ and there exists $a \in (0,\infty)$ such that $a \le -f(t,x,y)$ for a.e. $t \in J$ and each $(x,y) \in \mathbb{R}_0^2$;

(H_4) for a.e. $t \in J$ and each $(x,y) \in \mathbb{R}_0^2$,

$$-f(t,x,y) \le \phi(t) + q_0(t)\omega_0(|x|) + q_1(t)\omega_1(|y|) + h_0(t)|x| + h_1(t)|y|,$$

where $\phi,\ h_i \in L_1(J)$, $q_i \in L_\infty(J)$ are nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing,

$$\int\limits_0^T \omega_i(t)\,dt < \infty,$$

$\omega_i(uv) = \Lambda\omega_i(u)\omega_i(v)$ for $u,v \in \mathbb{R}_+$ with a positive constant Λ, $i = 0,1$, and (1.6) holds.

The paper is organized as follows. Section 2 deals with auxiliary regular two-parameters BVPs to problems (1.1), (1.2) and (1.1), (1.3) which depend on the parameters $\lambda \in [0,1]$ and $n \in \mathbb{N}$. We give bounds for their solutions. Applying the Borsuk antipodal theorem and the Leray – Schauder degree theory (see, e.g., [32]), we prove the existence of solutions for the above auxiliary problems with $\lambda = 1$. In addition, the uniform absolute continuity on J for some sets of functions formed by a superposition using solutions of the auxiliary problems with $\lambda = 1$ is proved. The main results for the solvability of BVPs (1.1), (1.2) and (1.1), (1.3) are given in Section 3. The proofs are based on the sequential technique and use the Arzelà – Ascoli theorem and the Vitali's convergence theorem (see, e.g., [33, 34]). Finally, two examples demonstrate general existence results.

2. Auxiliary regular BVPs. ***2.1. BVP (1.1), (1.2).*** Let assumptions (H_1) and (H_2) be satisfied. For each $n \in \mathbb{N}$, define $\chi_n \in C^0(\mathbb{R})$ and $f_n \in \mathrm{Car}\,(J \times \mathbb{R}^2)$ by the formulas

$$\chi_n(u) = \begin{cases} |u| & \text{for } |u| \geq \dfrac{1}{n}; \\ \dfrac{1}{n} & \text{for } |u| < \dfrac{1}{n}, \end{cases}$$

$$f_n(t,x,y) = \begin{cases} f(t,x,\chi_n(y)) & \text{for } (t,x,y) \in J \times \left(\left(-\infty, -\dfrac{1}{n}] \cup [\dfrac{1}{n}, \infty\right)\right) \times \mathbb{R}; \\ \dfrac{n}{2}\left[f\left(t,\dfrac{1}{n},\chi_n(y)\right)\left(x+\dfrac{1}{n}\right) + f\left(t,-\dfrac{1}{n},\chi_n(y)\right)\left(\dfrac{1}{n}-x\right)\right] & \\ \qquad\qquad \text{for } (t,x,y) \in J \times \left[\left(-\dfrac{1}{n},\dfrac{1}{n}\right)\right] \times \mathbb{R}. & \end{cases}$$

From (H_1) and (H_2) it follows

$$0 < \psi(t) \leq f_n(t,x,y) \quad \text{for a.e. } t \in J \text{ and each } (x,y) \in \mathbb{R}^2 \tag{2.1}$$

and

$$f_n(t,x,y) \leq \phi(t) + q_0(t)\omega_0\left(\max\left\{\frac{1}{n}, |x|\right\}\right) + q_1(t)\omega_1\left(\frac{1}{n}\right) +$$

$$+ \; h_0(t)(1+|x|) + h_1(t)(1+|y|) \tag{2.2}$$

for a.e. $t \in J$ and each $(x,y) \in \mathbb{R}^2$. It is obvious that

$$f_n(t,x,y) \leq \phi(t) + q_0(t)\omega_0(|x|) + q_1(t)\omega_1(|y|) + h_0(t)(1+|x|) + h_1(t)(1+|y|) \tag{2.3}$$

for a.e. $t \in J$ and each $(x,y) \in \mathbb{R}^2_0$.

Consider the two-parameter family of the differential equations

$$x''(t) = \lambda f_n(t,x(t),x'(t)) \tag*{$(2.4)_{\lambda,n}$}$$

depending on the parameters $\lambda \in [0,1]$ and $n \in \mathbb{N}$.

In our considerations we will use the following lemma.

Lemma 2.1 [1, 2]. *Let $\alpha \in \mathcal{A}_\varepsilon$ with $\varepsilon \in [0,T)$ and let $\alpha(x) = 0$ for some $x \in C^0([\varepsilon,T])$. Then there exists $\xi \in [\varepsilon,T]$ such that*

$$x(\xi) = 0.$$

Lemma 2.2. *Let assumptions* (H_1) *and* (H_2) *be satisfied and let* $n \in \mathbb{N}$. *Then any solution* x *of BVPs* $(2.4)_{\lambda,n}$, (1.2) *with* $\lambda \in [0,1]$ *satisfies the inequalities*

$$\|x\| \leq KT, \quad \|x'\| \leq K, \tag{2.5}$$

where $K > 0$ *is a constant independent of* λ *and* $\alpha \in \mathcal{A}_\varepsilon$.

Proof. Fix $\lambda \in [0,1]$. Let x be a solution of BVP $(2.4)_{\lambda,n}$, (1.2). By Lemma 2.1, $x(\xi) = 0$ for some $\xi \in [\varepsilon, T]$, and so

$$|x(t)| = \Big| \int_\xi^t x'(s)\,ds \Big| \leq T\|x'\|, \quad t \in J. \tag{2.6}$$

From $x'(0) = 0$ and (2.1) we see that $x'(t) \geq 0$ for $t \in J$ and then (2.2) gives

$$\begin{aligned} 0 \leq x'(t) \;&=\; \lambda \int_0^t f_n(s, x(s), x'(s))\,ds \leq \\ &\leq \int_0^T \Big[\phi(t) + q_0(t)\omega_0\left(\frac{1}{n}\right) + q_1(t)\omega_1\left(\frac{1}{n}\right) + \\ &\qquad + h_0(t)(1 + |x(t)|) + h_1(t)(1 + |x'(t)|) \Big]\,dt \leq \\ &\leq \|\phi\|_L + \omega_0\left(\frac{1}{n}\right)\|q_0\|_\infty + \omega_1\left(\frac{1}{n}\right)\|q_1\|_\infty + \\ &\qquad + (1 + T\|x'\|)\|h_0\|_L + (1 + \|x'\|)\|h_1\|_L \end{aligned}$$

for $t \in J$. Hence

$$\|x'\| \leq \frac{A}{1 - T\|h_0\|_L - \|h_1\|_L}$$

where

$$A = \|\phi\|_L + \omega_0\left(\frac{1}{n}\right)\|q_0\|_\infty + \omega\left(\frac{1}{n}\right)\|q_1\|_\infty + \|h_0\|_L + \|h_1\|_L$$

and then (2.6) yields

$$\|x\| \leq \frac{AT}{1 - T\|h_0\|_L - \|h_1\|_L}.$$

Consequently, (2.5) holds with $K = \dfrac{A}{1 - T\|h_0\|_L - \|h_1\|_L}$.

Lemma 2.3. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then for each* $n \in \mathbb{N}$*, there exists a solution of BVP* $(2.4)_{1,n}$, (1.2).

Proof. Fix $n \in \mathbb{N}$. By Lemma 2.2, there is a constant $K > 0$ such that (2.5) holds for any solution x of BVPs $(2.4)_{\lambda,n}$, (1.2) with $\lambda \in [0,1]$. Set

$$\Omega = \{(x,c) : (x,c) \in C^1(J) \times \mathbb{R},\ \|x\| < KT + 1,\ \|x'\| < K + 1,\ |c| < KT + 1\}.$$

Then Ω is an open ball in the Banach space $C^1(J)\times\mathbb{R}$ with the norm $\|(x,c)\|_* = \max\{\|x\|,\ \|x'\|\}+ +|c|$. Define the operator $\mathcal{K} : [0,1] \times \overline{\Omega} \to C^1(J) \times \mathbb{R}$ by

$$\mathcal{K}(\lambda, x, c) = \left(c + \lambda \int_0^t \int_0^s f_n(v, x(v), x'(v))\, dv\, ds,\ c + \alpha(x) - (1-\lambda)\alpha(-x) \right).$$

Since $f_n \in \mathrm{Car}\,(J \times \mathbb{R}^2)$ and α is a continuous and increasing functional on $C^0([\varepsilon, T])$, we see that $\mathcal{K}$ is a compact operator. Besides

$$\mathcal{K}(0, -x, -c) = (-c,\ -c + \alpha(-x) - \alpha(x)) = -(c,\ c + \alpha(x) - \alpha(-x)) = -\mathcal{K}(0, x, c)$$

for $(x,c) \in \overline{\Omega}$, and so $\mathcal{K}(0,\cdot,\cdot)$ is an odd operator. Assume that $\mathcal{K}(\lambda_0, x_0, c_0) = (x_0, c_0)$ for some $\lambda_0 \in [0,1]$ and $(x_0, c_0) \in \partial\Omega$. Then

$$x_0(t) = c_0 + \lambda_0 \int_0^t \int_0^s f_n(v, x_0(v)), x_0'(v))\, dv\, ds, \quad t \in J, \tag{2.7}$$

and

$$\alpha(x_0) = (1 - \lambda_0)\alpha(-x_0). \tag{2.8}$$

From (2.7) we deduce that $x_0'(0) = 0$ and $x_0(t)$ is a solution of $(2.4)_{\lambda_0,n}$. If $x_0(t) \neq 0$ for $t \in [\varepsilon, T]$, say $x_0 > 0$ on $[\varepsilon, T]$, then $\alpha(x_0) > 0$ and $\alpha(-x_0) < 0$, contrary to (2.8). Therefore $x_0(\eta) = 0$ for some $\eta \in [\varepsilon, T]$. Now Lemma 2.2 (with the functional α in Lemma 2.2 defined by $\alpha(x) = x(\eta)$ for $x \in C^0([\varepsilon, T])$) gives $\|x_0\| \leq KT$, $\|x_0'\| \leq K$ and then, by (2.7), $|c_0| = = |x_0(0)| \leq KT$. Hence $(x_0, c_0) \notin \partial\Omega$ and we have proved that

$$\mathcal{K}(\lambda, x, c) \neq (x, c) \quad \text{for } \lambda \in [0,1] \text{ and } (x,c) \in \partial\Omega.$$

Therefore, by the Leray – Schauder degree theory,

$$D(\mathcal{I} - \mathcal{K}(0,\cdot,\cdot), \Omega, 0) = D(\mathcal{I} - \mathcal{K}(1,\cdot,\cdot), \Omega, 0)$$

where "D" stands for the Leray – Schauder degree and $\mathcal{I}$ is the identity operator on $C^1(J) \times \mathbb{R}$. Since $D(\mathcal{I}-\mathcal{K}(0,\cdot,\cdot), \Omega, 0) \neq 0$ by the Borsuk antipodal theorem, we have $D(\mathcal{I}-\mathcal{K}(1,\cdot,\cdot), \Omega, 0) \neq \neq 0$, and consequently there exists a fixed point $(x_*, c_*) \in \Omega$ of the operator $\mathcal{K}(1,\cdot,\cdot)$. Then

$$x_*(t) = c_* + \int_0^t \int_0^s f_n(v, x_*(v), x_*'(v))\, dv\, ds, \quad t \in J,$$

and $\alpha(x_*) = 0$. It follows that x_* is a solution of BVP $(2.4)_{1,n}$, (1.2).

Lemma 2.4. *Let assumption* (H_1) *be satisfied and* $n \in \mathbb{N}$. *Let* x *be a solution of BVP* $(2.4)_{1,n}$, (1.2). *Then*

$$x'(t) \geq a t^{\gamma} \quad \text{for } t \in J \tag{2.9}$$

and

$$|x(t)| \geq \frac{a}{1+\gamma}|t-\xi|^{1+\gamma} \quad \text{for } t \in J \tag{2.10}$$

where $\xi \in [\varepsilon, T]$ *is a unique zero of* x *on* J.

Proof. From $x''(t) \geq \psi(t)$ for a.e. $t \in J$, $x'(0) = 0$ and (1.4) we deduce (2.9). Hence x is increasing on J and from $\alpha(x) = 0$ and Lemma 2.1 we see that x vanishes at a unique point $\xi \in [\varepsilon, T]$. Now from $x(\xi) = 0$ and (2.9) it may be concluded

$$|x(t)| \geq \begin{cases} \dfrac{a}{1+\gamma}(\xi^{1+\gamma} - t^{1+\gamma}) & \text{for } t \in [0,\xi] \text{ if } \xi > 0; \\ \dfrac{a}{1+\gamma}(t^{1+\gamma} - \xi^{1+\gamma}) & \text{for } t \in [\xi, T] \text{ if } \xi < T \end{cases}$$

and since $\xi^{1+\gamma} - t^{1+\gamma} \geq (\xi - t)^{1+\gamma}$ for $t \in [0,\xi]$ and $t^{1+\gamma} - \xi^{1+\gamma} \geq (t-\xi)^{1+\gamma}$ for $t \in [\xi, T]$, (2.10) is true.

Lemma 2.5. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exists a positive constant* A *such that*

$$\|x\| \leq AT, \quad \|x'\| \leq A, \tag{2.11}$$

for any solution x *of BVP* $(2.4)_{1,n}$, (1.2) *with* $n \in \mathbb{N}$.

Proof. Let x be a solution of BVP $(2.4)_{1,n}$, (1.2). We first show that

$$\int_0^T q_0(t)\omega_0(|x(t)|)\,dt \leq 2\|q_0\|_\infty \int_0^T \omega\left(\frac{a}{1+\gamma}t^{1+\gamma}\right)\,dt. \tag{2.12}$$

By Lemma 2.4, there exists a unique $\xi \in [\varepsilon, T]$ such that $x(\xi) = 0$ and (2.10) holds. Therefore

$$\begin{aligned}\int_0^T q_0(t)\omega_0(|x(t)|)\,dt &\le \|q_0\|_\infty\left[\int_0^\xi \omega_0\left(\frac{a}{1+\gamma}(\xi-t)^{1+\gamma}\right)dt + \right.\\ &\left.+\int_\xi^T \omega_0\left(\frac{a}{1+\gamma}(t-\xi)^{1+\gamma}\right)dt\right] =\\ &= \|q_0\|_\infty\left[\int_0^\xi \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right)dt + \int_0^{T-\xi}\omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right)dt\right] \le\\ &\le 2\|q_0\|_\infty\int_0^T \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right)dt.\end{aligned}$$

Using (2.3), (2.9), (2.12) and $|x(t)| \le \left|\int_\xi^t x'(s)\,ds\right| \le T\|x'\|$ for $t \in J$, we have

$$\begin{aligned}0 \le x'(t) &\le \int_0^T f_n(t, x(t), x'(t))\,dt \le\\ &\le \int_0^T \Big[\phi(t) + q_0(t)\omega_0(|x(t)|) + q_1(t)\omega_1(x'(t)) +\\ &+h_0(t)(1+T\|x'\|) + h_1(t)(1+\|x'\|)\Big]\,dt \le\\ &\le \|\phi\|_L + 2\|q_0\|_\infty\int_0^T \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right)dt + \|q_1\|_\infty\int_0^T \omega_1\left(at^\gamma\right)dt +\\ &+\|h_0\|_L(1+T\|x'\|) + \|h_1\|_L(1+\|x'\|)\end{aligned}$$

for $t \in J$, and so (2.11) holds with

$$A = \frac{\|\phi\|_L + 2\|q_0\|_\infty \int\limits_0^T \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt + \|q_1\|_\infty \int\limits_0^T \omega_1(at^\gamma)\, dt + \|h_0\|_L + \|h_1\|_L}{1 - T\|h_0\|_L - \|h_1\|_L}.$$

Lemma 2.6. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then for any at most countable set* $\{(a_j, b_j)\}_{j\in\mathbb{I}}$ *of mutually disjoint intervals* $(a_j, b_j) \subset J$ *and any solution* x *of BVP* $(2.4)_{1,n}$, (1.2) *with* $n \in \mathbb{N}$, *there exist measurable subsets* $\mathcal{M}_i$ *of* J, $\mu(\mathcal{M}_i) \le \sum_{j\in\mathbb{I}}(b_j - a_j)$, $i = 1, 2$, *such that*

$$\sum_{j\in\mathbb{I}} \int\limits_{a_j}^{b_j} q_0(t)\omega_0(|x(t)|)\, dt \le$$

$$\le \|q_0\|_\infty \left(\int\limits_{\mathcal{M}_1} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt + \int\limits_{\mathcal{M}_2} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt \right). \tag{2.13}$$

Proof. Let $\{(a_j, b_j)\}_{j\in\mathbb{I}}$ be at most countable set of mutually disjoint intervals $(a_j, b_j) \subset J$ and x be a solution of BVP $(2.4)_{1,n}$, (1.2). By Lemma 2.4, there exists a unique zero $\xi \in [\varepsilon, T]$ of x on J and (2.10) holds. Set

$$\mathbb{I}_1 = \{j : j \in \mathbb{I},\, (a_j, b_j) \subset (0, \xi)\}, \quad \mathbb{I}_2 = \{j : j \in \mathbb{I},\, (a_j, b_j) \subset (\xi, T)\}.$$

Then for $j \in \mathbb{I}_1$ and $i \in \mathbb{I}_2$ we have

$$\int\limits_{a_j}^{b_j} q_0(t)\omega_0(|x(t)|)\, dt \le \|q_0\|_\infty \int\limits_{a_j}^{b_j} \omega_0\left(\frac{a}{1+\gamma}(\xi - t)^{1+\gamma}\right) dt =$$

$$= \|q_0\|_\infty \int\limits_{\xi - b_j}^{\xi - a_j} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt, \tag{2.14}$$

$$\int\limits_{a_i}^{b_i} q_0(t)\omega_0(|x(t)|)\, dt \le \|q_0\|_\infty \int\limits_{a_i}^{b_i} \omega_0\left(\frac{a}{1+\gamma}(t - \xi)^{1+\gamma}\right) dt =$$

$$= \|q_0\|_\infty \int\limits_{a_i - \xi}^{b_i - \xi} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt. \tag{2.15}$$

If $\mathbb{I} \setminus (\mathbb{I}_1 \cup \mathbb{I}_2) = \{j_0\}$, that is $a_{j_0} < \xi < b_{j_0}$, then

$$\int_{a_{j_0}}^{b_{j_0}} q_0(t)\omega_0(|x(t)|)\,dt \le$$

$$\le \|q_0\|_\infty \left[\int_{a_{j_0}}^{\xi} \omega_0\left(\frac{a}{1+\gamma}(\xi - t)^{1+\gamma}\right) dt + \int_{\xi}^{b_{j_0}} \omega_0\left(\frac{a}{1+\gamma}(t-\xi)^{1+\gamma}\right) dt \right] =$$

$$= \|q_0\|_\infty \left[\int_0^{\xi - a_{j_0}} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt + \int_0^{b_{j_0}-\xi} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt \right]. \tag{2.16}$$

Hence

$$\sum_{j\in\mathbb{I}} \int_{a_j}^{b_j} q_0(t)\omega_0(|x(t)|)\,dt \le \|q_0\|_\infty \sum_{j\in\mathbb{I}} \int_{a_j}^{b_j} \omega_0(|x(t)|)\,dt =$$

$$= \|q_0\|_\infty \left[\sum_{j\in\mathbb{I}_1} \int_{\xi-b_j}^{\xi-a_j} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt + \sum_{j\in\mathbb{I}_2} \int_{a_j-\xi}^{b_j-\xi} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt + E \right]$$

where

$$E = \begin{cases} 0 & \text{if } \mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2; \\ \displaystyle\int_0^{\xi-a_{j_0}} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt + \int_0^{b_{j_0}-\xi} \omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right) dt & \text{if } \{j_0\} = \mathbb{I} \setminus (\mathbb{I}_1 \cup \mathbb{I}_2). \end{cases}$$

Set

$$\mathcal{M}_1 = \mathcal{M}_1^* \cup \bigcup_{j\in\mathbb{I}_1} (\xi - b_j, \xi - a_j), \quad \mathcal{M}_2 = \mathcal{M}_2^* \cup \bigcup_{j\in\mathbb{I}_2} (a_j - \xi, b_j - \xi)$$

where

$$\mathcal{M}_1^* = \begin{cases} \emptyset & \text{if } \mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2; \\ (0, \xi - a_{j_0}) & \text{if } \{j_0\} = \mathbb{I} \setminus (\mathbb{I}_1 \cup \mathbb{I}_2), \end{cases}$$

$$\mathcal{M}_2^* = \begin{cases} \emptyset & \text{if } \mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2; \\ (0, b_{j_0} - \xi) & \text{if } \{j_0\} = \mathbb{I} \setminus (\mathbb{I}_1 \cup \mathbb{I}_2). \end{cases}$$

Then $\mathcal{M}_k$ are measurable subsets of J, $\mu(\mathcal{M}_k) \le \sum_{j\in\mathbb{I}}(b_j - a_j)$, $k = 1, 2$, and (2.13) is true.

2.2. BVP (1.1), (1.3). Let assumptions (H_3) and (H_4) be satisfied. For each $n \in \mathbb{N}$, define $\hat{f}_n \in \mathrm{Car}\,(J \times \mathbb{R}^2)$ by the formula

$$\hat{f}_n(t,x,y) = \begin{cases} f(t,x,y) \quad \text{for } (t,x,y) \in J \times \left(\left(-\infty, -\frac{1}{n}\right) \bigcup \left(\frac{1}{n}, \infty\right)\right)^2; \\ \frac{n}{2}\left[f\left(t, \frac{1}{n}, y\right)\left(x + \frac{1}{n}\right) + f\left(t, -\frac{1}{n}, y\right)\left(\frac{1}{n} - x\right)\right] \\ \qquad \text{for } (t,x,y) \in J \times \left[-\frac{1}{n}, \frac{1}{n}\right] \times \left(\left(-\infty, -\frac{1}{n}\right) \bigcup \left(\frac{1}{n}, \infty\right)\right); \\ \frac{n}{2}\left[\hat{f}_n\left(t, x, \frac{1}{n}\right)(y + \frac{1}{n}\right) + \hat{f}_n\left(t, x, -\frac{1}{n}\right)\left(\frac{1}{n} - y\right)\right] \\ \qquad \text{for } (t,x,y) \in J \times \mathbb{R} \times \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases}$$

Then for a.e. $t \in J$ and each $(x,y) \in \mathbb{R}^2$ we have

$$a \le -\hat{f}_n(t,x,y) \tag{2.17}$$

and

$$-\hat{f}_n(t,x,y) \le \phi(t) + q_0(t)\omega_0\left(\max\left\{\frac{1}{n}, |x|\right\}\right) + \\ + q_1(t)\omega_1\left(\max\left\{\frac{1}{n}, |y|\right\}\right) + (1+|x|)h_0(t) + \\ + (1+|y|)h_1(t). \tag{2.18}$$

From (2.18) and the properties of ω_i, $i = 0, 1$, we see that

$$-\hat{f}_n(t,x,y) \le \phi(t) + q_0(t)\omega_0(|x|) + q_1(t)\omega_1(|y|) + \\ + (1+|x|)h_0(t) + (1+|y|)h_1(t) \tag{2.19}$$

for a.e. $t \in J$ and each $(x,y) \in \mathbb{R}_0^2$.

Consider the two-parameter family of the regular differential equations

$$x''(t) = \lambda \hat{f}_n(t, x(t), x'(t)) \tag*{$(2.20)_{\lambda,n}$}$$

depending on the parameters $\lambda \in [0,1]$ and $n \in \mathbb{N}$.

Lemma 2.7. *Let assumptions* (H_3) *and* (H_4) *be satisfied and let* $n \in \mathbb{N}$. *Then any solution* x *of BVPs* $(2.20)_{\lambda,n}$, (1.3) *with* $\lambda \in [0,1]$ *satisfies the inequalities*

$$\|x\| \leq LT, \quad \|x'\| \leq L \tag{2.21}$$

where $L > 0$ *is a constant independent of* λ *and* $\alpha \in \mathcal{A}_\varepsilon$.

Proof. Fix $\lambda \in [0,1]$ and let x be a solution of BVP $(2.20)_{\lambda,n}$, (1.3). By Lemma 2.1, $x(\xi) = 0$ for some $\xi \in [\varepsilon, T]$ and since $\varepsilon \in (0,T)$ and $x(0) = 0$, there exists $\tau \in (0,\xi)$ such that $x'(\tau) = 0$. Then for $t \in J$ we have

$$|x(t)| = \left|\int_0^t x'(s)\,ds\right| \leq T\|x'\| \tag{2.22}$$

and, by (2.18),

$$\begin{aligned}
|x'(t)| &= \lambda\left|\int_\tau^t \hat{f}_n(s, x(s), x'(s))\,ds\right| \leq \\
&\leq \int_0^T \Bigg[\phi(t) + q_0(t)\omega_0\left(\frac{1}{n}\right) + q_1(t)\omega_1\left(\frac{1}{n}\right) + \\
&\qquad + (1+|x(t)|)h_0(t) + (1+|x'(t)|)h_1(t)\Bigg]\,dt \leq \\
&\leq \|\phi\|_L + \omega_0\left(\frac{1}{n}\right)\|q_0\|_\infty + \omega\left(\frac{1}{n}\right)\|q_1\|_\infty + \\
&\qquad + (1+T\|x'\|)\|h_0\|_L + (1+\|x'\|)\|h_1\|_L.
\end{aligned}$$

Hence

$$\|x'\| \leq \frac{K}{1 - T\|h_0\|_L - \|h_1\|_L}$$

where

$$K = \|\phi\|_L + \omega_0\left(\frac{1}{n}\right)\|q_0\|_\infty + \omega\left(\frac{1}{n}\right)\|q_1\|_\infty + \|h_0\|_L + \|h_1\|_L$$

and then (2.22) yields

$$\|x\| \leq \frac{KT}{1 - T\|h_0\|_L - \|h_1\|_L}.$$

Consequently, (2.21) holds with $L = \dfrac{K}{1 - T\|h_0\|_L - \|h_1\|_L}$.

Lemma 2.8. *Let assumptions* (H_3) *and* (H_4) *be satisfied. Then for each* $n \in \mathbb{N}$*, BVP* $(2.20)_{1,n}$*, (1.3) has a solution.*

Proof. Fix $n \in \mathbb{N}$. Lemma 2.7 guarantees the existence of a positive constant L such that (2.21) holds for any solution x of BVP $(2.20)_{\lambda,n}$, (1.3) with $\lambda \in [0,1]$. Let

$$\Omega = \{(x,c) : (x,c) \in C^1(J) \times \mathbb{R},\ \|x\| < LT + 1,\ \|x'\| < L + 1,\ |c| < L + 1\}.$$

Then Ω is an open ball in the Banach space $C^1(J) \times \mathbb{R}$. Define the operator $\mathcal{F} : [0,1] \times \overline{\Omega} \to$ $\to C^1(J) \times \mathbb{R}$ by

$$\mathcal{F}(\lambda, x, c) = \left(ct + \lambda \int_0^t \int_0^s \hat{f}_n(v, x(v), x'(v))\, dv\, ds,\ c + \alpha(x) - (1-\lambda)\alpha(-x) \right).$$

Then $\mathcal{F}$ is a compact operator since $\hat{f}_n \in \operatorname{Car}(J \times \mathbb{R}^2)$ and α is a continuous and increasing functional on $C^0([\varepsilon, T])$. Besides, $\mathcal{F}(0,\cdot,\cdot)$ is an odd operator, that is $\mathcal{F}(0,-x,-c) = -\mathcal{F}(0,x,c)$ for $(x,c) \in \overline{\Omega}$, as it is easy to check.

Suppose that $\mathcal{F}(\lambda_0, x_0, c_0) = (x_0, c_0)$ for some $(\lambda_0, x_0, c_0) \in [0,1] \times \partial\Omega$. Then

$$x_0(t) = c_0 t + \lambda_0 \int_0^t \int_0^s \hat{f}_n(v, x_0(v), x_0'(v))\, dv\, ds, \quad t \in J, \tag{2.23}$$

and

$$\alpha(x_0) = (1-\lambda_0)\alpha(-x_0). \tag{2.24}$$

From (2.23) we conclude that x_0 is a solution of $(2.20)_{\lambda_0,n}$ and $x_0'(0) = c_0$. If $x_0(t) \neq 0$ for $t \in [\varepsilon, T]$, say $x_0 < 0$ on $[\varepsilon, T]$, then $\alpha(x_0) < 0$ and $\alpha(-x_0) > 0$, contrary to (2.24). Therefore $x_0(\xi) = 0$ for some $\xi \in [\varepsilon, T]$ and then Lemma 2.7 (with the functional α in the boundary condition (1.3) defined by $\alpha(x) = x(\xi)$ for $x \in C^0([\varepsilon, T])$) gives $\|x_0\| \leq LT$, $\|x_0'\| \leq L$. Hence $|c_0| = |x_0'(0)| \leq L$, so $(x_0, c_0) \notin \partial\Omega$ and we have proved that

$$\mathcal{F}(\lambda, x, c) \neq (x, c) \quad \text{for } \lambda \in [0,1],\ (x,c) \in \partial\Omega.$$

Consequently, by the Leray – Schauder degree theory,

$$D(\mathcal{I} - \mathcal{F}(0,\cdot,\cdot), \Omega, 0) = D(\mathcal{I} - \mathcal{F}(1,\cdot,\cdot), \Omega, 0)$$

where $\mathcal{I}$ is the identity operator on $C^1(J) \times \mathbb{R}$. According to the Borsuk antipodal theorem, $D(\mathcal{I} - \mathcal{F}(0,\cdot,\cdot), \Omega, 0) \neq 0$, and so $D(\mathcal{I} - \mathcal{F}(1,\cdot,\cdot), \Omega, 0) \neq 0$. Therefore there exists a fixed point $(\bar{x}, \bar{c}) \in \Omega$ of the operator $\mathcal{F}(1,\cdot,\cdot)$ and it is easy to check that $\bar{x}$ is a solution of BVP $(2.20)_{1,n}$, (1.3).

Lemma 2.9. *Let assumption* (H_3) *be satisfied. Let* $n \in \mathbb{N}$ *and* x *be a solution of* $(2.20)_{1,n}$, (1.3). *Then*

$a)$ *there exist a unique zero* $\xi \in [\varepsilon, T]$ *of* x *in* $(0, T]$ *and a unique zero* $\tau \in (0, \xi)$ *of* x' *in* J, *b*) *x satisfies the inequalities*

$$x(t) \geq \begin{cases} \dfrac{a\xi}{4}t & for\ \ t \in \left[0, \dfrac{\xi}{2}\right]; \\[2ex] \dfrac{a\xi}{4}(\xi - t) & for\ \ t \in \left(\dfrac{\xi}{2}, \xi\right], \end{cases} \tag{2.25}$$

$$x(t) \leq -\frac{a}{2}(\xi - \tau)(t - \xi) \quad for\ t \in (\xi, T]\ \ if\ \xi < T \tag{2.26}$$

and

$$x'(t) \geq a(\tau - t)\ \ for\ t \in [0, \tau], \quad x'(t) \leq -a(t - \tau)\ \ for\ t \in (\tau, T]. \tag{2.27}$$

Proof. First we see from (2.17) that $x''(t) \leq -a$ for a.e. $t \in J$. Next, by Lemma 2.1, $x(\xi) = 0$ for some $\xi \in [\varepsilon, T]$. Since $x(0) = 0$ and x' is decreasing on J, there is a unique $\tau \in (0, \xi)$ such that $x'(\tau) = 0$. Then

$$-x'(t) = \int\limits_t^\tau x''(s)\, ds \leq -a(\tau - t), \quad t \in [0, \tau],$$

and

$$x'(t) = \int\limits_\tau^t x''(s)\, ds \leq -a(t - \tau), \quad t \in (\tau, T],$$

which proves (2.27). In addition, ξ is a unique zero of x in $(0, T]$.

By (2.27), we have

$$x(t) = \int\limits_0^t x'(s)\, ds \geq a \int\limits_0^t (\tau - s)\, ds = \frac{a}{2}t(2\tau - t)$$

for $t \in [0, \tau]$ and

$$x(t) = -\int\limits_t^\xi x'(s)\, ds \geq a \int\limits_t^\xi (s - \tau)\, ds = \frac{a}{2}(\xi - t)(\xi - 2\tau + t)$$

for $t \in (\tau, \xi]$. Hence $x(\xi/2) \geq a\xi(4\tau - \xi)/8 \geq a\xi^2/8$ provided $\tau \geq \xi/2$ and $x(\xi/2) \geq a\xi(3\xi - 4\tau)/8 > a\xi^2/8$ provided $\tau < \xi/2$. Consequently, $x(\xi/2) \geq a\xi^2/8$ and since x is concave on J which follows from $x'' \leq -a < 0$ a.e. on J and $x(0) = x(\xi) = 0$, (2.25) is true.

Finally, assume that $\xi < T$. Then

$$x(t) = \int\limits_\xi^t x'(s)\, ds \leq -a \int\limits_\xi^t (s - \tau)\, ds =$$

$$= -\frac{a}{2}(\xi - 2\tau + t)(t - \xi) \leq -\frac{a}{2}(\xi - \tau)(t - \xi)$$

for $t \in (\xi, T]$ which proves (2.26).

Lemma 2.10. *Let assumptions* (H_3) *and* (H_4) *be satisfied. Then there exists a positive constant* A *such that*

$$\|x\| \le AT, \quad \|x'\| \le A \tag{2.28}$$

for any solution x *of BVP* $(2.20)_{1,n}$, (1.3) *with* $n \in \mathbb{N}$.

Proof. Let x be a solution of BVP $(2.20)_{1,n}$, (1.3). By Lemma 2.9, x satisfies inequalities (2.25) – (2.27) where $\xi \in [\varepsilon, T]$ is the unique zero of x in $(0, T]$ and $\tau \in (0, \xi)$ is the unique zero of x' in J. We now prove the existence of a positive constant B independent of $n \in \mathbb{N}$ such that

$$|x(t)| \le BT, \ \ |x'(t)| \le B \quad \text{for } t \in [0, \xi]. \tag{2.29}$$

From (H_4) and (2.25) – (2.29) we obtain

$$\int_0^{\xi} q_0(t)\omega_0(|x(t)|)\,dt \le \|q_0\|_\infty \left[\int_0^{\xi/2} \omega_0\left(\frac{a\xi}{4}t\right)\,dt + \int_{\xi/2}^{\xi} \omega_0\left(\frac{a\xi}{4}(\xi - t)\right)\,dt\right] \le$$

$$\le 2\Lambda\|q_0\|_\infty \omega_0\left(\frac{a\xi}{4}\right)\int_0^{\xi/2} \omega_0(t)\,dt \le$$

$$\le 2\Lambda\|q_0\|_\infty \omega_0\left(\frac{a\varepsilon}{4}\right)\int_0^{T} \omega_0(t)\,dt \tag{2.30}$$

and

$$\int_0^{\xi} q_1(t)\omega_1(|x'(t)|)\,dt \le \|q_1\|_\infty \left[\int_0^{\tau} \omega_1(a(\tau - t))\,dt + \int_{\tau}^{\xi} \omega_1(a(t-\tau))\,dt\right] \le$$

$$\le \Lambda\|q_1\|_\infty \omega_1(a)\left[\int_0^{\tau} \omega_1(t)\,dt + \int_0^{\xi-\tau} \omega_1(t)\,dt\right] \le$$

$$\le 2\Lambda\|q_1\|_\infty \omega_1(a)\int_0^{T} \omega_1(t)\,dt. \tag{2.31}$$

Using (2.19), (2.30), (2.31) and $|x(t)| = \left|\int\limits_0^t x'(s)\,ds\right|$, we get (for $t \in [0,\xi]$)

$$|x'(t)| = \left|\int\limits_\tau^t \hat{f}_n(s,x(s),x'(s))\,ds\right| \le \int\limits_0^\xi \Big[\phi(t) + q_0(t)\omega_0(|x(t)|) + q_1(t)\omega_0(|x'(t)|) +$$

$$+ (1+|x(t)|)h_0(t) + (1+|x'(t)|)h_1(t)\Big]\,dt \le$$

$$\le \|\phi\|_L + 2\Lambda\|q_0\|_\infty\omega_0\left(\frac{a\varepsilon}{4}\right)\int\limits_0^T \omega_0(t)\,dt + 2\Lambda\|q_1\|_\infty\omega_1(a)\int\limits_0^T \omega_1(t)\,dt +$$

$$+ (1 + T\max\{|x'(t)| : 0 \le t \le \xi\})\|h_0\|_L +$$

$$+ (1 + \max\{|x'(t)| : 0 \le t \le \xi\})\|h_1\|_L.$$

Hence

$$\max\{|x'(t)| : 0 \le t \le \xi\} \le \frac{\Psi}{1 - T\|h_0\|_L - \|h_1\|_L}$$

where

$$\Psi = \|\phi\|_L + 2\Lambda\|q_0\|_\infty\omega_0\left(\frac{a\varepsilon}{4}\right)\int\limits_0^T \omega_0(t)\,dt + 2\Lambda\|q_1\|_\infty\omega_1(a)\int\limits_0^T \omega_1(t)\,dt + \|h_0\|_L + \|h_1\|_L.$$

Thus (2.29) is true with

$$B = \frac{\Psi}{1 - T\|h_0\|_L - \|h_1\|_L}.$$

If $\xi = T$ then (2.28) holds with $A = B$.

Suppose that $\xi < T$. We claim that

$$\xi - \tau \ge \frac{a\varepsilon^2}{8B}, \tag{2.32}$$

so the difference $\xi - \tau$ is independent of $n \in \mathbb{N}$. Indeed, from (2.25) it follows that $x(\xi/2) \ge a\xi^2/8 \ge a\varepsilon^2/8$ and then $\max\{x(t) : 0 \le t \le \xi\} = x(\tau) \ge a\varepsilon^2/8$. Since $x(\tau) = x(\tau) - x(\xi) = -x'(\eta)(\xi - \tau) \le B(\xi - \tau)$ where $\eta \in (\tau, \xi)$, we have $a\varepsilon^2/8 \le B(\xi - \tau)$ which proves (2.32). Now from (2.26) and (2.32) we get

$$x(t) \le -\frac{a^2\varepsilon^2}{16B}(t - \xi) \quad \text{for } t \in (\xi, T], \tag{2.33}$$

and so (for $t \in [\xi, T]$)

$$|x'(t)| \;= \left| x'(\xi) + \int\limits_{\xi}^{t} \hat{f}_n(s, x(s), x'(s))\, ds \right| \le$$

$$\le B + \int\limits_{\xi}^{t} \Big[\phi(t) + q_0(s)\omega_0(|x(s)|) + q_1(s)\omega_0(|x'(s)|) +$$

$$+ \; (1 + |x(s)|)h_0(s) + (1 + |x'(s)|)h_1(s) \Big]\, ds \le$$

$$\le B + \|\phi\|_L + \Lambda \|q_0\|_\infty \omega_0\left(\frac{a^2\varepsilon^2}{16B}\right) \int\limits_{\xi}^{t} \omega_0(s - \xi)\, ds +$$

$$+ \; \Lambda \|q_1\|_\infty \omega_1(a) \int\limits_{\xi}^{t} \omega_1(s - \tau)\, ds +$$

$$+ \; (1 + T \max\{|x'(t)| : \xi \le t \le T\}) \|h_0\|_L +$$

$$+ \; (1 + \max\{|x'(t)| : \xi \le t \le T\}) \|h_1\|_L.$$

Hence

$$\max\{|x'(t)| : \xi \le t \le T\} \le \frac{\Phi}{1 - T\|h_0\|_L - \|h_1\|_L}$$

and

$$\max\{|x(t)| : \xi \le t \le T\} \le \frac{\Phi T}{1 - T\|h_0\|_L - \|h_1\|_L}$$

where

$$\Phi \,=\, B + \|\phi\|_L + \Lambda \|q_0\|_\infty \omega_0\left(\frac{a^2\varepsilon^2}{16B}\right) \int\limits_{0}^{T} \omega_0(t)\, dt + \Lambda \|q_1\|_\infty \omega_1(a) \int\limits_{0}^{T} \omega_1(t)\, dt + \|h_0\|_L + \|h_1\|_L.$$

Since $B \le \Phi$, (2.28) holds with $A = \Phi/(1 - T\|h_0\|_L - \|h_1\|_L)$.

Lemma 2.11. *Let assumptions (H_3) and (H_4) be satisfied. Then for any at most countable set $\{(a_j, b_j)\}_{j \in \mathbb{J}}$ of mutually disjoint intervals $(a_j, b_j) \subset J$ and any solution x of BVP $(2.20)_{1,n}$, (1.3) with $n \in \mathbb{N}$, there exist positive constants S_0, S_1 independent of x and measurable subsets $\mathcal{M}_k$,*

$\mathcal{N}_i$ of J, $\mu(\mathcal{M}_k) \le \sum_{j\in\mathbb{J}}(b_j - a_j)$, $\mu(\mathcal{N}_i) \le \sum_{j\in\mathbb{J}}(b_j - a_j)$, $k = 1,2,3$, $i = 1,2$, such that

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} q_0(t)\omega_0(|x(t)|)\,dt \le S_0 \sum_{k=1}^{3} \int_{\mathcal{M}_k} \omega_0(t)\,dt \tag{2.34}$$

and

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} q_1(t)\omega_1(|x'(t)|)\,dt \le S_1 \sum_{i=1}^{2} \int_{\mathcal{N}_i} \omega_1(t)\,dt. \tag{2.35}$$

Proof. Let x be a solution of BVP $(2.20)_{1,n}$, (1.3) and $\{(a_j, b_j)\}_{j\in\mathbb{J}}$ be at most countable set of mutually disjoint intervals $(a_j, b_j) \subset J$. By Lemma 2.9, there exists the unique zero $\xi \in [\varepsilon, T]$ of x in $(0, T]$ and the unique zero $\tau \in (0, \xi)$ of x' in J and x satisfies the inequalities (2.25) – (2.27). Besides, it follows from the proof of Lemma 2.10 that

$$\xi - \tau \ge C \tag{2.36}$$

(see (2.32) where $C = a\varepsilon^2/(8B)$ is a positive constant independent of x and $n \in \mathbb{N}$. Set

$$\mathbb{J}_1 = \left\{ j : j \in \mathbb{J},\, (a_j, b_j) \subset \left(0, \frac{\xi}{2}\right) \right\}, \quad \mathbb{J}_2 = \left\{ j : j \in \mathbb{J},\, (a_j, b_j) \subset \left(\frac{\xi}{2}, \xi\right) \right\},$$

$$\mathbb{J}_3 = \begin{cases} \{j : j \in \mathbb{J},\, (a_j, b_j) \subset (\xi, T)\} & \text{if } \xi < T; \\ \emptyset & \text{if } \xi = T \end{cases}$$

and

$$\mathbb{I}_1 = \{j : j \in \mathbb{J},\, (a_j, b_j) \subset (0, \tau)\}, \quad \mathbb{I}_2 = \{j : j \in \mathbb{J},\, (a_j, b_j) \subset (\tau, T)\}.$$

Then for $j \in \mathbb{J}_1$, $i \in \mathbb{J}_2$ and $k \in \mathbb{J}_3$ we have

$$\int_{a_j}^{b_j} q_0(t)\omega_0(|x(t)|)\,dt \le \|q_0\|_\infty \int_{a_j}^{b_j} \omega_0\left(\frac{a\xi}{4}t\right)\,dt \le$$

$$\le \Lambda\|q_0\|_\infty \omega_0\left(\frac{a\xi}{4}\right) \int_{a_j}^{b_j} \omega_0(t)\,dt \le \Lambda\|q_0\|_\infty \omega_0\left(\frac{a\varepsilon}{4}\right) \int_{a_j}^{b_j} \omega_0(t)\,dt, \tag{2.37}$$

$$\int_{a_i}^{b_i} q_0(t)\omega_0(|x(t)|)\,dt \le \|q_0\|_\infty \int_{a_i}^{b_i} \omega_0\left(\frac{a\xi}{4}(\xi - t)\right)\,dt \le$$

$$\le \Lambda\|q_0\|_\infty \omega_0\left(\frac{a\xi}{4}\right) \int_{a_i}^{b_i} \omega_0(\xi - t)\,dt \le \Lambda\|q_0\|_\infty \omega_0\left(\frac{a\varepsilon}{4}\right) \int_{\xi - b_i}^{\xi - a_i} \omega_0(t)\,dt \tag{2.38}$$

and

$$\int_{a_k}^{b_k} q_0(t)\omega_0(|x(t)|)\,dt \le \|q_0\|_\infty \int_{a_k}^{b_k} \omega_0\left(\frac{a}{2}(\xi-\tau)(t-\xi)\right)\,dt \le$$

$$\le \|q_0\|_\infty \int_{a_k}^{b_k} \omega_0\left(\frac{aC}{2}(t-\xi)\right)\,dt \le \Lambda\|q_0\|_\infty \omega_0\left(\frac{aC}{2}\right)\int_{a_k-\xi}^{b_k-\xi} \omega_0(t)\,dt. \quad (2.39)$$

Suppose that $\mathbb{J}\setminus(\mathbb{J}_1\cup\mathbb{J}_2\cup\mathbb{J}_3)=\{j_0\}$, that is $a_{j_0}<\xi<b_{j_0}$. If $a_{j_0}<\xi/2$, we have

$$\int_{a_{j_0}}^{b_{j_0}} q_0(t)\omega_0(|x(t)|)\,dt \le \|q_0\|_\infty \left[\int_{a_{j_0}}^{\xi/2} \omega_0(|x(t)|)\,dt + \int_{\xi/2}^{\xi} \omega_0(|x(t)|)\,dt + \int_{\xi}^{b_{j_0}} \omega_0(|x(t)|)\,dt\right] \le$$

$$\le \Lambda\|q_0\|_\infty \left[\omega_0\left(\frac{a\varepsilon}{4}\right)\int_{a_{j_0}}^{\xi/2} \omega_0(t)\,dt + \omega_0\left(\frac{a\varepsilon}{4}\right)\int_{0}^{\xi/2} \omega_0(t)\,dt + \right.$$

$$\left. + \ \omega_0\left(\frac{aC}{2}\right)\int_{0}^{b_{j_0}-\xi} \omega_0(t)\,dt\right] \quad (2.40)$$

and if $a_{j_0}\ge\xi/2$, we have

$$\int_{a_{j_0}}^{b_{j_0}} q_0(t)\omega_0(|x(t)|)\,dt \le \|q_0\|_\infty \left[\int_{a_{j_0}}^{\xi} \omega_0(|x(t)|)\,dt + \int_{\xi}^{b_{j_0}} \omega_0(|x(t)|)\,dt\right] \le$$

$$\le \Lambda\|q_0\|_\infty \left[\omega_0\left(\frac{a\varepsilon}{4}\right)\int_{0}^{\xi-a_{j_0}} \omega_0(t)\,dt + +\omega_0\left(\frac{aC}{2}\right)\int_{0}^{b_{j_0}-\xi} \omega_0(t)\,dt\right]. \quad (2.41)$$

Let

$$S_0 = \Lambda\|q_0\|_\infty \max\left\{\omega_0\left(\frac{a\varepsilon}{4}\right),\, \omega_0\left(\frac{aC}{2}\right)\right\}$$

and

$$\mathcal{M}_1 = \mathcal{E}_1 \cup \bigcup_{j\in\mathbb{J}_1}(a_j,b_j), \quad \mathcal{M}_2 = \mathcal{E}_2 \cup \bigcup_{j\in\mathbb{J}_2}(\xi-b_j,\xi-a_j),$$

$$\mathcal{M}_3 = \mathcal{E}_3 \cup \bigcup_{j\in\mathbb{J}_3}(a_j-\xi,b_j-\xi)$$

where $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \emptyset$ provided $\mathbb{J} = \mathbb{J}_1 \cup \mathbb{J}_2 \cup \mathbb{J}_3$,

$$\mathcal{E}_1 = \left(a_{j_0}, \frac{\xi}{2}\right), \quad \mathcal{E}_2 = \left(0, \frac{\xi}{2}\right), \quad \mathcal{E}_3 = (0, b_{j_0} - \xi)$$

provided $\{j_0\} = \mathbb{J} \setminus (\mathbb{J}_1 \cup \mathbb{J}_2 \cup \mathbb{J}_3)$ and $a_{j_0} < \xi/2 < \xi < b_{j_0}$,

$$\mathcal{E}_1 = \emptyset, \quad \mathcal{E}_2 = (0, \xi - a_{j_0}), \quad \mathcal{E}_3 = (0, b_{j_0} - \xi)$$

provided $\{j_0\} = \mathbb{J} \setminus (\mathbb{J}_1 \cup \mathbb{J}_2 \cup \mathbb{J}_3)$ and $\xi/2 \le a_{j_0} < \xi < b_{j_0}$. Then $\mu(\mathcal{M}_k) \le \sum_{j \in \mathbb{J}} (b_j - a_j)$ for $k = 1, 2, 3$ and from (2.37)–(2.41) we obtain (2.34). Finally, for $j \in \mathbb{I}_1$ and $i \in \mathbb{I}_2$ we have

$$\int_{a_j}^{b_j} q_1(t)\omega_1(|x'(t)|)\,dt \le \|q_1\|_\infty \int_{a_j}^{b_j} \omega_0(a(\tau - t))\,dt \le$$

$$\le \Lambda \|q_1\|_\infty \omega_1(a) \int_{a_j}^{b_j} \omega_0(\tau - t)\,dt = \Lambda \|q_1\|_\infty \omega_0(a) \int_{\tau - b_j}^{\tau - a_j} \omega_0(t)\,dt, \qquad (2.42)$$

and

$$\int_{a_i}^{b_i} q_1(t)\omega_1(|x'(t)|)\,dt \le \|q_1\|_\infty \int_{a_i}^{b_i} \omega_1(a(t - \tau))\,dt \le$$

$$\le \Lambda \|q_1\|_\infty \omega_1(a) \int_{a_i - \tau}^{b_i - \tau} \omega_1(t)\,dt. \qquad (2.43)$$

If $a_{i_0} < \tau < b_{i_0}$ for some $i_0 \in \mathbb{J}$, that is $\mathbb{J} \setminus (\mathbb{I}_1 \cup \mathbb{I}_2) = \{i_0\}$, then

$$\int_{a_{i_0}}^{b_{i_0}} q_1(t)\omega_1(|x'(t)|)\,dt \le \|q_1\|_\infty \left[\int_{a_{i_0}}^{\tau} \omega_1(a(\tau - t))\,dt + \int_{\tau}^{b_{i_0}} \omega_1(a(t - \tau))\,dt \right] \le$$

$$\le \Lambda \|q_1\|_\infty \omega_1(a) \left[\int_0^{\tau - a_{i_0}} \omega_0(t)\,dt + \int_0^{b_{i_0} - \tau} \omega_1(t)\,dt \right]. \qquad (2.44)$$

Set

$$S_1 = \Lambda \|q_1\|_\infty \omega_1(a),$$

$$\mathcal{N}_1 = \mathcal{B}_1 \cup \bigcup_{j \in \mathbb{I}_1} (\tau - b_j, \tau - a_j), \quad \mathcal{N}_2 = \mathcal{B}_2 \cup \bigcup_{j \in \mathbb{I}_2} (a_j - \tau, b_j - \tau)$$

where $\mathcal{B}_1 = (0, \tau - a_{i_0}), \mathcal{B}_2 = (0, b_{i_0} - \tau)$ provided $a_{i_0} < \tau < b_{i_0}$ and $\mathcal{B}_1 = \mathcal{B}_2 = \emptyset$ if $\mathbb{J} = \mathbb{I}_1 \cup \mathbb{I}_2$. Then $\mu(\mathcal{N}_k) \leq \sum_{j \in \mathbb{J}}(b_j - a_j)$ for $k = 1, 2$ and (2.35) holds which follows from (2.42) – (2.44).

3. Existence results. ***3.1. BVP (1.1, (1.2).***

Theorem 3.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exists a solution of BVP* (1.1), (1.2) *for each* $\alpha \in \mathcal{A}_\varepsilon$ *with* $\varepsilon \in [0, T)$.

Proof. Fix $\varepsilon \in [0, T)$ and $\alpha \in \mathcal{A}_\varepsilon$. By Lemma 2.3, for each $n \in \mathbb{N}$, there exists a solution x_n of BVP $(2.4)_{1,n}$, (1.2). Consider the sequence $\{x_n(t)\}$. By Lemmas 2.1 and 2.4,

$$x'_n(t) \geq at^\gamma \quad \text{for } t \in J,\ n \in \mathbb{N}, \tag{3.1}$$

x_n has a unique zero $\xi_n \in [\varepsilon, T]$ in J and Lemma 2.6 guarantees the existence of a positive constant A such that

$$\|x_n\| \leq AT, \ \ \|x'_n\| \leq A \quad \text{for } n \in \mathbb{N}. \tag{3.2}$$

In addition, for $0 \leq t_1 < t_2 \leq T$ we deduce from the proof of Lemma 2.6 (see (2.14) – (2.16))

$$\int\limits_{t_1}^{t_2} q_0(t)\omega_0(|x'_n(t)|)\, dt \leq$$

$$\leq \begin{cases} \|q_0\|_\infty \displaystyle\int\limits_{\xi_n - t_2}^{\xi_n - t_1} \omega_0\left(\frac{a}{1+\gamma} t^{1+\gamma}\right) dt & \text{if } t_2 \leq \xi_n; \\ \|q_0\|_\infty \displaystyle\int\limits_{t_1 - \xi_n}^{t_2 - \xi_n} \omega_0\left(\frac{a}{1+\gamma} t^{1+\gamma}\right) dt & \text{if } t_1 \leq \xi_n; \\ \|q_0\|_\infty \left[\displaystyle\int\limits_0^{\xi_n - t_1} \omega_0\left(\frac{a}{1+\gamma} t^{1+\gamma}\right) dt + \int\limits_0^{t_2 - \xi_n} \omega_0\left(\frac{a}{1+\gamma} t^{1+\gamma}\right) dt\right] & \text{if } t_1 < \xi_n < t_2, \end{cases}$$

and so

$$\int\limits_{t_1}^{t_2} q_0(t)\omega_0(|x'_n(t)|)\, dt \leq 2\|q_0\|_\infty \int\limits_{\tau_{n,1}}^{\tau_{n,2}} \omega_0\left(\frac{a}{1+\gamma} t^{1+\gamma}\right) dt, \quad n \in \mathbb{N}, \tag{3.3}$$

where $0 \le \tau_{n,1} < \tau_{n,2} \le T$, $\tau_{n,2} - \tau_{n,1} \le t_2 - t_1$. Using (2.3) and (3.1) – (3.3) we have

$$0 \le x_n'(t_2) - x_n'(t_1) = \int_{t_1}^{t_2} f_n(t, x_n(t), x_n'(t))\, dt \le$$

$$\le \int_{t_1}^{t_2} [\phi(t) + q_0(t)\omega_0(|x_n(t)|) + q_1(t)\omega_1(at^\gamma) +$$

$$+ \ (1 + AT)h_0(t) + (1 + A)h_1(t)]\, dt \le$$

$$\le \int_{t_1}^{t_2} \phi(t)\, dt + 2\|q_0\|_\infty \int_{\tau_{n,1}}^{\tau_{n,2}} \omega_0\left(\frac{a}{1+\gamma} t^{1+\gamma}\right) dt + \|q_1\|_\infty \int_{t_1}^{t_2} \omega_1(at^\gamma)\, dt +$$

$$+ \ (1 + AT) \int_{t_1}^{t_2} h_0(t)\, dt + (1 + A) \int_{t_1}^{t_2} h_1(t)\, dt \tag{3.4}$$

for $0 \le t_1 < t_2 \le T$ where $0 \le \tau_{n,1} < \tau_{n,2} \le T$, $\tau_{n,2} - \tau_{n,1} \le t_2 - t_1$. From this and using the properties of the functions ϕ, ω_j, h_j, $j = 1, 2$, given in (H_2), we conclude that $\{x_n'(t)\}$ is equicontinuous on J and then (3.2) and the Arzelà – Ascoli theorem guarantee the existence of a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ converging in $C^1(J)$, say $\lim_{n\to\infty} x_{k_n} = x$. Hence $x \in C^1(J)$, $x'(0) = 0$, $x'(t) \ge at^\gamma$, $0 = \lim_{n\to\infty} \alpha(x_{k_n}) = \alpha(x)$ and $x(\xi) = 0$ for a unique $\xi \in [\varepsilon, T]$. Now from the definition of the functions $f_n \in \mathrm{Car}\,(J \times \mathbb{R}^2)$ it follows that there exists $\mathcal{V} \subset J$, $\mu(\mathcal{V}) = 0$, such that $f_n(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$ for each $t \in J \setminus \mathcal{V}$ and $n \in \mathbb{N}$, and so

$$\lim_{n\to\infty} f_{k_n}(t, x_{k_n}(t), x_{k_n}'(t)) = f(t, x(t), x'(t), \quad t \in J \setminus (\mathcal{V} \cup \{0, \xi\}).$$

Let $\{(a_j, b_j)\}_{j \in \mathbb{I}}$ be at most countable set of mutually disjoint intervals $(a_j, b_j) \subset J$. Then, by

(2.3), (3.1), (3.2) and Lemma 2.6,

$$\sum_{j\in\mathbb{I}}\int\limits_{a_j}^{b_j} f_{k_n}(t,x_{k_n}(t),x'_{k_n}(t))\,dt \le \sum_{j\in\mathbb{I}}\Bigg[\int\limits_{a_j}^{b_j}\phi(t)\,dt+\|q_1\|_\infty\int\limits_{a_j}^{b_j}\omega_1(at^\gamma)\,dt+$$

$$+\ (1+AT)\int\limits_{a_j}^{b_j}h_0(t)\,dt+(1+A)\int\limits_{a_j}^{b_j}h_1(t)\,dt\Bigg]+$$

$$+\ \|q_0\|_\infty\left(\int\limits_{\mathcal{M}^1_{k_n}}\omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right)dt+\int\limits_{\mathcal{M}^2_{k_n}}\omega_0\left(\frac{a}{1+\gamma}t^{1+\gamma}\right)dt\right) \tag{3.5}$$

for $n\in\mathbb{N}$, where $\mathcal{M}^i_{k_n}$ are measurable subsets of J, $\mu(\mathcal{M}^i_{k_n})\le\sum_{j\in\mathbb{I}}(b_j-a_j)$, $i=1,2$. From (3.5) and the properties of the functions ϕ, ω_i, h_i, $i=0,1$, given in (H_2) it follows that $\{f_{k_n}(t,x_{k_n}(t),x'_{k_n}(t))\}$ is uniformly absolutely continuous on J, that is for every $\kappa>0$ there exists $\delta>0$ such that

$$\int\limits_{\mathcal{N}} f_{k_n}(t,x_{k_n}(t),x'_{k_n}(t))\,dt<\kappa$$

for any measurable $\mathcal{N}\subset J$, $\mu(\mathcal{N})<\delta$. Then using the Vitali's convergence theorem we have $f(t,x(t),x'(t))\in L_1(J)$ and

$$\lim_{n\to\infty}\int\limits_0^t f_{k_n}(s,x_{k_n}(s),x'_{k_n}(s))\,ds=\int\limits_0^t f(s,x(s),x'(s))\,ds,\quad t\in J.$$

Taking the limit as $n\to\infty$ in the equalities

$$x'_{k_n}(t)=\int\limits_0^t f_{k_n}(s,x_{k_n}(s),x'_{k_n}(s))\,ds,\quad t\in J,\ n\in\mathbb{N},$$

we get

$$x'(t)=\int\limits_0^t f(s,x(s),x'(s))\,ds,\quad t\in J.$$

Consequently, $x\in AC^1(J)$ and x is a solution of BVP (1.1), (1.2).

Remark 3.1. If the function f in (1.1) is continuous on $J_1\times\mathbb{R}_0\times\mathbb{R}_+$ with $J_1=J\backslash\{t_1,\dots,t_n\}$ and satisfies assumptions (H_1) and (H_2), then from our above considerations it follows that any solution x of BVP (1.1), (1.2) (whose existence is guaranteed by Theorem 3.1) is increasing on

J, x vanishes on J at a unique point $\xi \in [\varepsilon, T]$ depending on $\alpha \in \mathcal{A}_\varepsilon$ in the boundary condition (1.2), $x \in AC^1(J) \cap C^2(J \setminus \{0, \xi, t_1, \ldots, t_n\})$ and satisfies (1.1) for $t \in J \setminus \{0, \xi, t_1, \ldots, t_n\}$.

Example 3.1. Consider the BVP

$$x''(t) = \frac{1}{(\min\{1, |x(t)|\})^b} + \frac{1}{(x'(t))^c} + A\frac{|x(t)|}{\sqrt{t(T-t)}} + B\frac{x'(t)}{\sqrt{|2t-T|}}, \tag{3.6}$$

$$\int_0^T r(t)x^3(t)\,dt = 0, \ \ x'(t) = 0 \tag{3.7}$$

where $b \in (0, 1/2)$, $c \in (0,1)$, $A, B \in [0, \infty)$, $A\pi T + 2B\sqrt{T} < 1$ and $r \in L_1(J)$, $r > 0$ a.e. on J. BVP (3.6), (3.7) satisfies assumptions (H_1) and (H_2) with $\psi(t) = 1$, $\phi(t) = 0$, $a = 1$, $\gamma = 1$, $\omega_0(u) = 1/(\min\{1, u\})^b$, $\omega_1(u) = 1/u^c$, $q_0(t) = q_1(t) = 1$, $h_0(t) = A/\sqrt{t(T-t)}$ and $h_1(t) = B/\sqrt{|2t-T|}$. Since the functional α defined on $C^0(J)$ by $\alpha(x) = \int_0^T r(t)x^3(t)\,dt$ belongs to $\mathcal{A}_0$, there exists a solution x of BVP (3.6), (3.7) by Theorem 3.1. From Remark 3.1 it follows that $x \in AC^1(J) \cap C^2(J \setminus \{0, \xi, T/2, T\})$ where $\xi \in J$ is the unique zero of x.

3.2. BVP (1.1), (1.3).

Theorem 3.2. *Let assumptions (H_3) and (H_4) be satisfied. Then for each $\alpha \in \mathcal{A}_\varepsilon$ with $\varepsilon \in (0, T)$ there exists a solution of BVP (1.1), (1.3).*

Proof. Fix $\varepsilon \in (0, T)$ and $\alpha \in \mathcal{A}_\varepsilon$. By Lemma 2.9, for each $n \in \mathbb{N}$ there exists a solution x_n of BVP $(2.20)_{1,n}$, (1.3) satisfying the inequalities

$$x_n(t) \geq \begin{cases} \dfrac{a\xi_n}{4}t & \text{for } t \in \left[0, \dfrac{\xi_n}{2}\right]; \\ \dfrac{a\xi_n}{4}(\xi_n - t) & \text{for } t \in \left(\dfrac{\xi_n}{2}, \xi_n\right], \end{cases} \tag{3.8}$$

$$x_n(t) \leq -\frac{a}{2}(\xi_n - \tau_n)(t - \xi_n) \quad \text{for } t \in (\xi_n, T] \text{ if } \xi_n < T \tag{3.9}$$

and

$$x_n'(t) \geq a(\tau_n - t) \ \text{ for } t \in [0, \tau_n], \quad x_n'(t) \leq -a(t - \tau_n) \ \text{ for } t \in (\tau_n, T] \tag{3.10}$$

where $\xi_n \in [\varepsilon, T]$ and $\tau_n \in (0, \xi_n)$ is the unique zero of x_n in $(0, T)$ and x_n' in J, respectively. In addition, from the proof of Lemma 2.10 it follows that (see (2.32))

$$\xi_n - \tau_n \geq C \quad \text{for } n \in \mathbb{N} \tag{3.11}$$

where C is a positive constant. Next, by Lemma 2.10, there is a positive constant A such that

$$\|x_n\| \leq AT, \ \ \|x_n'\| \leq A \quad \text{for } n \in \mathbb{N}. \tag{3.12}$$

Consider the sequence $\{x_n(t)\}$. We are going to show that $\{\hat{f}_n(t, x_n(t), x_n'(t))\}$ is uniformly absolutely continuous on J. To prove this results let $\{(a_j, b_j)\}_{j\in\mathbb{J}}$ be a set of at most countable mutually disjoint intervals $(a_j, b_j) \subset J$. By (2.19),

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} |\hat{f}_n(t, x_n(t), x_n'(t))|\, dt \le$$

$$\le \sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} \Big[\phi(t) + q_0(t)\omega_0(|x_n(t)|) + q_1(t)\omega_0(|x_n'(t)|) +$$

$$+ (1 + |x_n(t)|)h_0(t) + (1 + |x_n'(t)|)h_1(t)\Big]\, dt. \tag{3.13}$$

Now Lemma 2.11 guarantees the existence of positive constants S_0, S_1 and measurable subsets $\mathcal{M}_{n,k}$, $\mathcal{N}_{n,i}$ of J, $\mu(\mathcal{M}_{n,k}) \le \sum_{j\in\mathbb{J}}(b_j - a_j)$, $\mu(\mathcal{N}_{n,i}) \le \sum_{j\in\mathbb{J}}(b_j - a_j)$, $k = 1, 2, 3$, $i = 1, 2$ and $n \in \mathbb{N}$ such that

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} q_0(t)\omega_0(|x_n(t)|)\, dt \le S_0 \sum_{k=1}^{3} \int_{\mathcal{M}_{n,k}} \omega_0(t)\, dt,$$

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} q_1(t)\omega_1(|x_n'(t)|)\, dt \le S_1 \sum_{i=1}^{2} \int_{\mathcal{N}_{n,i}} \omega_1(t)\, dt.$$

Since ϕ, ω_i, $h_i \in L_1(J)$, $i = 1, 2$, we see from (3.13) and the last two inequalities that for each $\kappa > 0$ there exists $\delta > 0$ such that

$$\sum_{j\in\mathbb{J}_1} \int_{a_j}^{b_j} |\hat{f}_n(t, x_n(t), x_n'(t))|\, dt < \kappa$$

for any at most countable set $\{(a_j, b_j)\}_{j\in\mathbb{J}}$ of mutually disjoint intervals $(a_j, b_j) \subset J$ such that $\sum_{j\in\mathbb{J}_1}(b_j - a_j) < \delta$. Consequently, for each $\kappa > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} |\hat{f}_n(t, x_n(t), x_n'(t))|\, dt < \kappa, \quad n \in \mathbb{N},$$

whenever $\mathcal{M} \subset J$ is a measurable set and $\mu(\mathcal{M}) \le \delta$, and so $\{\hat{f}_n(t, x_n(t), x_n'(t))\}$ is uniformly absolutely continuous on J. From this fact and from the inequalities

$$|x_n'(t_2) - x_n'(t_1)| = \int_{t_1}^{t_2} |\hat{f}_n(t, x_n(t), x_n'(t))|\, dt, \quad 0 \le t_1 < t_2 \le T,\ n \in \mathbb{N},$$

it also follows that $\{x_n'(t)\}$ is equicontinuous on J and since $\{x_n\}$ is bounded in $C^1(J)$ by (3.12), there exists a subsequence $\{x_{k_n}\}$ converging in $C^1(J)$, say $\lim_{n\to\infty} x_{k_n} = x$. Then $x \in C^1(J)$, $x(0) = 0$,

$$\alpha(x) = \alpha(\lim_{n\to\infty} x_{k_n}) = \lim_{n\to\infty} \alpha(x_{k_n}) = 0,$$

and so $x(\xi) = 0$ with a $\xi \in [\varepsilon, T]$ by Lemma 2.1. Because of $x(0) = x(\xi) = 0$ we have $x'(\tau) = 0$ for some $\tau \in (0, \xi)$. From $\lim_{n\to\infty} x_{k_n} = x$ in $C^1(J)$ we see that

$$\lim_{n\to\infty} x_{k_n}(\xi) = 0, \ \lim_{n\to\infty} x_{k_n}'(\tau) = 0.$$

We are going to show that $\lim_{n\to\infty} \xi_{k_n} = \xi$. If not, there exists a subsequence of $\{\xi_{k_n}\}$ — for simplicity of our notation denoted by $\{\xi_{k_n}\}$ again — such that $\lim_{n\to\infty} \xi_{k_n} = \sigma$, $\sigma \ne \xi$. If $\sigma < \xi$ then, by (3.9) and (3.11), $x_{k_n}(\xi) \le -aC(\xi - \xi_{k_n})/2$ for sufficiently large n, contrary to $\lim_{n\to\infty} x_{k_n}(\xi) = 0$. If $\sigma > \xi$ then, by (3.8), $x_{k_n}(\xi) \ge a\min\{\xi_{k_n}\xi,\ \xi_{k_n}(\xi_{k_n} - \xi)\}/4$ for sufficiently large n, contrary again to $\lim_{n\to\infty} x_{k_n}(\xi) = 0$.

Now we claim: $\lim_{n\to\infty} \tau_{k_n} = \tau$. If not, going if necessary to a subsequence, we can assume that $\lim_{n\to\infty} \tau_{k_n} = \chi$, $\chi \ne \tau$. If $\chi < \tau$ then, by (3.10), $x_{k_n}'(\tau) \le -a(\tau - \tau_{k_n})$ for sufficiently large n, contrary to $\lim_{n\to\infty} x_{k_n}'(\tau) = 0$. If $\chi > \tau$ then, by (3.10), $x_{k_n}'(\tau) \ge a(\tau_{k_n} - \tau)$ for sufficiently large n, contrary to $\lim_{n\to\infty} x_{k_n}'(\tau) = 0$. Hence $\lim_{n\to\infty} \tau_{k_n} = \tau$.

Now letting $n \to \infty$ in (3.8) – (3.10) with k_n instead of n and using (3.11), we get

$$x(t) \ge \begin{cases} \dfrac{a\xi}{4} t & \text{for } t \in \left[0, \dfrac{\xi}{2}\right]; \\ \dfrac{a\xi}{4}(\xi - t) & \text{for } t \in \left(\dfrac{\xi}{2}, \xi\right], \end{cases}$$

$$x(t) \le -\frac{a}{2} C(t - \xi) \quad \text{for } t \in (\xi, T] \text{ if } \xi < T$$

and

$$x'(t) \ge a(\tau - t) \ \text{ for } t \in [0, \tau], \quad x'(t) \le -a(t - \tau) \ \text{ for } t \in (\tau, T].$$

Consequently, ξ is the unique zero of x in $(0, T]$ and τ is the unique zero of x' in J.

From the definition of $\hat{f}_n(t, x, y) \in \operatorname{Car}(J \times \mathbb{R}^2)$ we conclude that there exists $\mathcal{U} \subset J$, $\mu(\mathcal{U}) = 0$ such that $\hat{f}_{k_n}(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$ for each $t \in J \setminus \mathcal{U}$ and each $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} \hat{f}_{k_n}(t, x_{k_n}(t), x_{k_n}'(t)) = f(t, x(t), x'(t)), \quad t \in J \setminus (\mathcal{U} \cup \{0, \tau, \xi\}).$$

Now the Vitali's convergence theorem gives $f(t, x(t), x'(t)) \in L_1(J)$ and

$$\lim_{n\to\infty} \int_0^t \hat{f}_{k_n}(s, x_{k_n}(s), x'_{k_n}(s))\, ds = \int_0^t f(s, x(s), x'(s))\, ds, \quad t \in J.$$

Taking the limit as $n \to \infty$ in the equalities

$$x'_{k_n}(t) = x'_{k_n}(0) + \int_0^t \hat{f}_{k_n}(s, x_{k_n}(s), x'_{k_n}(s))\, ds, \quad t \in J,\ n \in \mathbb{N},$$

we get

$$x'(t) = x'(0) + \int_0^t f(s, x(s), x'(s))\, ds, \quad t \in J.$$

Hence $x \in AC^1(J)$ and x is a solution of (1.1). Since $x(0) = 0$ and $\alpha(x) = 0$, we see that x is a solution of BVP (1.1), (1.3).

Remark 3.2. If we assume that f in (1.1) is continuous on $J_1 \times \mathbb{R}_0^2$ with $J_1 = J \setminus \{\xi_1, \ldots, \xi_n\}$ and satisfies assumptions (H_3) and (H_4), then there exists a solution x of BVP (1.1), (1.3) by Theorem 3.2. From our considerations we know that there exist $\xi \in [\varepsilon, T]$ and $\tau \in (0, \xi)$ such that $x(t) > 0$ for $t \in (0, \xi)$ and $x(t) < 0$ for $t \in (\xi, T]$ provided $\xi < T$ and $x'(t) > 0$ for $t \in [0, \tau)$, $x'(t) < 0$ for $t \in (\tau, T]$. Hence from the continuity of f on $J_1 \times \mathbb{R}_0^2$ it may be concluded that $x \in AC^1(J) \cap C^2(J \setminus \{0, \xi, \tau, \xi_1, \ldots, \xi_n\})$ and x satisfies (1.1) for each $t \in J \setminus \{0, \xi, \tau, \xi_1, \ldots, \xi_n\}$.

Example 3.2. Consider the BVP

$$x''(t) + \frac{1}{(\min\{1, |x(t)|\})^b} + \frac{1}{|x'(t)|^c} + A\frac{|x(t)|}{\sqrt{t(T-t)}} + B\frac{|x'(t)|}{\sqrt{|2t-T|}} = 0, \tag{3.14}$$

$$x(0) = 0, \ \ x(\varepsilon) + \lambda \min\{x(t) : \varepsilon \le t \le T\} = 0, \tag{3.15}$$

where $b, c \in (0,1)$, $A, B \in [0,\infty)$, $A\pi T + 2B\sqrt{T} < 1$, $\varepsilon \in (0,T)$ and $\lambda \in [0,\infty)$. Then assumptions (H_3) and (H_4) are satisfied with $a = 1$, $\phi(t) = 0$, $\omega_0(u) = 1/(\min\{1,u\})^b$, $\omega_1(u) = 1/u^c$, $q_0(t) = q_1(t) = 1$, $h_1(t) = A/\sqrt{t(T-t)}$ and $h_2(t) = B/\sqrt{|2t-T|}$. Since the functional α defined on $C^0([\varepsilon, T])$ by $\alpha(x) = x(\varepsilon) + \lambda \min\{x(t) : \varepsilon \le t \le T\}$ belongs to $\mathcal{A}_\varepsilon$, there exists a solution x of BVP (3.14), (3.15) by Theorem 3.2 for each $\varepsilon \in (0,T)$ and $\lambda \in [0,\infty)$ in (3.15).

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