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COATING THIN FILM FLOWS ON A SOLID SPHERE*
ТЕЧІЇ ТОНКОЇ ПЛІВКИ ВЗДОВЖ ПОВЕРХНІ ТВЕРДОЇ КУЛІ

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By using the Arzela – Ascoli theorem, we prove the existence of strong solutions of the thin film equation on a solid sphere in weighted Sobolev spaces.

За допомогою теореми Арцела – Асколі доведено існування сильних розв'язків рівняння течії тонкої плівки на сферичній поверхні у просторах Соболева з вагою.

Introduction. Many problems in industrial and natural settings involve the flow of thin liquid films driven by gravity on different types of surfaces including a spherical one [1]. For example, the flow of a thin liquid film on a flat surface such as an inclined plane in the presence of gravity has been the subject of numerous investigations over the years (see, e. g., [2 – 4]). Dynamics of viscous coating flows on an outer surface of a solid sphere has been studied by Kang, Nadim, and Chugunova [5] in situations where the draining of the film due to gravity was balanced by centrifugal forces arising from the rotation of the sphere about a vertical axis and by capillary forces due to surface tension. The time evolution of a thin liquid film coating of the outer surface of a sphere in the presence of gravity, surface tension, and thermal gradients was considered in [6]. The spherical coating model without the surface tension and Marangoni effects was studied in [7, 8]. Recently, in [5], the authors derived the following equation for the no-slip regime in dimensionless form

$$h_t + \frac{1}{\sin\theta}\left(h^3 \sin\theta J\right)_\theta = 0,$$

$$J := a\sin\theta + b\sin\theta\cos\theta + c\left[2h + \frac{1}{\sin\theta}\left(\sin\theta h_\theta\right)_\theta\right]_\theta,$$

where $h(\theta, t)$ represents the thickness of the thin film, $\theta \in (0, \pi)$ is the polar angle in spherical coordinates, with t denoting time; the dimensionless parameters a, b and c describe the effects of gravity, rotation and surface tension, respectively. After the change of variable $x = -\cos\theta$, this equation can be written in the form:

$$h_t + \left[h^3\left(1-x^2\right)\left(a - bx + c\left(2h + \left(\left(1-x^2\right)h_x\right)_x\right)_x\right)\right]_x = 0, \tag{1.1}$$

where $x \in (-1, 1)$.

The goal of this paper is to study an arbitrary slip (weak and Navier slippage) generalisation of (1.1) with $a = b = 0$:

$$u_t + c\left(\left(1-x^2\right)u^n\left(\left(1-x^2\right)u_x + 2du\right)_{xx}\right)_x = 0 \quad \text{in} \quad Q_T, \tag{1.2}$$

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where $Q_T = \Omega \times (0,T),\ n > 0,\ d > 0,\ T > 0$, and $\Omega = (-1,1)$. As a result, equation (1.2) for $n = 3$ is a particular case of (1.1) for no-slip regime. This is a nonlinear fourth-order parabolic equation that is doubly degenerate. This equation captures the dynamics of a thin viscous liquid film on the outer surface of a solid sphere without gravity.

In contrast to the classical thin film equation:

$$u_t + (|u|^n u_{xxx})_x = 0, \tag{1.3}$$

which describes the behavior of a thin viscous film on a flat surface under the effect of surface tension, the equation (1.2) is not yet well analysed. To the best of our knowledge for (1.2) with $d = 0$, in [9] the authors proved existence of nonnegative weak solutions in weighted Sobolev spaces, and in [10] the author proved existence of nonnegative strong solutions and its asymptotic convergence to a flat profile. Note that (1.2) loses its parabolicity not only at $u = 0$ (as in (1.3)) but also at $x = \pm 1$. For this reason, it is natural to seek solution in a Sobolev space with weight $1 - x^2$.

In 1990, Bernis and Friedman [11] constructed nonnegative weak solutions of the equation (1.3) for nonlinearity $n \geq 1$, and it was also shown that for $n \geq 4$, with uniformly positive initial data, there exists a unique positive classical solution. In 1994, Bertozzi et al. [12] generalised this positivity property for the case $n \geq \frac{7}{2}$. In 1995, Beretta et al. [13] proved the existence of nonnegative weak solutions for the equation (1.3) if $n > 0$, and the existence of strong ones for $0 < n < 3$. Also, they could show that this positivity-preserving property holds for almost every time t in the case $n \geq 2$. A similar result on a cylindrical surface was obtained in [14]. Regarding the long-time behaviour, Carrillo and Toscani [15] proved the convergence to a self-similar solution for equation (1.3) with $n = 1$ and Carlen and Ulusoy [16] gave an upper bound on the distance from the self-similar solution. A similar result on a cylindrical surface was obtained in [17].

In the present article, using energy and entropy estimates, we obtain the existence of nonnegative strong solutions for (1.2) with $c = d = 1$ and $n \geq 1$.

2. Existence of strong solutions. We study the following thin film equation

$$u_t + \Big(\left(1 - x^2\right) |u|^n \left[\left(\left(1 - x^2\right) u_x \right)_x + 2u \right]_x \Big)_x = 0 \quad \text{in} \quad Q_T \tag{2.1}$$

with the no-flux boundary conditions

$$\left(1 - x^2\right) u_x = \left(1 - x^2\right) \left(\left(1 - x^2\right) u_x \right)_{xx} = 0 \quad \text{at} \quad x = \pm 1, \quad t > 0, \tag{2.2}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{2.3}$$

Here $n > 0,\ Q_T = \Omega \times (0,T),\ \Omega := (-1,1)$, and $T > 0$. Integrating the equation (2.1) by using boundary conditions (2.2), we obtain the mass conservation property

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx =: M > 0.$$

Consider initial data $u_0(x) \geq 0$ for all $x \in \bar{\Omega}$ satisfying

$$\int_\Omega \left\{ u_0^2(x) + \left(1 - x^2\right) u_{0,x}^2(x) \right\} dx < \infty. \tag{2.4}$$

Definition 2.1 (weak solution)**.** *Let $n > 0$. A function u is a weak solution of the problem (2.1)–(2.3) with initial data u_0 satisfying (2.4) if $u(x,t)$ has the following properties:*

$$\left(1-x^2\right)^{\beta/2} u \in C^{\alpha/2,\alpha/8}_{x,t}\left(\bar{Q}_T\right), \qquad 0<\alpha<\beta\leq\frac{2}{n}, \qquad u_t \in L^2\left(0,T;\left(H^1(\Omega)\right)^*\right),$$

$$\left(1-x^2\right)^{1/2} u_x \in L^{\infty}\left(0,T;L^2(\Omega)\right),$$

$$\left(1-x^2\right)^{1/2}|u|^{n/2}\left[\left(\left(1-x^2\right)u_x\right)_x+2u\right]_x \in L^2(P),$$

and $u(x,t)$ satisfies (2.1) in the following sense:

$$\int\limits_0^T \langle u_t,\phi\rangle\,dt - \iint\limits_P \left(1-x^2\right)|u|^n\left[\left(\left(1-x^2\right)u_x\right)_x+2u\right]_x \phi_x\,dxdt = 0$$

for all $\phi \in L^2\big(0,T;H^1(\Omega)\big)$, where $P := \bar{Q}_T \setminus \{\{u=0\}\cup\{t=0\}\}$,

$$u(.,t)+\left(1-x^2\right)^{1/2}u_x(.,t) \to u_0(.)+\left(1-x^2\right)^{1/2}u_{0,x}(.) \quad \textit{strongly in} \quad L^2(\Omega)$$

as $t\to 0$, and boundary conditions (2.2) hold at all points of the lateral boundary, where $\{u\neq 0\}$.

Let us denote by

$$\mathcal{E}_0(z) := \frac{1}{2}\int\limits_\Omega \left[\left(1-x^2\right)z_x^2 - 2z^2\right]\,dx,$$

$$0 \leq G_0(z) := \begin{cases} \dfrac{z^{2-n}-A^{2-n}}{(n-1)(n-2)} - \dfrac{A^{1-n}}{1-n}(z-A) & \text{if} \quad n\neq 1,2,\\ z\ln z - z(\ln A+1)+A & \text{if} \quad n=1,\\ \ln\left(\dfrac{A}{z}\right)+\dfrac{z}{A}-1 & \text{if} \quad n=2,\end{cases}$$

where $A\geq 0$ if $n\in(1,2)$ and $A>0$ if else. Next, we establish existence of a more regular solution u of the problem (2.1)–(2.3) than a weak solution in the sense of Definition 2.1.

Theorem 2.1 (strong solution)**.** *Assume that $n\geq 1$ and initial data u_0 satisfies*

$$\int\limits_\Omega G_0(u_0)\,dx < +\infty, \qquad \textit{and} \qquad \mathcal{E}_0(u_0) \geq -\frac{M^2}{|\Omega|} - 2M^2C_N^4,$$

where $C_N>0$ is from (3.20), then the problem (2.1)–(2.3) has a nonnegative weak solution, u, in the sense of Definition 2.1, such that

$$\left(1-x^2\right)u_x \in L^2(0,T;H^1(\Omega)), \qquad \left(1-x^2\right)^{\gamma/2}u_x \in L^2(Q_T), \quad \gamma\in(0,1],$$

$$u\in L^{\infty}\left(0,T;L^2(\Omega)\right), \qquad \left(1-x^2\right)^{\mu/2}u\in L^2(Q_T), \quad \mu\in(-1,\beta]$$

for all $T>0$.

3. Proof of Theorem 2.1. ***3.1. Approximating problems.*** Let us denote the energy functional and its variation by

$$\mathcal{E}_\delta(u(t)) := \frac{1}{2}\int\limits_\Omega \left[\left(1 - x^2 + \delta\right) u_x^2 - 2u^2\right] dx,$$

$$\frac{\delta \mathcal{E}_\delta(u)}{\delta u} := -\left[\left(\left(1 - x^2 + \delta\right) u_x\right)_x + 2u\right].$$

Equation (2.1) is doubly degenerate when $u = 0$ and $x = \pm 1$. For this reason, for any $\epsilon > 0$ and $\delta > 0$ we consider two-parametric regularised equations

$$u_{\epsilon\delta,t} - \left[\left(1 - x^2 + \delta\right)\left(|u_{\epsilon\delta}|^n + \epsilon\right)\left(\frac{\delta \mathcal{E}_\delta(u_{\epsilon\delta})}{\delta u}\right)_x\right]_x = 0 \quad \text{in} \quad Q_T \tag{3.1}$$

with boundary conditions

$$u_{\epsilon\delta,x} = \left(\left(1 - x^2 + \delta\right) u_{\epsilon\delta,x}\right)_{xx} = 0 \quad \text{at} \quad x = \pm 1, \tag{3.2}$$

and initial data

$$u_{\epsilon\delta}(x,0) = u_{0,\epsilon\delta}(x) \in C^{4+\gamma}(\bar{\Omega}) \quad \text{for some} \quad \gamma > 0, \tag{3.3}$$

where

$$u_{0,\epsilon\delta}(x) \geq u_{0\delta}(x) + \epsilon^\theta, \quad \theta \in \left(0, \frac{1}{2(n-1)}\right), \tag{3.4}$$

$$u_{0,\epsilon\delta} \to u_{0\delta} \quad \text{strongly in} \quad H^1(\Omega) \quad \text{as} \quad \epsilon \to 0, \tag{3.5}$$

$$\left(1 - x^2 + \delta\right)^{1/2} u_{0x,\delta} \to \left(1 - x^2\right)^{1/2} u_{0,x} \quad \text{strongly in} \quad L^2(\Omega) \quad \text{as} \quad \delta \to 0, \tag{3.6}$$

$$u_{0,\delta} \to u_0 \quad \text{strongly in} \quad L^2(\Omega) \quad \text{as} \quad \delta \to 0. \tag{3.7}$$

The parameters $\epsilon > 0$ and $\delta > 0$ in (3.1) make the problem regular up to the boundary (i.e., uniformly parabolic). The existence of a solution of (3.1) in a small time interval is guaranteed by the Schauder estimates in [18]. Now suppose that $u_{\epsilon\delta}$ is a solution of equation (3.1) and that it is continuously differentiable with respect to the time variable and fourth order continuously differentiable with respect to the spatial variable.

3.2. Limit process as $\boldsymbol{\epsilon \to 0}$. In order to get an *a priori* estimate of $u_{\epsilon\delta}$, we multiply both sides of equation (3.1) by $\dfrac{\delta \mathcal{E}_\delta(u_{\epsilon\delta})}{\delta u}$ and integrate over Ω by (3.2). This gives us

$$\frac{d}{dt}\,\mathcal{E}_\delta(u_{\epsilon\delta}) + \int\limits_\Omega \left(1 - x^2 + \delta\right)\left(|u_{\epsilon\delta}|^n + \epsilon\right)\left[\frac{\delta \mathcal{E}_\delta(u_{\epsilon\delta})}{\delta u}\right]_x^2 dx = 0. \tag{3.8}$$

Integrating (3.8) in time, we get

$$\mathcal{E}_\delta(u_{\epsilon\delta}) + \iint\limits_{Q_T} \left(1 - x^2 + \delta\right)\left(|u_{\epsilon\delta}|^n + \epsilon\right)\left[\frac{\delta \mathcal{E}_\delta(u_{\epsilon\delta})}{\delta u}\right]_x^2 dx\,dt = \mathcal{E}_\delta(u_{0,\epsilon\delta}). \tag{3.9}$$

Multiplying (3.1) by $-\big((1-x^2+\delta)u_{\epsilon\delta,x}\big)_x + u_{\epsilon\delta}$, integrating over Ω, and using the boundary conditions (3.2) we find

$$\frac{1}{2}\,\frac{d}{dt}\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u^2_{\epsilon\delta,x}+u^2_{\epsilon\delta}\right]\,dx+$$

$$+\int\limits_{\Omega}\left(1-x^2+\delta\right)\left(|u_{\epsilon\delta}|^n+\epsilon\right)\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}\right]^2_{xx}\,dx=$$

$$=-\int\limits_{\Omega}\left(1-x^2+\delta\right)\left(|u_{\epsilon\delta}|^n+\epsilon\right)\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}\right]_{xx}u_{\epsilon\delta,x}\,dx+$$

$$+\,2\int\limits_{\Omega}\left(1-x^2+\delta\right)\left(|u_{\epsilon\delta}|^n+\epsilon\right)u^2_{\epsilon\delta,x}\,dx\leq$$

$$\leq\frac{1}{2}\int\limits_{\Omega}\left(1-x^2+\delta\right)\left(|u_{\epsilon\delta}|^n+\epsilon\right)\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}\right]^2_{xx}\,dx+$$

$$+\,\frac{5}{2}\int\limits_{\Omega}\left(1-x^2+\delta\right)\left(|u_{\epsilon\delta}|^n+\epsilon\right)u^2_{\epsilon\delta,x}\,dx,$$

whence

$$\frac{d}{dt}\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u^2_{\epsilon\delta,x}+u^2_{\epsilon\delta}\right]dx+$$

$$+\int\limits_{\Omega}\left(1-x^2+\delta\right)\left(|u_{\epsilon\delta}|^n+\epsilon\right)\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}\right]^2_{xx}\,dx\leq$$

$$\leq 5\left(\|u_{\epsilon\delta}\|^n_{\infty}+\epsilon\right)\int\limits_{\Omega}\left(1-x^2+\delta\right)u^2_{\epsilon\delta,x}\,dx. \tag{3.10}$$

By the mass conservation

$$\int\limits_{\Omega}u_{\epsilon\delta}\,dx=\int\limits_{\Omega}u_{0,\epsilon\delta}\,dx=:M_{\epsilon\delta}>0,$$

we find that

$$\|u_{\epsilon\delta}\|_{\infty}\leq\left(\frac{|\Omega|}{\delta}\right)^{1/2}\left(\int\limits_{\Omega}\left(1-x^2+\delta\right)u^2_{\epsilon\delta,x}\,dx\right)^{1/2}+\frac{M_{\epsilon\delta}}{|\Omega|}. \tag{3.11}$$

Using (3.11), from (3.10) we get

$$\frac{d}{dt}\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u^2_{\epsilon\delta,x}+u^2_{\epsilon\delta}\right]dx+$$

$$+\int\limits_{\Omega}\left(1-x^2+\delta\right)\left(|u_{\epsilon\delta}|^n+\epsilon\right)\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}\right]_{xx}^2\,dx\le$$

$$\le C_{\epsilon\delta}\left(\max\left\{1,\int\limits_{\Omega}\left(1-x^2+\delta\right)u_{\epsilon\delta,x}^2\,dx\right\}\right)^{\frac{n+2}{2}},$$

where

$$C_{\epsilon\delta}:=2^{n+1}5\left[\left(\frac{|\Omega|}{\delta}\right)^{n/2}+\left(\frac{M_{\epsilon\delta}}{|\Omega|}\right)^n+\epsilon\right].$$

Applying the nonlinear Grönwall inequality to

$$y(T)\le y(0)+C_{\epsilon\delta}\int\limits_0^T\max\left\{1,y^{\frac{n+2}{2}}(t)\right\}dt,$$

where

$$y(t):=\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}^2+u_{\epsilon\delta}^2\right]dx,$$

yields

$$\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}^2+u_{\epsilon\delta}^2\right]dx\le 2^{2/n}\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u_{0\epsilon\delta,x}^2+u_{0,\epsilon\delta}^2\right]dx\le C_\delta$$

for all $T\in[0,T_{\epsilon\delta,\mathrm{loc}}]$, where

$$T_{\epsilon\delta,\mathrm{loc}}:=\frac{1}{nC_{\epsilon\delta}}\min\left\{1,\left(\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u_{0\epsilon\delta,x}^2+u_{0,\epsilon\delta}^2\right]dx\right)^{-n/2}\right\}.$$

The times $T_{\epsilon\delta,\mathrm{loc}}$ converge to a positive limit as $\epsilon\to0$, and tends to 0 as $\delta\to0$. Taking ϵ smaller if necessary, the time of existence is defined as

$$T_{\delta,\mathrm{loc}}=\frac{9}{10}\lim_{\epsilon\to0}T_{\epsilon\delta,\mathrm{loc}}=$$

$$=\frac{9}{10}\frac{1}{nC_{0\delta}}\min\left\{1,\left(\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u_{0\delta,x}^2+u_{0,\delta}^2\right]dx\right)^{-n/2}\right\}<T_{\epsilon\delta,\mathrm{loc}}.$$

As a result, the bound

$$\int\limits_{\Omega}\left[\left(1-x^2+\delta\right)u_{\epsilon\delta,x}^2+u_{\epsilon\delta}^2\right]dx\le C_\delta \tag{3.12}$$

holds for all $T\in[0,T_{\delta,\mathrm{loc}}]$, where $C_\delta>0$ is independent of ϵ. From (3.12) and (3.9) it follows that

$$\{u_{\epsilon\delta}\}_{\epsilon>0}\quad\text{is uniformly bounded in}\quad L^\infty\left(0,T;H^1(\Omega)\right), \tag{3.13}$$

$$\left\{ \left(1 - x^2 + \delta\right)^{1/2} (|u_{\epsilon\delta}|^n + \epsilon)^{1/2} \left[\frac{\delta \mathcal{E}_\delta(u_{\epsilon\delta})}{\delta u} \right]_x \right\}_{\epsilon > 0} \quad \text{is uniformly bounded in} \quad L^2(Q_T) \quad (3.14)$$

for all $T \in [0, T_{\delta,\mathrm{loc}}]$. By (3.13) and (3.14), using the same method as in [11], we can prove that solutions $u_{\epsilon\delta}$ have uniformly (in ϵ) bounded $C^{1/2,1/8}_{x,t}$-norms. By the Arzelà – Ascoli theorem, this equicontinuous property, together with the uniformly boundedness shows that every sequence $\{u_{\epsilon\delta}\}_{\epsilon>0}$ has a subsequence such that

$$u_{\epsilon\delta} \to u_\delta \quad \text{uniformly in} \quad Q_T \quad \text{as} \quad \epsilon \to 0.$$

As a result, we obtain a local (in time) solution u_δ of the problem (3.1) – (3.3) with $\epsilon = 0$ in the sense of [11, p. 185 – 186] (Theorem 3.1).

3.3. Non-negativity of u_δ. Let us denote by $G_\epsilon(z)$ the following function

$$G_\epsilon(z) \geq 0 \quad \forall z \in \mathbb{R}, \qquad G''_\epsilon(z) = \frac{1}{|s|^n + \epsilon}.$$

Now we multiply equation (3.1) by $G'_\epsilon(u_{\epsilon\delta})$ and integrate over Ω to get

$$\frac{d}{dt} \int\limits_\Omega G_\epsilon\left(u_{\epsilon\delta}(x,t)\right) dx + \int\limits_\Omega \left(1 - x^2 + \delta\right) (|u_{\epsilon\delta}|^n + \epsilon) \left[\frac{\delta \mathcal{E}_\delta(u_{\epsilon\delta})}{\delta u} \right]_x G''_\epsilon(u_{\epsilon\delta}) u_{\epsilon\delta,x}\, dx = 0,$$

whence by (3.12) we obtain

$$\frac{d}{dt} \int\limits_\Omega G_\epsilon(u_{\epsilon\delta}(x,t))\, dx + \int\limits_\Omega \left[\left(1 - x^2 + \delta\right) u_{\epsilon\delta,x}\right]_x^2 dx = 2 \int\limits_\Omega \left(1 - x^2 + \delta\right) u^2_{\epsilon\delta,x}\, dx \leq 2C_\delta. \quad (3.15)$$

After integration in time, equation (3.15) becomes

$$\int\limits_\Omega G_\epsilon(u_{\epsilon\delta}(x,T))\, dx + \iint\limits_{Q_T} \left[\left(1 - x^2 + \delta\right) u_{\epsilon\delta,x}\right]_x^2 dx\, dt \leq \int\limits_\Omega G_\epsilon(u_{0,\epsilon\delta}(x))\, dx + 2C_\delta T \qquad (3.16)$$

for all $T \in [0, T_{\delta,\mathrm{loc}}]$. We compute

$$G''_0(z) - G''_\epsilon(z) = \frac{\epsilon}{|z|^n\,(|z|^n + \epsilon)},$$

and consequently

$$G_0(z) - G_\epsilon(z) = \epsilon \int\limits_A^z \int\limits_A^v \frac{ds dv}{|s|^n\,(|s|^n + \epsilon)},$$

where A is some positive constant. As $u_{0,\epsilon\delta}(x)$ is bounded then by (3.4) it follows that

$$\left|G_0(u_{0,\epsilon\delta}(x)) - G_\epsilon(u_{0,\epsilon\delta}(x))\right| \leq C\, \epsilon^{1-2\theta(n-1)} \to 0 \quad \text{as} \quad \epsilon \to 0,$$

and therefore, due to (3.5), we have

$$\int\limits_\Omega G_\epsilon(u_{0,\epsilon}(x))\, dx \to \int\limits_\Omega G_0(u_{0\delta}(x))\, dx \quad \text{as} \quad \epsilon \to 0. \qquad (3.17)$$

As a result, by (3.16), (3.17) we deduce that

$$\int_{\Omega} G_{\epsilon}(u_{\epsilon\delta}(x,T))\,dx \le C_1(\delta), \tag{3.18}$$

$$\left\{\left(1-x^2+\delta\right)u_{\epsilon\delta,x}\right\}_{\epsilon>0} \quad \text{is uniformly bounded in} \quad L^2\left(0,T;H^1(\Omega)\right)$$

for all $T \in [0, T_{\delta,\mathrm{loc}}]$, where $C_1(\delta) > 0$ is independent of $\epsilon > 0$. Similar to [11, p. 190] (Theorem 4.1), using (3.13) and (3.18), we can show that the limit solution u_δ is nonnegative if $n \in [1,4)$ and positive if $n \ge 4$.

3.4. Limit process as $\delta \to 0$. Next, we show that the family of solutions $\{u_\delta\}_{\delta>0}$ is uniformly bounded in some weighted space. Using non-negativity of u_δ, we have to clarify a priori estimate (3.12).

Next, we will use the mass conservation property

$$\int_{\Omega} u_\delta(x,t)\,dx = M_\delta > 0, \tag{3.19}$$

and the following interpolation inequality:

Lemma 3.1 [19]. *Let $p,\ q,\ r,\ \alpha,\ \beta,\ \gamma,\ \sigma$ and θ be real numbers satisfying $p,q \ge 1,\ r > 0,$ $0 \le \theta \le 1,\ \gamma = \theta\sigma + (1-\theta)\beta,\ \dfrac{1}{p} + \dfrac{\alpha}{N} > 0,\ \dfrac{1}{q} + \dfrac{\beta}{N} > 0$ and $\dfrac{1}{r} + \dfrac{\gamma}{N} > 0$. There exists a positive constant C such that the following inequality holds for all $v \in C_0^\infty\left(\mathbb{R}^N\right),\ N \ge 1$*

$$\left\||x|^{\gamma} v\right\|_{L^r} \le C \left\||x|^{\alpha}|\nabla v|\right\|_{L^p}^{\theta} \left\||x|^{\beta} v\right\|_{L^q}^{1-\theta}$$

if and only if

$$\frac{1}{r} + \frac{\gamma}{N} = \theta\left(\frac{1}{p} + \frac{\alpha-1}{N}\right) + (1-\theta)\left(\frac{1}{q} + \frac{\beta}{N}\right)$$

and

$$\begin{cases} 0 \le \alpha - \sigma \quad \textit{if} \quad a > 0, \\ \alpha - \sigma \le 1 \quad \textit{if} \quad a > 0 \quad \textit{and} \quad \dfrac{1}{p} + \dfrac{\alpha-1}{N} = \dfrac{1}{r} + \dfrac{\gamma}{N}. \end{cases}$$

Applying Lemma 3.1 to $v = u_\delta - \dfrac{M_\delta}{|\Omega|}$ with $\Omega = (-1,1),\ \gamma = \beta = 0,\ \alpha = \dfrac{1}{2},\ r = p = 2,$ $q = 1,\ N = 1$, and $\theta = \dfrac{1}{2}$, we have

$$\left\|u_\delta - \frac{M_\delta}{|\Omega|}\right\|_2 \le C_N \left\|\left(1-x^2\right)^{1/2} u_{\delta,x}\right\|_2^{\theta} \left\|u_\delta - \frac{M_\delta}{|\Omega|}\right\|_1^{1-\theta}, \tag{3.20}$$

whence for $u_\delta \ge 0$ we deduce that

$$\int_{\Omega} \left(u_\delta - \frac{M_\delta}{|\Omega|}\right)^2 dx \le 2M_\delta C_N^2 \left(\int_{\Omega} \left(1-x^2\right) u_{\delta,x}^2\,dx\right)^{1/2}. \tag{3.21}$$

By (3.9) with $\epsilon = 0,$ due to (3.21), we find that

$$\frac{1}{2}\int\limits_{\Omega}\left(1-x^2\right)u_{\delta,x}^2dx \leq \int\limits_{\Omega}\left(u_\delta - \frac{M_\delta}{|\Omega|}\right)^2 dx + \frac{M_\delta^2}{|\Omega|} + \mathcal{E}_\delta(u_{0,\delta}) \leq$$

$$\leq 2M_\delta C_N^2\left(\int\limits_{\Omega}\left(1-x^2\right)u_{\delta,x}^2\,dx\right)^{1/2} + \frac{M_\delta^2}{|\Omega|} + \mathcal{E}_\delta(u_{0,\delta}).$$

As a result, we get

$$\int\limits_{\Omega}\left(1-x^2\right)u_{\delta,x}^2dx \leq \left[2M_\delta C_N^2 + \sqrt{4M_\delta^2 C_N^4 + 2\left[\frac{M_\delta^2}{|\Omega|} + \mathcal{E}_\delta(u_{0,\delta})\right]}\,\right]^2 \tag{3.22}$$

provided

$$\mathcal{E}_\delta(u_{0,\delta}) \geq -\frac{M_\delta^2}{|\Omega|} - 2M_\delta^2 C_N^4. \tag{3.23}$$

Taking into account (3.6) and (3.7), from (3.22) and (3.15) we arrive at

$$\int\limits_{\Omega}\left[\left(1-x^2\right)u_{\delta,x}^2 + u_\delta^2\right]dx \leq C_2 \tag{3.24}$$

for all $T > 0,$ where C_2 is independent of $\delta > 0,$ provided (3.23).

Using (3.19), we find that

$$\left|u_\delta - \frac{M_\delta}{|\Omega|}\right| = \left|\int\limits_{x_0}^{x} u_{\delta,x}\,dx\right| \leq \left(\int\limits_{\Omega}\left(1-x^2\right)u_{\delta,x}^2\,dx\right)^{1/2}\left|\int\limits_{x_0}^{x}\frac{dx}{1-x^2}\right|^{1/2}. \tag{3.25}$$

Multiplying (3.25) by $(1-x^2)^{\beta/2}$ for any $\beta > 0,$ by (3.24) we deduce that

$$(1-x^2)^{\beta/2}\left|u_\delta - \frac{M_\delta}{|\Omega|}\right| \leq \left(\frac{C_2}{2}\right)^{1/2}\left(\left(1-x^2\right)^\beta \ln\left(\frac{(1+x)(1-x_0)}{(1-x)(1+x_0)}\right)\right)^{1/2} \leq C_3 \tag{3.26}$$

for all $x \in \bar{\Omega},$ where $C_3 > 0$ is independent of $\delta > 0.$ From (3.26) we find that

$$\left\{\left(1-x^2\right)^{\beta/2}u_\delta\right\}_{\delta>0} \quad \text{is uniformly bounded in} \quad Q_T \quad \text{for any} \quad \beta > 0. \tag{3.27}$$

In particular, by (3.24) we get

$$\left(1-x^2\right)^{\beta/2}|u_\delta(x_1,t) - u_\delta(x_2,t)| \leq C_4|x_1-x_2|^{\alpha/2} \quad \forall x_1,x_2 \in \Omega, \quad \alpha \in (0,\beta). \tag{3.28}$$

By (3.14), (3.27) and (3.28) with $\beta \in \left(0, \frac{2}{n}\right],$ using the same method as in [11, p. 183] (Lemma 2.1), we can prove similarly that

$$\left(1-x^2\right)^{\beta/2}|u_\delta(x,t_1) - u_\delta(x,t_2)| \leq C_5|t_1-t_2|^{\alpha/8} \quad \forall t_1,t_2 \in (0,T). \tag{3.29}$$

The inequalities (3.28) and (3.29) show the uniform (in δ) boundedness of a sequence $\left\{\left(1-x^2\right)^{\beta/2}u_\delta\right\}_{\delta>0}$ in the $C_{x,t}^{\alpha/2,\alpha/8}$-norm.

By the Arzelà – Ascoli theorem, this a priori bound together with (3.27) shows that as $\delta\to 0,$ every sequence $\left\{\left(1-x^2\right)^{\beta/2}u_\delta\right\}_{\delta>0}$ has a subsequence $\left\{\left(1-x^2\right)^{\beta/2}u_{\delta_k}\right\}_{\delta_k>0}$ such that

$$\left(1-x^2\right)^{\beta/2}u_{\delta_k}\to\left(1-x^2\right)^{\beta/2}u \quad \text{uniformly in} \quad \bar{Q}_T \quad \text{as} \quad \delta_k\to 0.$$

Following the idea of proof [11] (Theorem 3.1), we obtain a global (in time) solution u of the problem (3.1) – (3.3) in the sense of Definition 2.1.

From (3.22) and (3.15) we have

$$\int\limits_\Omega G_0(u_{0\delta}(x,T))\,dx\le C_6, \tag{3.30}$$

$$\left\{\left(1-x^2+\delta\right)u_{\delta,x}\right\}_{\delta>0} \quad \text{is uniformly bounded in} \quad L^2\left(0,T;H^1(\Omega)\right) \tag{3.31}$$

for all $T>0,$ where C_6 is independent of $\delta>0,$ provided (3.24). The estimates (3.30), (3.31) allow us to construct a strong solution.

Note that the energy functional $\mathcal{E}_0(u(t))$ is decaying (by (3.8)), bounded from below and lower semi-continuous (by (3.24)) it must have a minimizer, $u_{\min}(x),$ which is continuous on Ω. Taking into account the mass conservation, we find (see [5]) that $u_{\min}(x)=\dfrac{M}{|\Omega|},$ $\mathcal{E}_0(u_{\min})=-\dfrac{M^2}{|\Omega|},$ and $\mathcal{E}_0(u(t))\to\mathcal{E}_0(u_{\min})$ as $t\to+\infty.$

Theorem 2.1 is proved.

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