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NONLINEAR BOUNDARY-LAYER PROBLEMS AND LAMINAR VORTICAL STREAM GENERATED BY RESONANT SLOSHING IN A CIRCULAR-BASE TANK

НЕЛІНІЙНІ ГРАНИЧНІ ЗАДАЧІ ТА ЛАМІНАРНА ВИХРОВА ТЕЧІЯ, ЩО ПОРОДЖЕНА РЕЗОНАНСНИМ ХЛЮПАННЯМ, У РЕЗЕРВУАРІ З КРУГОВОЮ ОСНОВОЮ

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For a viscous incompressible liquid with laminar flows, nonlinear boundary-layer problems on the wetted tank surface (wall and bottom) of a rigid circular-base tank partly filled with a finite depth are derived assuming known the resonant steady-state inviscid liquid sloshing due to a horizontal translatory orbital tank motion with the forcing frequency close to the lowest natural sloshing frequency. By adopting a Narimanov – Moiseev-type approximation of the above-mentioned inviscid sloshing, an analytical asymptotic solution of the derived boundary-layer problems is constructed to prove that the inviscid flows must contain a global stationary vortex component. A new nonlinear boundary-value problem governing this component is derived.

За припущення в'язкої нестисливої рідини з ламінарною течією виведено нелінійні крайові задачі пришарових течій біля змоченої поверхні бака (стінки та дна) вертикального кругового циліндричного бака, який частково заповнений рідиною із скінченною глибиною. Допускаючи, що відомо резонансні усталені нев'язкі течії рідини (хлюпання), які збурюються горизонтальними поступальними орбітальними рухами бака із частотою, яка є близькою до першої власної частоти коливання рідини, та використовуючи наближення Наріманова – Моісеєва вищезазначених усталених нев'язких режимів хлюпання, ми будуємо асимптотичний аналітичний розв'язок виведених задач пришарової течії. Доведено, що ці нев'язкі хлюпання повинні містити глобальну стаціонарну вихрову компоненту. Виведено нову нелінійну крайову задачу, розв'язок якої описує цю компоненту.

1. Introduction. Free-surface problems describing the liquid sloshing dynamics in a rigid circular-base tank have been studied, analytically and numerically [9, 16], starting from the 50's, originally, in context of spacecraft applications [1, 13]. Nonlinear analytical theories normally adopted the *inviscid irrotational* hydrodynamic model but the *viscous rotational* (Navier – Stokes) equations were in focus of the Computational Fluid Dynamics.

Ludwig Prandtl [15] was probably the first who observed, in his dedicated model tests of 1949, a slow rotation of liquid particles around the vertical tank axis that accompanies resonant angular progressive waves (swirling) in a circular-base container; the forcing frequency was close to the lowest natural sloshing frequency. In Prandtl's experiments, container moves horizontally and translatory along a circular orbit, but the same rotational stream was detected by Hutton [8] and Royon-Lebeaud, Hopfinger & Cartellier [20] for swirling due to the resonant longitudinal harmonic tank forcing. Being interested in life-science applications (bioreactors), the Prandtl-like experiments were reproduced, on a more sophisticated and systematic level in [3, 5, 17, 18, 21]. After pointing out that the Prandtl vortical stream can never be theoretically explained within the framework of the inviscid hydrodynamic model based on the Eulerian specification, the latter authors referred, as two sources for liquid particles to rotate, to the angular Stokes drift (comes from kinematic relations in the Lagrangian specification) or/and the so-called *steady streaming* [2, 19], whose occurrence is caused by the viscous boundary-layer flows. Comparing theoretical predictions of the Stokes drift with rotational stream measurements, Reclari [17] found out a satisfactory agreement *only* for the *non-resonant* sloshing. The same disagreement for the resonant steady-state swirling but due to the longitudinal forcing was reported by Hutton [8] who guessed that viscosity, most probably, then matters and must be included into analysis.

In the present paper, an attempt to extend the steady streaming theory onto sloshing problems is performed. The primary *goal* consists of creating a mathematical background for explaining the *Prandtl* stationary vortical stream *phenomenon*. Extending the steady streaming theory needs nonlinear boundary-layer equations and boundary conditions (boundary-layer problems) to be derived and solved under assumption that a "parental" inviscid solution of the hydrodynamic problem is *a priori* known, in a suitable analytical form. This faces the two *challenges*: (i) analogous nonlinear boundary-layer problems were mainly considered for two-dimensional external (infinite domain) statements but sloshing in a circular-base tank deals with internal (the limited liquid volume) three-dimensional free-surface boundary-value problems, that is, a naïve implementation of existing results [19] is doubtful; (ii) the above-mentioned parental inviscid solutions were derived [6, 7, 11] only for potential flows. In Sections 2–4, we demonstrate how to overcome the challenges.

In Section 2, after introducing a scaling (normalisation) and giving the necessary notations, we introduce inviscid $\boldsymbol{v}$ and viscous $\boldsymbol{V}$ velocity fields in the liquid domain $Q(t)$ (depends on the time), which are governed by the Euler and Navier–Stokes equations, respectively. The Reynolds number $R_s = \nu/(l_*\sigma)$ (ν is the kinematic viscosity, l_* is the chosen characteristic size, σ is the forcing frequency) determines the nondimensional boundary-layer thickness $\delta = 1/\sqrt{R_s} \ll 1$. Introducing an asymptotic solution by $O(\delta)$ makes it possible to derive (boundary-layer) equations and boundary conditions, which govern the $O(1)$-quantities of the velocity field difference $(\boldsymbol{V}-\boldsymbol{v})$. These equations are defined on the tank bottom and the wetted vertical wall. Projections of $\boldsymbol{v}$ (and its derivatives) on the wetted tank surface are assumed being known functions; they appear in both boundary conditions and coefficients of the boundary-layer equations. Solutions of the boundary-layer problems should tend to zero as $\xi \to +\infty$, where ξ is a local (boundary-layer) spatial variable whose $O(1)$-values specify the inner points of $Q(t)$, which belong to the δ-vicinity of the wetted tank surface.

The Narimanov–Moiseev asymptotic scheme is employed in Section 3 to get an analytical approximation of the inviscid velocity field $\boldsymbol{v}$ and solve the nonlinear boundary-layer problems derived in Section 2. The scheme introduces the leading $O(\epsilon^{1/3})$ term in $\boldsymbol{v}$ ($O(\epsilon)$ is the forcing amplitude) and the asymptotic relationship $\delta \lesssim \epsilon^{2/3}$ is required as the necessary applicability condition of the boundary-layer problems adopting the Narimanov–Moiseev-type inviscid solution for $\boldsymbol{v}$. The leading asymptotic term is associated with the two perpendicular natural sloshing modes; it is taken from [6]. Specifically, solutions of the boundary-layer problems

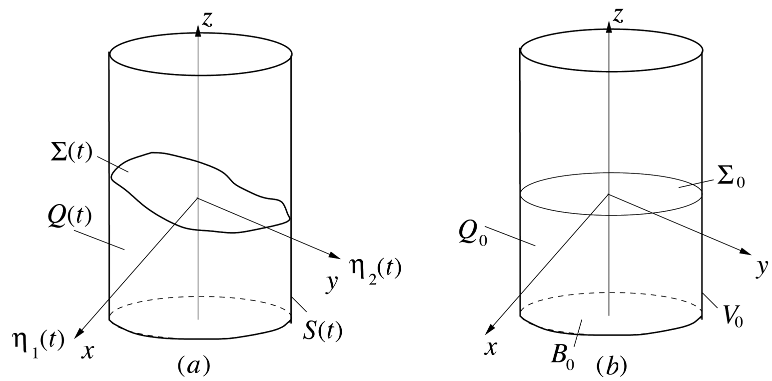

Fig. 1. An upright circular cylindrical rigid tank moves horizontally along an elliptic orbit and, thereby, excites a steady-state resonant sloshing (panel a). The coordinate system $Oxyz$ is rigidly fixed with the tank. The mean liquid depth is finite. Auxiliary problems of the asymptotic sloshing theories are defined in the the unperturbed (hydrostatic) liquid domain Q_0 confined by the mean free surface Σ_0, wetted wall V_0, and the bottom B_0 (panel b).

include a time-independent summand, which must decay away from the mean wetted tank surface. The latter is only possible, if $\boldsymbol{v}$ contains a non-zero stationary vortical stream component $\boldsymbol{w} = O(\epsilon^{2/3})$. The corresponding non-zero boundary conditions for $\boldsymbol{w}$ on the mean wetted tank surface are derived. To find a governing equation for $\boldsymbol{w}$ in the mean liquid domain Q_0 (in addition, to the non-zero boundary conditions), the vorticity equation is used in Section 4.

2. Boundary-layer problems. Our analysis suggests a nondimensional statement adopting the characteristic size, time and mass,

$$l_* = r_0/k, \quad t_* = 1/\sigma \quad \text{and} \quad m_* = \rho l_*^3, \tag{1}$$

respectively, where r_0 is the tank radius, σ is the forcing frequency, ρ is the liquid density, and $k = 1.84\ldots$ is the lowest positive real root of the transcedential equation $J_1'(k) = 0$ (J_1 is the Bessel function of the first kind); the normalisation factor k is used to simplify the forthcoming derivations. The nondimensional tank radius is equal to k. The mean nondimensional liquid depth is a finite value, i.e., of the order $O(1)$.

A *viscous incompressible liquid* partly filling a circular-base rigid tank is considered in the tank-fixed coordinate system $Oxyz$ as shown in Figure 1 (a). The tank moves along an elliptic horizontal orbit $\eta_1(t)\,\hat{\boldsymbol{x}} + \eta_2(t)\,\hat{\boldsymbol{y}}$ ($\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are the coordinate units of the tank-fixed coordinate system $Oxyz$), $\eta_1(t) = \eta_{1a}\cos t$, $\eta_2(t) = \eta_{2a}\sin t$. The nondimensional forcing amplitudes are small values,

$$\eta_{1a} \sim \eta_{2a} = O(\epsilon) \ll 1. \tag{2}$$

The *task* consists of describing the steady-state (time-periodic) liquid sloshing dynamics, i.e., finding the free-surface $\Sigma(t)$ governed by $z = f(r,\theta,t)$ (the liquid domain $Q(t) = \{(r,\theta,z,t)\colon 0 < r < k, -\pi \le \theta < \pi, -h < f(r,\theta,z,t)\}$, the velocity and pressure fields in $Q(t)$, which should be found from the corresponding free-surface problem whose actual mathematical formulation depends on the used hydrodynamic model [7].

In the present section, the steady-state *inviscid* $\boldsymbol{v}(r,\theta,z,t) = u\,\hat{\boldsymbol{r}} + v\,\hat{\boldsymbol{\theta}} + w\,\hat{\boldsymbol{z}}$ and *viscous* $\boldsymbol{V}(r,\theta,z,t) = U\hat{\boldsymbol{r}} + V\hat{\boldsymbol{\theta}} + W\hat{\boldsymbol{z}}$ velocity fields are considered, simultaneously. The cylindrical frame coordinate units are $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ and $R_s = \nu/(l_*\sigma)$ (ν is the kinematic viscosity) is the Reynolds number. We assume that $\boldsymbol{v}(r,\theta,z,t)$ (as well as the pressure $p_0(r,\theta,z,t)$) is a known analytical vector-function in $Q(t)$, which is found from the corresponding inviscid-flow free-surface problem. The viscous velocity field $\boldsymbol{V}(r,\theta,z,t)$ and the pressure $P(r,\theta,z,t)$ are governed by the continuity equation

$$(rU)_r + V_\theta + r\,W_z = 0 \tag{3}$$

(the subscripts denote the corresponding spatial derivative and the dot marks the time-derivative) and the nondimensional Navier – Stokes equation

$$\dot U + UU_r + \frac{VU_\theta}{r} - \frac{V^2}{r} + WU_z = -P_r + \delta^2\left[\frac{(rU_r)_r}{r} - \frac{U}{r^2} + \frac{U_{\theta\theta}}{r^2} - \frac{2V_\theta}{r^2} + U_{zz}\right] + \\ + \cos\theta\ddot\eta_1 + \sin\theta\ddot\eta_2, \tag{4a}$$

$$\dot V + UV_r + \frac{VV_\theta}{r} + \frac{UV}{r} + WV_z = -\frac{P_\theta}{r} + \delta^2\left[\frac{(rV_r)_r}{r} - \frac{V}{r^2} + \frac{V_{\theta\theta}}{r^2} + \frac{2U_\theta}{r^2} + V_{zz}\right] - \\ - \sin\theta\ddot\eta_1 + \cos\theta\ddot\eta_2, \tag{4b}$$

$$\dot W + UW_r + \frac{VW_\theta}{r} + WW_z = -P_z + \delta^2\left[\frac{r(W_r)_r}{r} + \frac{W_{\theta\theta}}{r^2} + W_{zz}\right], \tag{4c}$$

where $\delta^2 = 1/R_s$.

Because the known inviscid solution ($\boldsymbol{v}(r,\theta,z,t)$ and $p_0(r,\theta,z,t)$) should exactly satisfy (3) and (4) with $\delta = 0$ (the boundary layer thickness δ is negligibly small relative to other nondimensional parameters), one can focus on the difference fields

$$\boldsymbol{V} - \boldsymbol{v} = (R,\Theta,Z) = (U-u, V-v, W-w) \quad \text{and} \quad p = P - p_0, \tag{5}$$

which are governed by

$$(rR)_r + \Theta_\theta + rZ_z = 0, \tag{6a}$$

$$\dot R + RR_r + \frac{\Theta R_\theta}{r} - \frac{\Theta^2}{r} + ZR_z + [uR_r + Ru_r] + \frac{1}{r}[vR_\theta + \Theta u_\theta] - \\ - \frac{2\Theta v}{r} + [Zu_z + wR_z] = -p_r + \delta^2\left[\frac{(rR_r)_r}{r} - \frac{R}{r^2} + \frac{R_{\theta\theta}}{r^2} - \frac{2\Theta_\theta}{r^2} + R_{zz}\right] + \\ + \delta^2\left[\frac{(ru_r)_r}{r} - \frac{u}{r^2} + \frac{u_{\theta\theta}}{r^2} - \frac{2v_\theta}{r^2} + w_{zz}\right], \tag{6b}$$

$$\dot\Theta + R\Theta_r + \frac{\Theta\Theta_\theta}{r} + \frac{R\Theta}{r} + Z\Theta_z + [u\Theta_r + Rv_r] + \frac{1}{r}[v\Theta_\theta + \Theta v_\theta] + \\ + \frac{1}{r}[u\Theta + Rv] + [w\Theta_z + Zv_z] = -\frac{p_\theta}{r} + \delta^2\left[\frac{(r\Theta_r)_r}{r} - \frac{\Theta}{r^2} + \frac{\Theta_{\theta\theta}}{r^2} + \frac{2R_\theta}{r^2} + \Theta_{zz}\right] + \\ + \delta^2\left[\frac{(rv_r)_r}{r} - \frac{v}{r^2} + \frac{v_{\theta\theta}}{r^2} + \frac{2u_\theta}{r^2} + v_{zz}\right], \tag{6c}$$

$$\dot{Z} + RZ_r + \frac{\Theta Z_\theta}{r} + ZZ_z + [uZ_r + Rw_r] + \frac{1}{r}[vZ_\theta + \Theta w_\theta] + [wZ_z + Zw_z] =$$
$$= -\,p_z + \delta^2\left[\frac{r(Z_r)_r}{r} + \frac{Z_{\theta\theta}}{r^2} + Z_{zz}\right] + \delta^2\left[\frac{r(w_r)_r}{r} + \frac{w_{\theta\theta}}{r^2} + w_{zz}\right] \qquad (6d)$$

in all internal points of $Q(t)$. The Navier – Stokes equation traditionally requires the non-slip $\boldsymbol{V} = \boldsymbol{0}$ condition on $S(t)$, but $\boldsymbol{v}$ adopts the slip condition $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ ($\boldsymbol{n}$ is the outward normal vector). The difference velocity field by (5) and (6), therefore, satisfies

$$(R, \Theta, Z) = \boldsymbol{V} - \boldsymbol{v} = (-v, -u, -w) \quad \text{on} \quad S(t), \qquad (7)$$

where the right-hand side is a known vector-function. The velocity fields $\boldsymbol{V}$ and $\boldsymbol{v}$ are close to each other on the $O(\delta)$ scale away from the wetted tank surface. The latter condition can be formalised as

$$\|(R, \Theta, Z)\| = \|\boldsymbol{V} - \boldsymbol{v}\| = O(\delta) \quad \text{in} \quad Q(t) \quad \text{at a distance} \quad d \gg \delta \quad \text{away from} \quad S(t) \qquad (8)$$

and, as a consequence, the difference velocity field $(\boldsymbol{V} - \boldsymbol{v})$ has the order $O(1)$ only in a $O(\delta)$-vicinity of the wetted tank surface. Our goal is to derive the boundary-value problems, which govern this zero-order component. The problems consist of boundary-layer equations defined on the wall ($r = k$) and the bottom ($z = -h$), respectively, with appropriate boundary conditions following from (7). In addition, one should require that this component vanishes away from the $O(\delta)$ boundary-layer zone.

To derive the *boundary-layer problem on the vertical wall*, we introduce the boundary-layer spatial variable ξ and consider

$$r = k - \delta\xi, \quad R = \delta R_1 + \ldots, \quad \Theta = \Theta_0 + \delta\Theta_1 + \ldots, \quad Z = Z_0 + \delta Z_1 + \ldots, \quad p = \delta p_1 + \ldots, \qquad (9)$$

where R, Θ, Z and p are now functions of $\xi, t;\, \theta, z$. Because the normal velocity is zero at $r = k$, $\Rightarrow\ R_0 = 0$. Equation (6b) (rewritten in the ξ, t, θ, z coordinates) shows that the pressure difference field has no zero-order component. Furthermore, using the rule $(\cdot)_\xi = -\delta(\cdot)_r$ for R, Θ and Z and keeping only the $O(1)$ terms derive

$$R_{1\xi} = Z_{0z} + \Theta_{0\theta}/k \qquad (10)$$

from (6a), but (6c) and (6d) transform to the two equations

$$\dot{\Theta}_0 - \Theta_{0\xi\xi} - R_1\Theta_{0\xi} + \frac{\Theta_0\Theta_{0\theta}}{k} + Z_0\Theta_{0z} + \xi\bar{u}_r\Theta_{0\xi} + \frac{1}{k}[\bar{v}\Theta_{0\theta} + \bar{v}_\theta\Theta_0] + [\bar{w}\Theta_{0z} + \bar{v}_z Z_0] = 0, \qquad (11a)$$

$$\dot{Z}_0 - Z_{0\xi\xi} - R_1 Z_{0\xi} + \frac{\Theta_0 Z_{0\theta}}{k} + Z_0 Z_{0z} + \xi\bar{u}_r Z_{0\xi} + \frac{1}{k}[\bar{v}Z_{0\theta} + \bar{w}_\theta\Theta_0] + [\bar{w}Z_{0z} + \bar{w}_z Z_0] = 0, \qquad (11b)$$

in which the bars denote projections of the known $\boldsymbol{v}$ and its derivatives on the wall (u, v and w and their derivatives are expanded in a Taylor series in δ) so that all coefficients in (11) are known time-depending functions, which parametrically depend on θ and z, i.e.,

$$\bar{u}_r(t;\theta,z) = u_r(k,\theta,z,t), \quad \bar{v}(t;\theta,z) = v(k,\theta,z,t), \quad \bar{v}_\theta(t;\theta,z) = v_\theta(k,\theta,z,t),$$
$$\bar{w}(t;\theta,z) = w(k,\theta,z,t), \quad \bar{v}_z(t;\theta,z) = w_z(k,\theta,z,t).$$

According to (7), the time-periodic (steady-state) solution of (11) satisfies the inhomogeneous boundary conditions

$$R_1(0,t;\theta,z) = 0, \quad \Theta_0(0,t;\theta,z) = -\bar{v}(t;\theta,z), \quad Z_0(0,t;\theta,z) = -\bar{w}(t;\theta,z). \tag{12}$$

Because Θ_0, $Z_0 = O(1)$, but R_1 corresponds to the first-order approximation in (9), the asymptotic condition (8) transforms to the form

$$|\Theta_0| + |Z_0| \to 0 \quad \text{and} \quad |R_1| \to O(1) \quad \text{as} \quad \xi \to +\infty. \tag{13}$$

The nonlinear boundary-value problem (10) – (13) is formulated with respect to Θ_0 and Z_0 on the inner points of the wetted tank wall, namely, for $-\infty < t < +\infty$, $\xi > 0$, and $-\pi \le \theta < \pi$, $-h < z < f(r,\theta,t)$.

Proceeding in a similar way for the bottom with

$$z = -h + \delta\xi,\ R = R_0 + \delta R_1 + \ldots,\ \Theta = \Theta_0 + \delta\Theta_1 + \ldots,\ Z = \delta Z_1 + \ldots,\ p = \delta p_1 + \ldots \tag{14}$$

leads to

$$Z_{1\xi} = -\frac{(rR_0)_r + \Theta_{0\theta}}{r}, \tag{15a}$$

$$\dot{R}_0 - R_{0\xi\xi} + R_0R_{0r} + \frac{\Theta_0 R_{0\theta}}{r} - \frac{\Theta_0^2}{r} + Z_1R_{0\xi} + [\bar{u}R_{0r} + R_0\bar{u}_r] + \\ + \frac{\bar{v}R_{0\theta} + \Theta_0\bar{u}_\theta}{r} - \frac{2\Theta_0\bar{v}}{r} + \xi\bar{w}_zR_{0\xi} = 0, \tag{15b}$$

$$\dot{\Theta}_0 - \Theta_{0\xi\xi} + R_0\Theta_{0r} + \frac{\Theta_0\Theta_{0\theta}}{r} + \frac{R_0\Theta_0}{r} + Z_1\Theta_{0\xi} + [\bar{u}\Theta_{0r} + R_0\bar{v}_r] + \\ + \frac{\bar{v}\Theta_{0\theta} + \Theta_0\bar{v}_\theta + \bar{u}\Theta_0 + R_0\bar{v}}{r} + \xi\bar{w}_z\Theta_{0\xi} = 0, \tag{15c}$$

where the bars denote projections of the known inviscid solution on the bottom ($z = -h$). We look for a time-periodic solution satisfying

$$R_0(0,t;r,\theta) = -\bar{u} = -u(r,\theta,-h,t),\ \Theta_0(0,t;r,\theta) = -\bar{v} = -v(r,\theta,-h,t),\ Z_1(0,t;r,\theta) = 0, \tag{16}$$

and

$$|\Theta_0| + |R_0| \to 0 \quad \text{and} \quad |Z_1| \to O(1) \quad \text{as} \quad \xi \to +\infty. \tag{17}$$

The nonlinear boundary-value problem (15) – (17) is formulated with respect to Θ_0 and R_0 as functions of $\xi > 0$, $-\infty < t < \infty$ and $r > 0$, $-\pi \le \theta < \pi$.

As matter of the above-done derivations, we have proved the following proposition.

Proposition 1. *Assuming $\delta = 1/\sqrt{R_s} \ll 1$, the $O(1)$-order difference between viscous $\boldsymbol{V}$ and inviscid $\boldsymbol{v}$ velocity fields of the steady-state sloshing problem in a circular-base container is* *localised in a $O(\delta)$-neighbourhood of the wetted tank surface and governed by the boundary-layer problems (10) – (13) and (15) – (17).*

3. An asymptotic solution of (10) – (13) and (15) – (17). Assuming an *inviscid irrotational* flow, a steady-state (periodic) solution of the corresponding free-surface problem was derived in [6] utilising the Narimanov – Moiseev asymptotic scheme [9, 10, 12]. According to the scheme, the lowest-order (dominant) solution component is associated with the primary excited natural sloshing modes and has the order $O(\epsilon^{1/3})$, unless a secondary resonance occurs [4, 6]; $O(\epsilon)$ characterises the nondimensinal forcing amplitude (2).

Because the natural sloshing modes imply potential flows, the Narimanov – Moiseev scheme for the inviscid *rotational* hydrodynamic model should not change the lowest-order asymptotic component (by primary-excited natural sloshing modes) and, therefore, the velocity field $\boldsymbol{v}(r,\theta,z,t)$ reads as (see [6] for the adopted normalization (1))

$$\boldsymbol{v} = u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{z}} = \underbrace{\cos t\,\nabla[J_1(r)\mathcal{Z}(z)\tau_c(\theta)] + \sin t\,\nabla[J_1(r)\mathcal{Z}(z)\tau_s(\theta)]}_{\boldsymbol{v}^{(1/3)}=\left(u^{(1/3)},v^{(1/3)},w^{(1/3)}\right)=O(\epsilon^{1/3})} +$$

$$+ \underbrace{\boldsymbol{w}(r,\theta,z) + \cos 2t\,\boldsymbol{w}_c(r,\theta,z) + \sin 2t\,\boldsymbol{w}_s(r,\theta,z)}_{\boldsymbol{v}^{(2/3)}=\left(u^{(2/3)},v^{(2/3)},w^{(2/3)}\right)=O(\epsilon^{2/3})} + O(\epsilon) \qquad (18)$$

in the unperturbed liquid domain Q_0 (see Figure 1 (b)), where

$$\mathcal{Z}(z) = \cosh(z)/\sinh h, \quad \tau_c(\theta) = b\sin\theta + \bar{a}\cos\theta, \quad \tau_s(\theta) = -\bar{b}\sin\theta - a\cos\theta \qquad (19)$$

($h = O(1)$ is the nondimensional liquid depth) and the nondimensional amplitude parameters $a, \bar{a}, \bar{b}, b = O(\epsilon^{1/3})$ are taken from [6] as a solution of the corresponding (secular) system of nonlinear algebraic equations. Generally speaking, the second-order velocity component $\boldsymbol{v}^{(2/3)}$ is a quadratic function of the nondimensional amplitude parameters. Specifically, the stationary part

$$\boldsymbol{w}(r,\theta,z) = (w_1(r,\theta,z), w_2(r,\theta,z), w_3(r,\theta,z)) = O\left(\epsilon^{2/3}\right)$$

cannot appear within the framework of the potential flow theory [6, 7, 11]; it corresponds to a global vortex stream in Q_0, that is,

$$\boldsymbol{\omega}(r,\theta,z) = (\omega_1,\omega_2,\omega_3) = \nabla\times\boldsymbol{w} \neq \boldsymbol{0}. \qquad (20)$$

Utilising (18), an asymptotic solution of the boundary-value problems (10) – (13) and (15) – (17) can be constructed in term of $O(\epsilon^{1/3}) \ll 1$. One should remember that the boundary-layer problems neglect the $O(\delta)$ quantities which should, therefore, be asymptotically smaller than $\epsilon^{2/3}$,

$$\delta \lesssim \epsilon^{2/3}. \qquad (21)$$

To get an asymptotic solution of (10) – (13), we introduce

$$\Theta_0 = \Theta_0^{(1/3)} + \Theta_0^{(2/3)} + \ldots, \quad Z_0 = Z_0^{(1/3)} + Z_0^{(2/3)} + \ldots, \quad R_1 = R_1^{(1/3)} + R_1^{(2/3)} + \ldots \qquad (22)$$

and consider a sequence of linear boundary-value problems for $\xi > 0$, $-\infty < t < \infty$ and $-h < z < 0$, $-\pi \leq \theta < \pi$.

The first-order approximation of (10) – (12) gives the linear parabolic problems ($\xi > 0$, $-\infty < t < \infty$):

$$\dot{\Theta}_0^{(1/3)} - \Theta_{0\xi\xi}^{(1/3)} = 0, \quad \Theta_0^{(1/3)}(0,\theta,z,t) = -\frac{J_1(k)}{k}\mathcal{Z}(z)\left[\cos t\,\tau_c'(\theta) + \sin t\,\tau_s'(\theta)\right], \tag{23a}$$

$$\dot{Z}_0^{(1/3)} - Z_{0\xi\xi}^{(1/3)} = 0, \quad Z_0^{(1/3)}(0,\theta,z,t) = -J_1(k)\mathcal{Z}'(z)\left[\cos t\,\tau_c(\theta) + \sin t\,\tau_s(\theta)\right], \tag{23b}$$

which consists of the two independent linear Stokes boundary-layer equations [2] parametrically dependent on $-h < z < 0$ and $-\pi \le \theta < \pi$. General time-periodic solution of (23) reads, according to § 3.1.1 in [14], as

$$\Theta_0^{(1/3)}(\xi,t;\theta,z) = -\frac{J_1(k)}{k}\,\mathcal{Z}(z)\exp(-\alpha\xi)\left[\tau_c'(\theta)\cos(t-\alpha\xi) + \tau_s'(\theta)\sin(t-\alpha\xi)\right], \tag{24a}$$

$$Z_0^{(1/3)}(\xi,t;\theta,z) = -J_1(k)\,\mathcal{Z}'(z)\exp(-\alpha\xi)\left[\tau_c(\theta)\cos(t-\alpha\xi) + \tau_s(\theta)\sin(t-\alpha\xi)\right] \tag{24b}$$

($\alpha = 1/\sqrt{2}$). Substituting (24) into the continuity equation (10) and using the first boundary condition of (12) give

$$R_1^{(1/3)}(\xi,t;\theta,z) = \int\limits_0^{\xi}\left(Z_{0z}^{(1/3)} + \Theta_{0\theta}^{(1/3)}/k\right)d\xi = -\frac{1}{2\alpha}\,J_1(k)\mathcal{Z}(z)\left(1 - \frac{1}{k^2}\right)\times$$
$$\times\Big\{\tau_c(\theta)\big[\sin t + \cos t - e^{-\alpha\xi}(\sin(t-\alpha\xi) + \cos(t-\alpha\xi))\big] +$$
$$+\,\tau_s(\theta)\big[\sin t - \cot t - e^{-\alpha\xi}(\sin(t-\alpha\xi) - \cos(t-\alpha\xi))\big]\Big\}. \tag{25}$$

One can see that $\left|R_1^{(1/3)}\right| \to O\left(\epsilon^{1/3}\right)$ and $\Theta_0^{(1/3)} \sim Z_0^{(1/3)} \to 0$ as $\xi \to +\infty$ that is consistent with (13) on the $O(\delta)$ asymptotic scale restricted to (21).

Expressions (24) and (25) should be inserted into (11) that leads to the following inhomogeneous parabolic equations with respect to $\Theta_0^{(2/3)}$ and $Z_0^{(2/3)}$:

$$\dot{\Theta}_0^{(2/3)} - \Theta_{0\xi\xi}^{(2/3)} = R_1^{(1/3)}\Theta_{0\xi}^{(1/3)} - \frac{\Theta_0^{(1/3)}\Theta_{0\theta}^{(1/3)}}{k} - Z_0^{(1/3)}\Theta_{0z}^{(1/3)} - \xi\bar{u}_r^{(1/3)}\Theta_{0\xi}^{(1/3)} -$$
$$-\frac{1}{k}\left[\bar{v}^{(1/3)}\Theta_{0\theta}^{(1/3)} + \bar{v}_\theta^{(1/3)}\Theta_0^{(1/3)}\right] - \left[\bar{w}^{(1/3)}\Theta_{0z}^{(1/3)} + \bar{v}_z^{(1/3)}Z_0^{(1/3)}\right] =$$
$$= G_\Theta(\xi,t;\theta,z) = e^{-\alpha\xi}\left[G_\Theta^{(0)}(\xi;\theta,z) + \cos 2t\,G_\Theta^{(c)}(\xi;\theta,z) + \sin 2t G_\Theta^{(s)}(\xi;\theta,z)\right], \tag{26a}$$

$$\dot{Z}_0^{(2/3)} - Z_{0\xi\xi}^{(2/3)} = R_1^{(1/3)}Z_{0\xi}^{(1/3)} - \frac{\Theta_0^{(1/3)}Z_{0\theta}^{(1/3)}}{k} - Z_0^{(1/3)}Z_{0z}^{(1/3)} - \xi\bar{u}_r^{(1/3)}Z_{0\xi}^{(1/3)} -$$
$$-\frac{1}{k}\left[\bar{v}^{(1/3)}Z_{0\theta}^{(1/3)} + \bar{w}_\theta^{(1/3)}\Theta_0^{(1/3)}\right] - \left[\bar{w}^{(1/3)}Z_{0z}^{(1/3)} + \bar{w}_z^{(1/3)}Z_0^{(1/3)}\right] =$$
$$= G_Z(\xi,t;\theta,z) = e^{-\alpha\xi}\left[G_Z^{(0)}(\xi;\theta,z) + \cos 2tG_Z^{(c)}(\xi;\theta,z) + \sin 2t\,G_Z^{(s)}(\xi;\theta,z)\right], \tag{26b}$$

where the right-hand sides being the known functions

$$
\begin{aligned}
G_\Theta(\xi,t;\theta,z) \;=\; & -\frac{J_1^2(k)}{2k}\,e^{-\alpha\xi}\Bigg\{\Big[(\tau_c'\tau_c+\tau_s'\tau_s)\,(2\cos\alpha\xi-e^{-\alpha\xi})\Big]+\\
& +\mathcal{Z}\Bigg[\frac{k^2-1}{k^2}\Big((\tau_c'(\tau_c+\tau_s)-\tau_s'(\tau_c-\tau_s))e^{-\alpha\xi}+\\
& +(-\tau_c'(2\tau_c+\tau_s)+\tau_s'(\tau_c-2\tau_s))\cos\alpha\xi+\big(\tau_c'\tau_c+\tau_s'\tau_s\big)\sin\alpha\xi\Big)+\\
& +\frac{J_1''(k)}{J_1(k)}\,\alpha\xi\Big((\tau_c'(\tau_c-\tau_s)+\tau_s'(\tau_c+\tau_s))\cos\alpha\xi+\big(\tau_c'(\tau_c+\tau_s)-\tau_s'(\tau_c-\tau_s)\big)\sin\alpha\xi\Big)\Bigg]+\\
& +\cos 2t\,\Bigg\{\Big[(\tau_c'\tau_c-\tau_s'\tau_s)(e^{-\alpha\xi}(1-2\cos^2\alpha\xi)+2\cos\alpha\xi)+\\
& +2(\tau_s'\tau_c+\tau_c'\tau_s)\sin\alpha\xi(e^{-\alpha\xi}\cos\alpha\xi-1)\Big]+\\
& +\mathcal{Z}\Bigg[\frac{k^2-1}{k^2}\Big((\tau_s'\tau_c+\tau_c'\tau_s)\sin\alpha\xi+(\tau_s'\tau_s-\tau_c'\tau_c)\cos\alpha\xi\Big)+\\
& +\frac{J_1''(k)}{J_1(k)}\,\alpha\xi\,\Big((\tau_c'(\tau_c+\tau_s)+\tau_s'(\tau_c-\tau_s))\cos\alpha\xi+(\tau_c'(\tau_c-\tau_s)-\tau_s'(\tau_c+\tau_s))\sin\alpha\xi\Big)\Bigg]\Bigg\}+\\
& +\sin 2t\Bigg\{\Bigg[(\tau_s'\tau_c+\tau_c'\tau_s)(e^{-\alpha\xi}(1-2\cos^2\alpha\xi)+2\cos\alpha\xi)-\\
& -2\,\big(\tau_c'\tau_c-\tau_s'\tau_s\big)\sin\alpha\xi\,\Big(e^{-\alpha\xi}\cos\alpha\xi-1\Big)\Bigg]+\\
& +\mathcal{Z}\Bigg[\frac{k^2-1}{k^2}\Bigg((\tau_s'\tau_s-\tau_c'\tau_c)\sin\alpha\xi-(\tau_s'\tau_c+\tau_s'\tau_s)\cos\alpha\xi\Bigg)-\frac{J_1''(k)}{J_1(k)}\,\alpha\xi\times\\
& \times\Bigg((\tau_c'(\tau_c-\tau_s)-\tau_s'(\tau_c+\tau_s))\cos\alpha\xi-(\tau_c'(\tau_c+\tau_s)+\tau_s'(\tau_c-\tau_s))\sin\alpha\xi\Bigg)\Bigg]\Bigg\}\Bigg\},
\end{aligned}
\tag{27a}
$$

$$
\begin{aligned}
G_Z(\xi,t;\theta,z) = & -\frac{J_1^2(k)}{2k^2}\,\mathcal{Z}\mathcal{Z}'\,e^{-\alpha\xi}\Bigg\{\Bigg[((\tau_c^2+\tau_s^2)k^2+\tau_c'^2+\tau_s'^2)(e^{-\alpha\xi}-2\cos\alpha\xi)+\\
& +(k^2-1)(\tau_c^2+\tau_s^2)\sin\alpha\xi+\frac{k^2J_1''(k)}{J_1(k)}\alpha\xi(\tau_c^2+\tau_s^2)(\sin\alpha\xi+\cos\alpha\xi)\Bigg]+\\
& +\cos 2t\Bigg[(-\tau_c^2-\tau_c'^2+\tau_s^2+\tau_s'^2)(e^{-\alpha\xi}-2\cos^2\alpha\xi)+
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\alpha\xi((k^2+1)(-\tau_c^2+\tau_s^2)-2(\tau_c'^2-\tau_s'^2))+\\
&+2\sin\alpha\xi(-2e^{-\alpha\xi}\cos\alpha\xi(\tau_c\tau_s+\tau_c'\tau_s')+\tau_s(k^2+1)\tau_c+2\tau_c'\tau_s')+\\
&+\left.\frac{k^2\alpha J_1''(k)}{J_1(k)}\,\xi\,\left((\tau_c^2-2\tau_c\tau_s-\tau_s^2)\sin\alpha\xi+(\tau_c^2+2\tau_c\tau_s-\tau_s^2)\cos\alpha\xi\right)\right]-\\
&-\sin 2t\left[(2\tau_c\tau_s+2\tau_c'\tau_s')(e^{-\alpha\xi}-2\cos^2\alpha\xi)+2\left(\tau_s\tau_c(k^2+1)+2\tau_c'\tau_s'\right)\cos\alpha\xi+\right.\\
&+\sin\alpha\xi\Big(2(-\tau_c^2-\tau_c'^2+\tau_s^2+\tau_s'^2)e^{-\alpha\xi}\cos\alpha\xi+(k^2+1)(\tau_c^2-\tau_s^2)+2\tau_c'^2-2\tau_s'^2\Big)+\\
&+\left.\left.\frac{k^2\alpha J_1''(k)}{J_1(k)}\,\xi\left((\tau_c^2-2\tau_c\tau_s-\tau_s^2)\cos\alpha\xi-(\tau_c^2+2\tau_c\tau_s-\tau_s^2)\sin\alpha\xi\right)\right]\right\}. \qquad (27b)
\end{aligned}
$$

The time-periodic solution of (26) should decay at the infinity,

$$
\Theta_0^{(2/3)}(\xi,t;\theta,z,t)\to 0 \quad \text{and} \quad \mathbb{Z}_0^{(2/3)}(\xi,\theta,z,t)\to 0 \quad \text{as} \quad \xi\to+\infty, \qquad (28)
$$

and satisfy the boundary conditions

$$
\Theta_0^{(2/3)}(0,t;\theta,z)=-\cos 2t\,w_{c2}(k,\theta,z)-\sin 2t\,w_{s2}(k,\theta,z)-w_2(k,\theta,z), \qquad (29a)
$$

$$
Z_0^{(2/3)}(0,t;\theta,z)=-\cos 2t\,w_{c3}(k,\theta,z)-\sin 2t\,w_{s3}(k,\theta,z)-w_3(k,\theta,z). \qquad (29b)
$$

Huge and very tedious derivations (Maple$^{(\mathrm{TM})}$ was employed to simplify them) make it possible to get an exact analytical solution of (26) – (29) in the form

$$
\begin{aligned}
\Theta_0^{(2/3)}(\xi,t;\theta,z)&=\Theta_{00}^{(2/3)}(\xi;\theta,z)+\underbrace{\Theta_{0c}^{(2/3)}(\xi;\theta,z)\,\cos 2t+\Theta_{0s}^{(2/3)}(\xi;\theta,z)\,\sin 2t}_{\Theta_{0p}^{(2/3)}(\xi,t;\theta,z)},\\
Z_0^{(2/3)}(\xi,t;\theta,z)&=Z_{00}^{(2/3)}(\xi;\theta,z)+\underbrace{Z_{0c}^{(2/3)}(\xi;\theta,z)\,\cos 2t+Z_{0s}^{(2/3)}(\xi;\theta,z)\,\sin 2t}_{Z_{0p}^{(2/3)}(\xi,t;\theta,z)}.
\end{aligned} \qquad (30)
$$

The oscillatory component is the sum

$$
\begin{aligned}
\Theta_{0p}^{(2/3)}(\xi,t;\theta,z)&=\hat\Theta_{0p}^{(2/3)}(\xi,t;\theta,z)+\bar\Theta_{0p}^{(2/3)}(\xi,t;\theta,z),\\
Z_{0p}^{(2/3)}(\xi,t;\theta,z)&=\hat Z_{0p}^{(2/3)}(\xi,t;\theta,z)+\bar Z_{0p}^{(2/3)}(\xi,t;\theta,z),
\end{aligned}
$$

where, according to § 3.1.1 in [14],

$$
\begin{aligned}
\hat\Theta_{0p}^{(2/3)}(\xi,t;\theta,z)&=-e^{-\xi}\left[w_{c2}(k,\theta,z)\cos(-\xi+2t)+w_{s2}(k,\theta,z)\sin(-\xi+2t)\right],\\
\hat Z_{0p}^{(2/3)}(\xi,t;\theta,z)&=-e^{-\xi}\left[w_{c3}(k,\theta,z)\cos(-\xi+2t)+w_{s3}(k,\theta,z)\sin(-\xi+2t)\right]
\end{aligned} \qquad (31)
$$

(corresponds to $G_\Theta = G_Z = 0$ in (26)) and ([14], § 4)

$$\bar{\Theta}_{0p}^{(2/3)}(\xi, t; \theta, z) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} e^{-\alpha\xi}\left[\cos 2(t-t_1)G_\Theta^{(c)}(x;\theta,z) + \sin 2(t-t_1)G_\Theta^{(s)}(x;\theta,z)\right]\times$$
$$\times\, \mathcal{K}(\xi, x, t_1)dxdt_1, \tag{32a}$$

$$\bar{Z}_{0p}^{(2/3)}(\xi, t; \theta, z) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} e^{-\alpha\xi}\left[\cos 2(t-t_1)\,G_Z^{(c)}(x;\theta,z) + \sin 2(t-t_1)G_Z^{(s)}(x;\theta,z)\right]\times$$
$$\times\, \mathcal{K}(\xi, x, t_1)dxdt_1, \tag{32b}$$

where

$$\mathcal{K}(\xi, x, t_1) = \frac{1}{2\sqrt{\pi t_1}}\left[\exp\left(-\frac{(\xi-x)^2}{4t_1}\right) - \exp\left(-\frac{(\xi+x)^2}{4t_1}\right)\right] \tag{33}$$

($\bar{\Theta}_{0p}^{(2/3)}$ and $\bar{Z}_{0p}^{(2/3)}$ exactly satisfy the zero boundary condition on $\xi = 0$).

Direct analysis of the obtained analytical expressions shows that

$$\left|\bar{\Theta}_{0p}^{(2/3)}\right| \sim \left|\bar{Z}_{0p}^{(2/3)}\right| \sim O\left(\xi e^{-\alpha\xi}\right) \quad \text{and} \quad \left|\hat{\Theta}_{0p}^{(2/3)}\right| \sim \left|\hat{Z}_{0p}^{(2/3)}\right| \sim O\left(e^{-\xi}\right) \quad \text{as} \quad \xi \to +\infty.$$

This means that $\Theta_{0p}^{(2/3)}$, $Z_{0p}^{(2/3)}$ *automatically satisfy* (28) for any oscillatory velocity field components $\boldsymbol{w}_c$ and $\boldsymbol{w}_s$ in (18).

The time-independent component of (30) comes from the equations

$$\left(\Theta_{00}^{(2/3)}\right)_{\xi\xi} = \frac{J_1^2(k)}{2k}\,e^{-\alpha\xi}\Bigg\{(\bar{a}\bar{b} - ab)\mathcal{Z}(z)\Bigg[\frac{k^2-1}{k^2}\,\left(e^{-\alpha\xi} - \cos\alpha\xi\right) +$$
$$+\,\frac{J_1''(k)}{J_1(k)}\alpha\xi\big(\sin\alpha\xi - \cos\alpha\xi\big)\Bigg] + \Bigg[2\cos\alpha\xi - e^{-\alpha\xi} +$$
$$+\,\mathcal{Z}(z)\left(\frac{k^2-1}{k^2}\,\left(e^{-\alpha\xi} - 2\cos\alpha\xi + \sin\alpha\xi\right) + \frac{J_1''(k)}{J_1(k)}\,\alpha\xi(\cos\alpha\xi + \sin\alpha\xi)\right)\Bigg]\times$$
$$\times\left[(a\bar{b} + \bar{a}b)\cos 2\theta + \frac{1}{2}\,\left(b^2 + \bar{b}^2 - a^2 - \bar{a}^2\right)\sin 2\theta\right]\Bigg\}, \tag{34a}$$

$$\left(Z_{00}^{(2/3)}\right)_{\xi\xi} = \frac{J_1^2(k)}{2k^2}\,\mathcal{Z}(z)\mathcal{Z}'(z)\,e^{-\alpha\xi}\Bigg\{\frac{1}{2}\,\left(a^2+\bar{a}^2+b^2+\bar{b}^2\right)\times$$
$$\times\left(\left(k^2+1\right)\left(e^{-\alpha\xi}-2\cos\alpha\xi\right)+\left(k^2-1\right)\sin\alpha\xi+\frac{k^2J_1''(k)}{J_1(k)}\,\alpha\xi(\sin\alpha\xi+\cos\alpha\xi)\right)+$$
$$+\left(\frac{1}{2}\,\left(a^2+\bar{a}^2-b^2-\bar{b}^2\right)\cos 2\theta+(a\bar{b}+\bar{a}b)\sin 2\theta\right)\times$$
$$\times\left(\left(k^2-1\right)\left(e^{-\alpha\xi}-2\cos\alpha\xi+\sin\alpha\xi\right)+\frac{k^2J_1''(k)}{J_1(k)}\,\alpha\xi(\sin\alpha\xi+\cos\alpha\xi)\right)\Bigg\}, \tag{34b}$$

whose fundamental solution is

$$\Theta_{00}^{(2/3)} = \frac{J_1^2(k)}{4k\alpha^2}\,e^{-\alpha\xi}\Bigg\{(\bar{a}\bar{b}-ab)\mathcal{Z}^2(z)\left[\frac{k^2-1}{2k^2}\,\left(e^{-\alpha\xi}+2\sin\alpha\xi\right)+\right.$$
$$\left.+\frac{J_1''(k)}{J_1(k)}\,(2\cos\alpha\xi+\alpha\xi(\sin\alpha\xi+\cos\alpha\xi))\right]-\left[-\frac{1}{2}\,\left(e^{-\alpha\xi}+4\sin\alpha\xi\right)+\right.$$
$$+\mathcal{Z}^2(z)\left(\frac{k^2-1}{2k^2}\,\left(e^{-\alpha\xi}+2\cos\alpha\xi+4\sin\alpha\xi\right)+\right.$$
$$\left.\left.+\frac{J_1''(k)}{J_1(k)}\,(\alpha\xi(\cos\alpha\xi-\sin\alpha\xi)-2\sin\alpha\xi)\right)\right]\times$$
$$\times\left[(a\bar{b}+\bar{a}b)\cos 2\theta+\frac{1}{2}\,\left(b^2+\bar{b}^2-a^2-\bar{a}^2\right)\sin 2\theta\right]\Bigg\}+C_0(\theta,z)+\xi\,C_1(\theta,z), \tag{35a}$$

$$Z_{00}^{(2/3)} = \frac{J_1^2(k)}{4k^2\alpha^2}\,\mathcal{Z}(z)\mathcal{Z}'(z)\,e^{-\alpha\xi}\Bigg\{\frac{1}{2}\,\left(a^2+\bar{a}^2+b^2+\bar{b}^2\right)\times$$
$$\times\left[\frac{1}{2}\left(\left(k^2+1\right)\left(4\sin\alpha\xi+e^{-\alpha\xi}\right)+2(k^2-1)\cos\alpha\xi\right)+\right.$$
$$\left.+\frac{k^2J_1''(k)}{J_1(k)}\,(\alpha\xi(\cos\alpha\xi-\sin\alpha\xi)-2\sin\alpha\xi)\right]+$$
$$+\left(\frac{1}{2}\,\left(a^2+\bar{a}^2-b^2-\bar{b}^2\right)\cos 2\theta+(a\bar{b}+\bar{a}b)\sin 2\theta\right)\times$$
$$\times\left(\frac{1}{2}\,\left(k^2-1\right)\left(e^{-\alpha\xi}+2\cos\alpha\xi+4\sin\alpha\xi\right)-\right.$$

$$- \frac{k^2 J_1''(k)}{J_1(k)} \left(2\sin\alpha\xi + \alpha\xi(\sin\alpha\xi - \cos\alpha\xi)\right)\Bigg)\Bigg\} + C_2(\theta, z) + \xi\, C_3(\theta, z). \tag{35b}$$

Obviously C_1, C_2, C_3 and C_4 must be zero to satisfy (28). A use (29) and (35) with $C_i = 0$ leads to the *necessary boundary condition* for the stationary vortical component on the wall

$$w_1(k, \theta, z) = 0, \tag{36a}$$

$$w_2(k,\theta,z) = -\frac{J_1^2(k)}{2k}\Bigg\{(\bar{a}\bar{b} - ab)\left(\frac{k^2-1}{2k^2} + 2\,\frac{J_1''(k)}{J_1(k)}\right)\mathcal{Z}^2(z) - \\ - \left(-\frac{1}{2} + \frac{3}{2}\,\frac{k^2-1}{k^2}\,\mathcal{Z}^2(z)\right)\left[(a\bar{b} + \bar{a}b)\cos 2\theta + \frac{1}{2}\,\left(b^2 + \bar{b}^2 - a^2 - \bar{a}^2\right)\sin 2\theta\right]\Bigg\}, \tag{36b}$$

$$w_3(k,\theta,z) = -\frac{J_1^2(k)}{4k^2}\,\mathcal{Z}(z)\mathcal{Z}'(z)\Bigg\{\frac{1}{2}\,\left(a^2 + \bar{a}^2 + b^2 + \bar{b}^2\right)\left(3k^2 - 1\right) + \\ + 3\left(k^2 - 1\right)\left[\frac{1}{2}\,\left(a^2 + \bar{a}^2 - b^2 - \bar{b}^2\right)\cos 2\theta + (a\bar{b} + \bar{a}b)\sin 2\theta\right]\Bigg\}. \tag{36c}$$

Proceeding in a similar way with the boundary-layer problem (15)–(17) we get another *necessary boundary condition* on the bottom:

$$w_1(r,\theta,-h) = -\frac{1}{8}\,\mathcal{Z}^2(-h)\Bigg\{\left(a^2 + \bar{a}^2 + b^2 + \bar{b}^2\right)\frac{-r^2 J_1'^2(r) + rJ_1'(r)J_1(r)(2 - 3r^2) - J_1^2(r)}{r^3} + \\ + \left[\left(a^2 + \bar{a}^2 - b^2 - \bar{b}^2\right)\cos 2\theta + 2(a\bar{b} + \bar{a}b)\sin 2\theta\right]\frac{-r^2 J_1'^2(r) - 3r^3 J_1'(r)J_1(r) + J_1^2(r)}{r^3}\Bigg\}, \tag{37a}$$

$$w_2(r,\theta,-h) = \frac{1}{4}\,\mathcal{Z}^2(-h)\Bigg\{3(ab - \bar{a}\bar{b})\,\frac{J_1^2(r)}{r} + \\ + \left[-(a\bar{b} + \bar{a}b)\cos 2\theta + \frac{1}{2}\,\left(a^2 + \bar{a}^2 - b^2 - \bar{b}^2\right)\sin 2\theta\right]\frac{r^2 J_1'^2(r) - J_1^2(r)(1 + 2r^2)}{r^3}\Bigg\}, \tag{37b}$$

$$w_3(r,\theta,-h) = 0. \tag{37c}$$

Remark. The necessary solvability condition of the boundary-layer problems (10)–(13) and (15)–(17), which are based on the inviscid resonant Narimanov–Moisev-type solution (18), consists of satisfying the inhomogeneous boundary conditions (36) and (37) for the stationary vortical stream component $\boldsymbol{w} = O(\epsilon^{2/3})$. To find the stationary stream by $\boldsymbol{w}$, one must also derive the corresponding governing equation (with respect to $\boldsymbol{w}$) in the mean liquid domain Q_0.

4. Governing equation for $\boldsymbol{w}$. Because the velocity field $\boldsymbol{v}$ can always be restored from the vortex $\boldsymbol{\Omega} = \nabla \times \boldsymbol{v}$ by using the Biot–Savart law, we employ the vorticity equation (inviscid incompressible flows)

$$\dot{\boldsymbol{\Omega}} = \nabla \times [\boldsymbol{v} \times \boldsymbol{\Omega}] \quad \text{with} \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_{1/3} + \boldsymbol{\Omega}_{2/3} + \ldots \tag{38}$$

and (18). Since $\boldsymbol{\Omega}_{1/3} = \mathbf{0}$,

$$\begin{aligned}\boldsymbol{\Omega}_{3/3}(r,\theta,z,t) = (a\cos t + \bar{a}\sin t)\,\nabla \times [\boldsymbol{v}_a(r,\theta,z) \times \boldsymbol{\omega}(r,\theta,z)] + \\ + \left(\bar{b}\cos t + b\sin t\right)\nabla \times [\boldsymbol{v}_b(r,\theta,z) \times \boldsymbol{\omega}(r,\theta,z)]\,,\end{aligned} \tag{39}$$

where

$$\begin{aligned}\boldsymbol{v}_a(r,\theta,z) &= \left(J_1'(r)\cos\theta\,\mathcal{Z}(z), -\frac{J_1(r)}{r}\,\sin\theta\mathcal{Z}(z), J_1(r)\cos\theta\mathcal{Z}'(z)\right),\\ \boldsymbol{v}_b(r,\theta,z) &= \left(J'(r)\sin\theta\mathcal{Z}(z), \,\frac{J_1(r)}{r}\,\cos\theta\mathcal{Z}(z), J_1(r)\sin\theta\mathcal{Z}'(z)\right).\end{aligned} \tag{40}$$

Furthermore, inserting (39) into (38) gives the time-averaged (4/3)-approximation

$$\mathbf{0} = \left\langle \dot{\boldsymbol{\Omega}}_{4/3} \right\rangle = \left\langle \nabla \times \left[\boldsymbol{v}_{1/3} \times \boldsymbol{\Omega}_{3/3}\right]\right\rangle + \left\langle \nabla \times \left[\boldsymbol{v}_{2/3} \times \boldsymbol{\Omega}_{2/3}\right]\right\rangle, \tag{41}$$

which leads to the *necessary solvability* condition

$$\nabla \times [\boldsymbol{w} \times \boldsymbol{\omega}] + \frac{1}{2}\,(ab - \bar{a}\bar{b})\,\nabla \times [\boldsymbol{v}_b \times \operatorname{rot}(\boldsymbol{v}_a \times \boldsymbol{\omega}) - \boldsymbol{v}_a \times \operatorname{rot}(\boldsymbol{v}_b \times \boldsymbol{\omega})] \quad \text{in } Q_0, \tag{42}$$

where $\operatorname{div}\boldsymbol{\omega} = \nabla \cdot [\nabla \times \boldsymbol{w}] \equiv 0$ and $\boldsymbol{v}_b, \boldsymbol{v}_a$ are defined by (40). This condition plays the role of a governing equation for $\boldsymbol{w}$ within the framework of the Narimanov–Moiseev asymptotic approximation (18).

5. Conclusions. The boundary-layer problems describing a local viscous flow at the wetted tank surface are derived assuming $\delta = 1/\sqrt{R_s} \ll 1$, where R_s is the sloshing-related Reynolds number. The problems govern the $O(1)$ difference between viscous and inviscid solutions, which only exists in the $O(\delta)$ neighbourhood of the wetted tank surface. By constructing and analysing the analytical asymptotic solution of the derived boundary-layer problems within the framework of the Narimanov–Moiseev-type approximation of the inviscid velocity field, we proved the following *main result*:

Proposition 2. *The inviscid Narimanov–Moiseev steady-state asymptotic solution (18) of the resonant sloshing problem in a circular-base tank [6] contains a non-zero global stationary vortical stream component* $\boldsymbol{w}$ ($\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \nabla \times \boldsymbol{w} \neq \mathbf{0}$), *which is governed by the nonlinear equation (rewriting (42))*

$$\begin{aligned}\nabla \times [\boldsymbol{w} \times \boldsymbol{\omega}] = \frac{1}{2}(ab - \bar{a}\bar{b})\Bigg\{ &\left[\mathcal{Z}^2(z)\,f(r) - \frac{J_1^2(r)}{r^2}\,\mathcal{Z}^2(-h)\right]\boldsymbol{\omega}_\theta + \\ &+ \hat{\boldsymbol{\theta}}\left(2\,\omega_1\left[3g(r)\,\mathcal{Z}^2(z) + g_1(r)\,\mathcal{Z}^2(-h)\right] - 2rf(r)\mathcal{Z}(z)\mathcal{Z}'(z)\omega_3\right)\Bigg\} \quad \textit{in } Q_0,\end{aligned}$$

where

$$g(r) = \frac{(rJ_1'(r) - J_1(r))^2}{r^4}, \quad f(r) = g(r) + \frac{J_1^2(r)}{r^2}, \quad g_1(r) = \frac{J_1(r)(rJ_1'(r) - J_1(r))}{r^2},$$

restricted to the inhomogeneous boundary conditions (36) and (37) on V_0 (the wall) and B_0 (the bottom), respectively.

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