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APPROXIMATION OF SOLUTIONS TO THE OPTIMAL CONTROL PROBLEMS FOR SYSTEMS WITH MAXIMUM

НАБЛИЖЕННЯ РОЗВ'ЯЗКІВ ОПТИМАЛЬНИХ ЗАДАЧ КЕРУВАННЯ ДЛЯ СИСТЕМ ІЗ МАКСИМУМОМ

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We consider optimal control problems for nonlinear systems which dynamics depends on the maximum of the control function and the maximum of the state over some prehistory time interval. We are interested in approximation of solutions for such kind of problems. Averaging method is developed for this purpose.

Розглянуто задачі оптимального керування нелінійними системами, динаміка яких залежить від максимуму функції керування та максимуму стану на деякому часовому інтервалі передісторії. Досліджено апроксимацію розв'язків таких задач. З цією метою обґрунтовано метод усереднення.

Introduction. In certain applications modeling controlled processes leads to functional differential equations with small parameters. Especially, if the corresponding system is nonlinear, an analytic solution is hardly possible to derive. In this case approximation methods can be helpful. There are two categories of approximation methods for analyzing nonlinear systems. One of them includes numerical methods, another one is a category of asymptotic methods. When we deal with an optimal control problem the choice of one or another asymptotic method depends on the structure of the differential equation. One possible option can be application of an averaging method. Initially, averaging methods were developed in [1, 2]. Its further generalization to the functional differential equation can be found in [3, 4]. However, the first time an averaging method was applied to controlled systems in [5]. There are two approaches for solving an optimal control problem by the averaging method:

– using the necessary condition of optimality one reduces the original optimal control problem to a boundary-value problem, which is solved by the averaging method;

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– applying averaging method directly to the original controlled problem and then solving the simplified averaged problem.

The first approach can be justified under an assumption that the control function is smooth enough. In the second approach one associates a nonautonomous controlled system with an autonomous one via the algorithm of averaging. It consists of the following steps:

(i) apply the averaging method to the controlled system;

(ii) establish a correspondence between controlled functions of both (averaged and original) systems;

(iii) estimate the quality of the control function of the averaged problem by a functional of the original problem.

The averaging method is used extensively for different classes of optimal control problems, for instance for ordinary differential equations [6], for systems with Hukuhara derivative [7], systems with impulsive actions [8], for differential inclusions [9] etc. In this paper we use the second approach mentioned above to obtain an averaging method for the optimal control problem described by differential equations with maximum of the state and input. Differential equations with max-operator is a particular case of the state dependent delay differential equations. We deal with the situation when maximum of the state is taken on the prehistory but only for $t \geq 0$. For the first time, such kind of equations appeared in applications to electric engineering problem in [10], and later in bioscience [11]. The issues of existence of a unique solution, stability of systems without input and oscillation theory are investigated in [12]. To the best of authors knowledge the problems with maximum and external input are not studied extensively. The stability properties of these systems with input in the linear form were studied in [13, 14]. In [15] the Lagrange approach is presented and the Pontryagin-like Minimum Principle is proved for optimal control problem with max-operator in the linear form. In this paper the averaged problem includes the maximum of the state however it is controlled by a control of the current time. Hence, the sets of control functions of the averaged and the original problems are different, that is why we need to have some tools to recover one control function from the other. For this reason, in the proposed algorithm we work with integral equations which include maximum.

The paper is organized as follows. In Section 2 we give some notations, in Section 3 an optimal control problem is introduced, then in Section 4 we develop an algorithm for the correspondence between control functions in the averaged and in the original system. In Section 5 we justify the application of the averaging method and, in the last section, functionals of the averaged and the original optimal control problems are compared.

2. Notation and preliminaries. For a piecewise continuous function $u\colon [0,\infty) \to \mathbb{R}^r$ and continuous functions $g, \gamma\colon [0,\infty) \to \mathbb{R}$, such that $0 \leq g(t) \leq \gamma(t) \leq t$ for any $t \geq 0$ we denote the componentwise "maximal" value of u over the time interval $[g(t), \gamma(t)]$ by

$$\widetilde{u}(t) = \left(\sup_{s \in [g(t),\gamma(t)]} u_1(s), \ldots, \sup_{s \in [g(t),\gamma(t)]} u_r(s) \right)^{\mathrm{T}} \tag{1}$$

and

$$\breve{x}(t) = \left(\max_{s \in [g(t),\gamma(t)]} x_1(s), \ldots, \max_{s \in [g(t),\gamma(t)]} x_n(s) \right)^{\mathrm{T}}.$$

For any function $f \in C([0,\infty);\mathbb{R}^n)$ and any matrix $A\colon [0,\infty) \to \mathbb{R}^{n\times n}$ we introduce

$$\|f(t)\| = \sup_{t\geq 0} |f(t)|, \qquad \|A(t)\| = \max_{1\leq i\leq n} \sum_{j=1}^{m} \sup_{t\geq 0} |a_{ij}(t)|.$$

Let X and Y be two nonempty subsets of $\mathbb{R}^n$. We define the Hausdorff distance between them by

$$d_H(X,Y) = \max\left\{ \sup_{x\in X} \inf_{y\in Y} \|x-y\|, \sup_{y\in Y} \inf_{x\in X} \|x-y\| \right\}.$$

The following notion of average will be used in this paper.

Definition 1 [16]. *A continuous bounded function $f\colon [0,\infty)\times D \to \mathbb{R}^n$ is said to have an average $\overline{f}(x)$ if the limit*

$$\overline{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,x)\,dt, \tag{2}$$

exists and for all $(t,x) \in [0,\infty)\times D'\times D'$

$$\left\| \frac{1}{T}\int_0^T f(t,x)dt - \overline{f}(x) \right\| \leq q\sigma(T),$$

for every compact set $D' \subset D$, where q is a positive constant (possibly dependent on D') and $\sigma\colon [0,\infty)\to[0,\infty)$ is a strictly decreasing, continuous, bounded function such that $\sigma(T)\to 0$ as $T\to\infty$. The function σ is called convergence function.

3. Optimal control by the system with maximum of control function. In this paper we consider the following optimal control problem with terminal functional:

$$\begin{gathered} \dot{x}(t) = \varepsilon\left[f(t,x(t),\check{x}(t)) + A(x(t))\zeta\left(t,u(t),\widetilde{u}(t)\right)\right], \quad t\geq 0, \\ x(0) = x_0, \end{gathered} \tag{3}$$

where $x\in\mathbb{R}^n$ is a phase vector; $\varepsilon > 0$ is a small parameter; $f\colon [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous, $L>0$ is some fixed constant; $A\in C([0,\infty);\mathbb{R}^{n\times m})$; $\zeta\colon [0,\infty)\times U\times U\to\mathbb{R}^r$ is a continuous function, u is a piecewise continuous function, with values $u(t)\in U\subset \mathrm{comp}\,(\mathbb{R}^r)$.

A solution to problem (3) is understood to be an absolutely continuous function $x = x(t)$, $t\geq 0$.

Let us consider the functional

$$J[u] = \Phi\left(x\left(L\varepsilon^{-1}\right)\right), \tag{4}$$

where $\Phi\colon \mathbb{R}^n\to\mathbb{R}$ is a continuous function.

A control function u^* which provides the minimum of functional (4) is called an optimal control function and the corresponding trajectory x^* is called an optimal trajectory. As an optimal solution to the problem (3), (4), we understand the pair $x^*,\ u^*$.

Let us consider the corresponding averaged problem for the problem (3),

$$\begin{gathered}\dot{y}(t) = \varepsilon\left(\overline{f}(y(t), \check{y}(t)) + A(y(t))v(t)\right), \quad t \ge 0, \\ y(t) = x_0\end{gathered} \tag{5}$$

with the functional

$$\overline{J}[v] = \Phi\left(y\left(L\varepsilon^{-1}\right)\right), \tag{6}$$

where $v \in V$ is a new control vector and the set V is defined as

$$V = \lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \zeta(t, U, U)dt = \left\{\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \zeta(t, u(t), \widetilde{u}(t))dt \colon u(t), \widetilde{u}(t) \in U\right\}. \tag{7}$$

In (7) we understand the integral of a set-valued function as Aumann integral [17], convergence we understand in the sense of Hausdorff metric.

Sufficient conditions for existence of a unique solution to the Cauchy problem with max-operator (without input signals) are given in [12, p. 65] (Theorem 3.13). We also refer to [3] (Sections 2.2 and 2.6) for the results for general functional differential equations.

4. An algorithm for the correspondence of control functions. The control functions in the original and the averaged systems are different and can belong to spaces of different dimensions. A natural question is, that how to obtain the controller u for the system (3) having the controller v for the system (5). By the following algorithm one can establish a correspondence between the control functions u and v.

1. For an admissible controller $v = v(t)$ we calculate the corresponding admissible controller $u = u(t)$ in the following way:

(a) first, calculate the points $v_i = \dfrac{1}{T_0}\displaystyle\int_{iT_0}^{(i+1)T_0} v(t)\,dt$, for $i = 0, 1, 2, \ldots$ ($T_0 > 0$ is an arbitrary constant);

(b) then define

$$u(t) = \{u_i(t),\ iT_0 \le t < (i+1)T_0,\ i = 0, 1, 2, \ldots\},$$

where $u_i = u_i(t)$ is such that

$$\min_{u(t)\in U}\left\|\frac{1}{T_0}\int\limits_{iT_0}^{(i+1)T_0} \zeta(t, u(t), \widetilde{u}(t))dt - v_i\right\| = \left\|\frac{1}{T_0}\int\limits_{iT_0}^{(i+1)T_0} \zeta(t, u_i(t), \widetilde{u}_i(t))dt - v_i\right\|. \tag{8}$$

The set-valued mapping $\zeta(t, U, U)$ is continuous and bounded, hence by the Lyapunov theorem [18] the set

$$V_{T_0}^i = \left\{\frac{1}{T_0}\int\limits_{iT_0}^{(i+1)T_0} \zeta(t, u_i(t), \widetilde{u}_i(t))dt, u_i(t), \widetilde{u}_i(t) \in U\right\}$$

is convex and compact. According to (7) $\lim_{T_0\to\infty} d_H\left(V^i_{T_0}, V\right) = 0$. Hence, there exists $\overline{v}_i \in V^i_{T_0}$ the nearest to the v_i, in other words, there exists a control function $u_i(t)$ in (8) such that

$$\frac{1}{T_0}\int\limits_{iT_0}^{(i+1)T_0} \zeta(t, u_i(t), \widetilde{u}_i(t))\,dt = \overline{v}_i. \tag{9}$$

2. For an admissible controller $u = u(t)$ we define the corresponding admissible controller $v = v(t)$ in the following way:

(a) calculate

$$w_i(t) = \frac{1}{T_0}\int\limits_{iT_0}^{(i+1)T_0} \zeta(t, u_i(t), \widetilde{u}_i(t))\,dt, \quad i = 0, 1, 2, \ldots$$

(T_0 is an arbitrary constant);

(b) assign the control

$$v(t) = \big\{v_i(t),\ iT_0 \le t < (i+1)T_0,\ i = 0, 1, 2, \ldots\big\},$$

where v_i is obtained from the condition

$$\operatorname*{argmin}_{v\in V} \|w_i - v\| = \|w_i - v_i\|.$$

There exists v_i as a minimum of the continuous function $\|w_i - v_i\|$ on a compact set V.

Remark 1. The second part in the algorithm is not necessary for practical purposes but we use it in the proof of the theorem below.

Remark 2. Control functions $u = u(t)$ in 1(b) and $v = v(t)$ in 2(b) are determined ambiguously.

5. Averaging method for controlled system with maximum of control function and of the state. The following theorem provides a justification for the averaging method for the controlled system (3).

Theorem 1. *Suppose that for the domain*

$$Q = \{t \ge 0, x \in D \subset \mathbb{R}^n, u \in U \subset \text{comp}\ (\mathbb{R}^r)\}$$

the following conditions hold:

(i) $f = f(t, x, \breve{x})$ is a continuous function in t and there exist positive constants K and λ s. t.,

$$\|f(t, x, \breve{x})\| \le K,$$

$$\left\|f\left(t, x', \breve{x}'\right) - f\left(t, x'', \breve{x}''\right)\right\| \le \lambda\left(\left\|x' - x''\right\| + \left\|\breve{x}' - \breve{x}''\right\|\right);$$

(ii) A is a continuous matrix and there exist K and λ such that

$$\|A(x)\| \le K,$$

$$\left\|A(x') - A(x'')\right\| \le \lambda\left\|x' - x''\right\|;$$

(iii) $\zeta = \zeta(t, u, \widetilde{u})$ *is continuous with respect to* $t,\ u,\ \widetilde{u}$*;*

(iv) uniformly exists (2);

(v) there exists $\rho > 0$ *s. t. the solution* $y = y(t)$ *to the averaged system (5), where* $y(0) = x(0) \in D' \subset D$ *defined for any* $t \geq 0$*, belongs together with its* ρ*-neighborhood to the domain* D*.*

Then for any $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon^* = \varepsilon^*(\eta, L) > 0$ *s. t. for any* $\varepsilon \in (0, \varepsilon^*]$ *and* $t \in \left[0, L\varepsilon^{-1}\right]$ *the following statements hold:*

(i) for any admissible controller u *of system (3) there exists a controller* v *of system (5), s. t.*

$$\|x(t) - y(t)\| \leq \eta; \tag{10}$$

(ii) for any admissible controller v *of system (5) there exists a controller* u *of system (3), s. t. (10) holds.*

Remark 3. By assumption (iii) the function ζ is continuous so we denote $M := \max_{t,u,\widetilde{u}} |\zeta(t, u, \widetilde{u})|$.

Proof. Let us notice that the function $\overline{f}(\cdot)$ is a bounded function and satisfies the Lipschitz condition. Indeed, according to the assumption (iv) and Definition 1 there is a function $\sigma = \sigma(T)$ such that the following estimation holds:

$$\left\|\overline{f}(x) - \overline{f}(x^1)\right\| \leq \left\|\overline{f}(x) - \frac{1}{T}\int\limits_0^T f(t,x)dt\right\| + \left\|\frac{1}{T}\int\limits_0^T \left[f(t,x)f(t,x^1)\right] dt\right\| +$$

$$+ \left\|\frac{1}{T}\int\limits_0^T f(t,x^1)\,dt - \overline{f}(t,x^1)\right\| \leq 2\sigma(T) + \frac{1}{T}\int\limits_0^T \left\|f(t,x) - f\left(t,x^1\right)\right\| dt \leq$$

$$\leq 2\sigma(T) + \lambda\left\|x - x^1\right\| \quad \forall x, x^1 \in Q,$$

since $\sigma(T) \to 0$ as $T \to \infty$ one obtains

$$\left\|\overline{f}(x) - \overline{f}(x^1)\right\| \leq \lambda\left\|x - x^1\right\| \quad \forall x, x^1 \in Q.$$

Let us use the integral form of (3) and (5),

$$x(t) = x_0 + \varepsilon\int\limits_0^t \left(f(s, x(s), \check{x}(s)) + A(x(s))\zeta(s, u(s), \widetilde{u}(s))\right) ds,$$

$$y(t) = y_0 + \varepsilon\int\limits_0^t \left(\overline{f}(y(s), \check{y}(s)) + A(y(s))v(s)\right) ds$$

for $t \in \left[0, L\varepsilon^{-1}\right]$. Then estimate the difference

$$\|x(t) - y(t)\| \leq \varepsilon\int\limits_0^t \|f(s, x(s), \check{x}(s)) - f(y(s), \check{y}(s))\|\, ds +$$

$$+\varepsilon\int\limits_0^t \|[A(x(s))-A(y(s))]\,\zeta(s,u(s),\widetilde{u}(s))\|\,ds+$$

$$+\varepsilon\left\|\int\limits_0^t A(y(s))\,[\zeta(s,u(s),\widetilde{u}(s))-v(s)]\right\|ds\le$$

$$\le\varepsilon\left\{M\lambda\int\limits_0^t\delta(s)ds+\int\limits_0^t\|A(y(s))\,[\zeta(s,u(s),\widetilde{u}(s)))-v(s)]\|\;ds\right\},\qquad(11)$$

i.e., (11) holds for any $t\in[0,L\varepsilon^{-1}]$; then one gets

$$\delta(t)\le\varepsilon\left\{M\lambda\int\limits_0^t\delta(s)ds+I_1(t)+I_2(t)\right\},\qquad(12)$$

where $\delta(t)=\max_{\mu\in[0,t]}\|x(\mu)-y(\mu)\|$ and

$$I_1(t):=\left\|\int\limits_0^t(f(s,y(s))-\overline{f}(y(s))\,ds\right\|,$$

$$I_2(t):=\int\limits_0^t\|A(y(s))\,[\zeta(s,u(s),\widetilde{u}(s))-v(s)]\|\,ds.$$

Applying the Gronwall – Bellman lemma to (12), we obtain the inequality

$$\delta(t)\le\varepsilon(I_1(t)+I_2(t))e^{\varepsilon\int_0^t(\lambda+M\lambda)ds}=\varepsilon(I_1(t)+I_2(t))e^{(\lambda+K\lambda)L}.$$

We divide the interval $\left[0,L\varepsilon^{-1}\right]$ into m equal parts by the points $t_i=\dfrac{iL}{\varepsilon m}$, $i=0,1,2,\ldots,m-1$. Consider $t\in[t_p,t_{p+1})$ for some $p\in[0,m-1]$

$$I_2(t)\le\varepsilon\sum_{i=0}^{p-1}\left\{\int\limits_{t_i}^{t_{i+1}}\|(A(y(s))-A(y(t_i)))\,(\varphi(s,u(s),\widetilde{u}(s))-v(s))\|\,ds+\right.$$

$$\left.+\int\limits_{t_i}^{t_{i+1}}\|A(y(t_i))\,(\varphi(s,u(s),\widetilde{u}(s))-v(s))\|\,ds\right\}+$$

$$+\varepsilon\int\limits_{t_p}^{t}\|A(y(s))\,(\varphi(s,u(s),\widetilde{u}(s))-v(s))\|\,ds\le$$

$$\leq \varepsilon \sum_{i=0}^{p-1} \left\{ \lambda \int_{t_i}^{t_{i+1}} \|y(s) - y(t_i)\| \, \|\varphi(s, u(s), \widetilde{u}(s)) - v(s)\| \, ds \right\} +$$

$$+ \varepsilon \int_{t_p}^{t} \|A(y(s))\| \, \big\|\varphi(s, u(s), \widetilde{u}(s)) - v(s)\big\| ds \leq 2ML\Big(K + \frac{KL\lambda}{m^2}\Big).$$

Using estimation from [19], we have the following:

$$I_1(t) = \left[\frac{2KL}{m} \left(\lambda \frac{L}{2} + \lambda \max(\omega(\gamma, L), \omega(g, L)) + 1\right) + \right.$$

$$\left. +2\varepsilon\lambda ML\Big(\lambda \frac{L}{2} + \lambda \max\left(\omega(\gamma, L), \omega(g, L)\right) + 2m\varepsilon\Big)\right] e^{2\varepsilon\lambda t},$$

where $\omega(\cdot, \cdot)$ is a module of continuity. Let us collect the estimations for $I_1(t)$ and $I_2(t)$, by an appropriate choice of sufficiently big m and sufficiently small ε the estimation for $I_2(t)$ can be made sufficiently small. Thus,

$$\|x(t) - y(t)\| \leq \eta,$$

where

$$\eta = 2\varepsilon \left[MLK\Big(1 + \frac{L\lambda}{m^2}\Big) + KL + \chi\right] + \frac{KL\lambda}{m} \, e^{(\lambda + K\lambda)L}.$$

The second statements of the theorem can be proved analogously.

6. Comparison of the functionals of the original and the averaged optimal control problems. In the previous section we have shown that solutions to the original and the averaged systems are close. But from this fact one cannot come to the conclusion that the correspondi- ng functionals are the close as well. By this reason one needs to find an approximation to the functionals.

Theorem 2. *Let, for the domain*

$$Q = \{t \geq 0, x \in D \subset \mathbb{R}^n, u \in U \subset \mathrm{comp}\,(\mathbb{R}^r)\},$$

the assumptions of Theorem 1 hold. Moreover, we assume that

(i) there exists λ such that

$$\big\|\Phi(x) - \Phi(x')\big\| \leq \lambda \big\|x - x'\big\|;$$

(ii) there exists $u^* = u^*(t)$, an optimal control function for the problem (3), (4), and let $x^* = x^*(t)$ be the corresponding optimal trajectory and J^* the optimal value of the functional.

Then for any $L > 0$ there exist $\eta_1 > 0$ and $\varepsilon^*(L) > 0$ such that for any $\varepsilon \in [0, \varepsilon^*)$ the following inequalities hold:

$$\big|\overline{J}\,[v^*] - J\,[u^*]\big| \leq \eta_1, \tag{13}$$

$$J\,[u_{v^*}] - J\,[u^*] \leq \eta_1, \tag{14}$$

where $\overline{J}\left[v^*\right]$ *is the optimal value of the functional of the problem (5), (6),* $u_{v^*} = u_{v^*}(t)$ *is a control function to the problem (5), (6) constructed by the algorithm and corresponding to the optimal control function* $v^* = v^*(t)$ *of the problem (5), (6),* $v_{u^*} = v_{u^*}(t)$ *is the optimal control function of the problem (5), (6) constructed from* $u^* = u^*(t)$.

Proof. Since the set V is convex and compact [18], the set of attainability of the problem (5), (4) is compact [20]. So there exists an optimal solution to the problem (5), (6).

Let $u^* = u^*(t),\ v^* = v^*(t),$ be optimal control functions for problems (3), (4) and (5), (6), respectively.

Denote by $x\left(t, u^*\right)$ and $x\left(t, u_{v^*}\right)$ the trajectories of the system (3) with the controllers $u^* = u^*(t)$, and $u_{v^*} = u_{v^*}(t)$, respectively; by $y\left(t, v^*\right)$ and $y\left(t, v_{u^*}\right)$ denote the trajectories of the system (5) with control functions $v^* = v^*(t)$ and $v_{u^*} = v_{u^*}(t)$.

According to Theorem 1 we have

$$\left\|x\left(t, u^*\right) - y\left(t, v_{u^*}\right)\right\| \leq \eta, \tag{15}$$

$$\left\|x\left(t, u_{v^*}\right) - y\left(t, v^*\right)\right\| \leq \eta. \tag{16}$$

From (16) and condition (i) of Theorem 2 we obtain

$$\begin{aligned}\left|\overline{J}\left[v^*\right] - J\left[u_{v^*}\right]\right| &= \left|\Phi\left(y\left(L\varepsilon^{-1}, v^*\right)\right) - \Phi\left(x\left(L\varepsilon^{-1}, u_{v^*}\right)\right)\right| \leq \\ &\leq \lambda\left|y\left(t, v^*\right) - x\left(t, u_{v^*}\right)\right| \leq \lambda\eta \end{aligned} \tag{17}$$

and

$$\begin{aligned}\left|J\left[u^*\right] - \overline{J}\left[v_{u^*}\right]\right| &= \left|\Phi\left(x\left(L\varepsilon^{-1}, u^*\right)\right) - \Phi\left(y\left(L\varepsilon^{-1}, v_{u^*}\right)\right)\right| \leq \\ &\leq \lambda\left|x\left(t, u^*\right) - y\left(t, v_{u^*}\right)\right| \leq \lambda\eta. \end{aligned} \tag{18}$$

Moreover,

$$\begin{aligned} J\left[u_{v^*}\right] &\geq J\left[u^*\right] =: J^*, \\ \overline{J}\left[v_{u^*}\right] &\geq \overline{J}\left[v^*\right] =: \overline{J}^*. \end{aligned} \tag{19}$$

For the values of the functionals $J\left[u^*\right]$ and $\overline{J}\left[v^*\right]$, one of the following inequalities is valid:

$$J\left[u^*\right] > \overline{J}\left[v^*\right] \tag{20}$$

or

$$J\left[u^*\right] \leq \overline{J}\left[v^*\right]. \tag{21}$$

For the first case from (19), (20) and (17) one gets

$$J\left[u_{v^*}\right] \geq J\left[u^*\right] > \overline{J}\left[v^*\right] \geq J\left[u_{v^*}\right] - \lambda\eta$$

thus,

$$\left|J\left[u^*\right] - \overline{J}\left[v^*\right]\right| \leq \lambda\eta. \tag{22}$$

For the second case from (19), (21) and (18) we have

$$J\left[v_{u^*}\right] \geq \overline{J}\left[v^*\right] \geq J\left[u^*\right] \geq \overline{J}\left[v_{u^*}\right] - \lambda\eta,$$

hence, (22) holds.

Moreover,

$$J\left[u_{v^*}\right] - J\left[u^*\right] = J\left[u_{v^*}\right] - \overline{J}\left[v^*\right] + \overline{J}\left[v^*\right] - J\left[u^*\right] \leq$$

$$\leq \left|J\left[u_{v^*}\right] - \overline{J}\left[v^*\right]\right| + \left|\overline{J}\left[v^*\right] - J\left[u^*\right]\right| \leq 2\lambda\eta_1. \tag{23}$$

Choosing $\eta_1 = 2\lambda\eta$ and taking into account (22) and (23), we obtain the statement of the theorem.

Remark 4. The set of attainability of the system (3) is bounded but if the set $\zeta(t, U, U)$ is not convex then the set of attainability is not convex, see [20]. In this case optimal control function in problems (3), (4) cannot exist, but there exists $J_0 = \inf J[u]$, and inequalities (13), (14) take the form

$$\left|J_0 - \overline{J}\left[v^*\right]\right| \leq \eta_1,$$

$$J\left[u_{v^*}\right] - J_0 \leq \eta_1.$$

Remark 5. Note that Theorem 2 is valid if instead of the problem (3), (4) with unfixed right and we consider the problem with flexible one, i.e., with the restriction

$$\psi_j\left(x\left(L\varepsilon^{-1}\right)\right) \leq 0, \quad j = \overline{1, m}.$$

Finally, we can formulate a numerical-asymptotic algorithm for solving an optimal control problem with a small parameter and with supremum of the control function as follows:

1. For a given control problem with small parameter and supremum of control function (3), (4) we define the averaged problem (5), (6).
2. For the known set of admissible control functions of the original system we construct a set of admissible control functions for the averaged problem according to the algorithm of correspondence (described above).
3. Solve optimal control problem of the averaged equation (5) with criterion (6) and find $v^*(t),\ y^*(t),\ \overline{J}^*$.
4. According to the algorithm by using the found optimal control function $v^* = v^*(t)$ of the averaged problem we find the corresponding control function of the original problem $u_v^* = u_v^*(t)$ which is asymptotically optimal for the problem (3), (4).
5. For the found control function $u_v^* = u_v^*(t)$ we obtain a corresponding trajectory of system (3), $x(t) = x\left(t, u_v^*\right)$.
6. Calculate the value of the functional (4) on the trajectory from step (5).

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