

**STEEN – ERMAKOV – PINNEY EQUATION  
AND INTEGRABLE NONLINEAR DEFORMATION  
OF ONE-DIMENSIONAL DIRAC EQUATION**

**РІВНЯННЯ СТИНА – ЄРМАКОВА – ПІННЕЯ  
ТА ІНТЕГРОВНІ НЕЛІНІЙНІ ДЕФОРМАЦІЇ  
ОДНОВИМІРНОГО РІВНЯННЯ ДІРАКА**

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*The paper deals with nonlinear one-dimensional Dirac equation. We describe its invariants set by means of the deformed linear Dirac equation, using the fact that two ordinary differential equations are equivalent if their sets of invariants coincide.*

*Розглянуто одновимірне нелінійне рівняння Дірака та описано множину його інваріантів за допомогою деформованого лінійного рівняння Дірака з використанням того факту, що два звичайних диференціальних рівняння є еквівалентними, якщо множини їх інваріантів збігаються між собою.*

**1. Introduction.** In 1874 Danish mathematician Adolph Steen wrote a paper [1] where he introduced the system of two equations

$$r'' + qr = \frac{1}{r^3}, \quad (1.1)$$

and

$$g'' + qg = 0, \quad (1.2)$$

where  $q = q(u)$  is a continuous function on a real interval. He discovered that these two equations above are in some sense equivalent, that is the general solution to the second equation (1.2) gives rise to that to the first one (1.1) and vice-versa. Unluckily, the paper was published in Danish and his research was lost. Later many authors have been rediscovering these equations and mentioning too the property above. In 1880 V. Yermakov [2] gave a novel derivation and generalization of the Steen's equations. This generalization was actively studied and developed by others researches. Later, in 1950 Edmund Pinney [3] showed that the solution of the first equation (1.1) is

$$r(t) = \sqrt{Au^2 + 2Buw + Cv^2}, \quad (1.3)$$

where  $u(t)$  and  $v(t)$  are two arbitrary linearly independent solutions of the second equation of (1.1),  $A$ ,  $B$  and  $C$  are constants which satisfy the equality

$$B^2 - AC = \frac{1}{W^2}, \quad (1.4)$$

with  $W$  being the constant Wronskian of the two independent solutions  $u$ ,  $v$  to (1.2).

Nowadays the Steen's contribution is forgotten too and nobody calls systems of equations (1.1) Steen's equations. The name of Ermakov is commonly used or they also often called Ermakov–Pinney equations. Raymond Redheffer and Irene Redheffer wrote [4] an historical survey on the Steen's equations with English translation of the original Steen's paper.

It is worth highlighting the following quite simple fact. Let us take the two equations

$$y'' = \omega(t)y \quad (1.5)$$

and

$$z'' = \omega(t)z + \frac{k}{z^3}, \quad (1.6)$$

which are, in fact, rewritten equations from (1.1). These equations can be easily transformed to the more generalized form  $(py')' = \omega(t)y$  and  $(pz')' = \omega(t)z + \frac{k}{z^3}$  for some smooth function  $p(z)$  by changing the variables on which we will not stop here.

Let us denote  $\alpha := y_1 y_2$ , where  $y_1$  and  $y_2$  are two arbitrary solutions to the equation (1.5), and  $\beta := z^2$ . After differentiation and substituting into (1.5) and (1.6) one can obtain the following expressions:

$$\alpha''' - 2\omega'\alpha - 4\omega\alpha' = 0, \quad (1.7)$$

or

$$\left[ \frac{d^3}{dt^3} - \left( 2\omega \frac{d}{dt} + 2 \frac{d}{dt} \omega \right) \right] \alpha = 0, \quad (1.8)$$

and

$$\beta''' - 2\omega'\beta - 4\omega\beta' = 0, \quad (1.9)$$

or

$$\left[ \frac{d^3}{dt^3} - \left( 2\omega \frac{d}{dt} + 2 \frac{d}{dt} \omega \right) \right] \beta = 0. \quad (1.10)$$

It is worth mentioning that the differential expression  $\eta = \left[ \frac{d^3}{dt^3} - \left( 2\omega \frac{d}{dt} + 2 \frac{d}{dt} \omega \right) \right]$  is the second Poisson operator [5, 6] for the KdV equation  $u_t = -u_{xxx} - 6uu_x = K[u]$  in its Hamiltonian form  $u_t = -\eta \text{grad} H$ , where  $H$  is the corresponding [7] Hamiltonian functional.

Taking into account that the equations (1.8) and (1.10) are linear and the same, one can infer that their sets of the solutions coincide to each other. It means that the solution to the equation (1.10) is the sum of the solutions to the equation (1.8), whereas the expression

$$z^2 = Ay_1^2 + By_1y_2 + Cy_2^2$$

is the solution of the (1.8), and then the function

$$z = \sqrt{Ay_1^2 + By_1y_2 + Cy_2^2}$$

is the solution of the (1.6). Taking into account that the Wronskian of any two different solutions to (1.5) is constant, one easily obtains the relationship (1.4).

In the next section we will use the above mentioned trick in finding solutions to the one-dimensional Dirac equations [5].

**2. Dirac equation, its invariants and integrable nonlinear deformation.** Let us consider the one-dimensional Dirac equation

$$\frac{df}{dx} = l(\lambda; q)f, \quad l(\lambda; q) := \begin{pmatrix} \lambda & q_1(x) \\ q_2(x) & -\lambda \end{pmatrix}, \quad (2.1)$$

where  $x \in \mathbb{R}$ ,  $\lambda \in \mathbb{C}$  is a complex parameter,  $(q_1, q_2)^T \in M \simeq C^\infty(\mathbb{R}; \mathbb{C}^2)$  is a functional vector of potentials, and  $f \in (L_\infty(\mathbb{R}; \mathbb{C}^2))$  is found solution to (2.1). The whole solution set to the equation (2.1) is completely described [8] by means of the corresponding fundamental solution  $F := \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in C^\infty(\mathbb{R} \times \mathbb{R}, \text{End } \mathbb{C}^2)$ , satisfying the matrix equation

$$\frac{d}{dx} F(x, x_0) = l(\lambda; q)F(x, x_0), \quad F(x, x_0)|_{x=x_0} = \mathbf{1} \quad (2.2)$$

at any point  $x_0 \in \mathbb{R}$ . Then an arbitrary solution  $f \in C^\infty(\mathbb{R}; \mathbb{C}^2)$  to (2.2), evidently, can be represented as

$$f(x) = F(x, x_0)f(x_0), \quad (2.3)$$

where  $f(x_0) \in \mathbb{C}^2$  is a suitable Cauchy data vector.

It is well known from the general theory of ordinary differential equations [8] that the solution set to the equation (2.1) can be equivalently described by means of its complete set of invariants. Moreover, the two ordinary differential equations are then considered to be equivalent if their sets of invariants coincide. From this point of view one can consider a suitable deformed Dirac equation

$$\frac{d\tilde{f}}{dx} = l(\lambda; q)\tilde{f} + \delta\tilde{f}, \quad (2.4)$$

where a vector  $\delta\tilde{f} \in C^\infty(\mathbb{R}\mathbb{C}^2)$  can depend on  $\tilde{f} \in C^\infty(\mathbb{R}; \mathbb{C}^2)$  and the vector  $q \in M$  coincides with that chosen in (2.1).

Now a problem consists in determining the vector  $\delta\tilde{f} \in C^\infty(\mathbb{R}; \mathbb{C}^2)$  in such a form which guarantees that the set of invariants of (2.4) will contain or coincide with that of the equation (2.1).

To approach a solution to this problem we need first to describe the invariants set to the equation (2.1). To do that we assume for simplicity that the functional vector  $q \in M$  is  $2\pi$ -periodic in  $x \in \mathbb{R}$ :  $q(x + 2\pi) = q(x)$  for all  $x \in \mathbb{R}$ . Then one can define [9] the monodromy matrix  $S(x) \in \text{End } \mathbb{C}^2$  as

$$S(x) := F(x + 2\pi, x), \quad (2.5)$$

satisfying the well known Novikov commutator equation [10]

$$\frac{dS(x)}{dx} = [l(\lambda; q), S(x)]. \quad (2.6)$$

As a consequence from (2.6) one easily obtains that all functions

$$\gamma_j := \text{tr} S^j(x), \quad \frac{d\gamma_j}{dx} = 0, \quad (2.7)$$

where  $j \in \mathbb{Z}_+$ , are invariants for (2.1) and form [11] its complete set. In particular, we can determine only two dependent invariants for (2.1)

$$\gamma_1 = \text{tr} S(x), \quad \gamma_2 = \det S(x) = 1, \quad (2.8)$$

and will now try to search for such a deformation vector  $\delta\tilde{f} \in C^\infty(\mathbb{R}; \mathbb{C}^2)$  which will give rise to the invariants set of (2.4) coinciding with (2.8) of (2.1). For this problem to be solved effectively, we need to find the determining equations for invariants (2.8) as for the independent gradient  $\text{grad } \gamma_1 \in T^*(M)$ ,

$$\left( \begin{array}{cc} q_2 D_x^{-1} q_2 & \frac{d}{dx} + \lambda - q_2 D_x^{-1} q_1 \\ \frac{d}{dx} - \lambda - q_1 D_x^{-1} q_2 & q_1 D_x^{-1} q_1 \end{array} \right) \text{grad } \gamma_1 = 0, \quad (2.9)$$

where, by definition, we put

$$D_x^{-1}(\cdot) := \frac{1}{2} \left( \int_{x_0}^x (\cdot) dy - \int_x^{x_0+2\pi} (\cdot) dy \right), \quad x \in \mathbb{R}.$$

It is now worth observing that the monodromy matrix  $S(x) \in \text{End } \mathbb{C}^2$  for any  $x \in \mathbb{R}$  allows the following matrix representation:

$$S(x) = F(x, x_0) C(x_0) F^{-1}(x, x_0) \quad (2.10)$$

for some specially chosen matrix  $C(x_0) \in \text{End } \mathbb{C}^2$ ,  $x_0 \in \mathbb{R}$ . As a simple consequence of the representation (2.10) one easily obtains that

$$\text{grad} \gamma_1 = \begin{pmatrix} c_{11} f_{21} f_{22} - c_{12} f_{21}^2 + c_{21} f_{22}^2 - c_{22} f_{22} f_{21} \\ c_{12} f_{11}^2 - c_{11} f_{11} f_{12} - c_{21} f_{12}^2 + c_{22} f_{12} f_{11} \end{pmatrix}, \quad (2.11)$$

where we used the matrix expression  $C(x_0) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \text{End } \mathbb{C}^2$ . Taking into account the arbitrariness of the matrix  $C(x_0) \in \text{End } \mathbb{C}^2$  entering (2.10), we can easily obtain from (2.9) and (2.11) that the equation

$$\begin{pmatrix} q_2 D_x^{-1} q_2 & \frac{d}{dx} + \lambda - q_2 D_x^{-1} q_1 \\ \frac{d}{dx} - \lambda - q_1 D_x^{-1} q_2 & q_1 D_x^{-1} q_1 \end{pmatrix} \begin{pmatrix} -f_{21}^2 \\ f_{11}^2 \end{pmatrix} = 0 \quad (2.12)$$

holds for all  $x \in \mathbb{R}$  and  $\lambda \in \mathbb{C}$ .

Now it is easy to infer that if the vector  $\tilde{f} := (\tilde{f}_1, \tilde{f}_2)^\tau \in \infty(\mathbb{C}; \mathbb{C}^2)$  of the equation (2.4) satisfies the same equation as (2.12)

$$\begin{pmatrix} q_2 D_x^{-1} q_2 & \frac{d}{dx} + \lambda - q_2 D_x^{-1} q_1 \\ \frac{d}{dx} - \lambda - q_1 D_x^{-1} q_2 & q_1 D_x^{-1} q_1 \end{pmatrix} \begin{pmatrix} -\tilde{f}_2^2 \\ \tilde{f}_1^2 \end{pmatrix} = 0, \quad (2.13)$$

then the corresponding set of invariants of the deformed equation (2.4) will possess that of invariants for (2.1). In particular, from (2.13) it follows that a partial solution to the deformed Dirac equation (2.4) can be represented as

$$\tilde{f} = \begin{pmatrix} (c_{12} f_{11}^2 - c_{21} f_{11} f_{12} - c_{21} f_{12}^2 + c_{22} f_{12} f_{11})^{1/2} \\ (c_{12} f_{21}^2 - c_{11} f_{21} f_{22} + c_{22} f_{22} f_{21} - c_{21} f_{22}^2)^{1/2} \end{pmatrix}, \quad (2.14)$$

depending only on the fundamental matrix  $F(x, x_0) \in \text{End } \mathbb{C}^2$  of the equation (2.1) and equivalently, on its set of invariants. It is clear that the deformed Dirac equation (2.4) can generate a new solutions to it, yet the problem of describing this set of invariants is much more complicated and will not be herewith discussed.

Let us proceed now to describing the deformed Dirac equation (2.4), taking into account that the vector  $(-\tilde{f}_2^2, \tilde{f}_1^2) \in C^\infty(\mathbb{R}; \mathbb{C}^2)$  satisfies in general the following equation:

$$\begin{aligned} & \begin{pmatrix} q_2 D_x^{-1} q_2 & \frac{d}{dx} + \lambda - q_2 D_x^{-1} q_1 \\ \frac{d}{dx} - \lambda - q_1 D_x^{-1} q_2 & q_1 D_x^{-1} q_1 \end{pmatrix} \begin{pmatrix} -\tilde{f}_2^2 \\ \tilde{f}_1^2 \end{pmatrix} = \\ & = \begin{pmatrix} -\delta \tilde{f}_2 \tilde{f}_2 + q_1 \frac{d^{-1}}{dx} (\delta \tilde{f}_1 \tilde{f}_2 + \delta \tilde{f}_2 \tilde{f}_1) \\ \delta \tilde{f}_1 \tilde{f}_1 - q_2 \frac{d^{-1}}{dx} (\delta \tilde{f}_1 \tilde{f}_2 + \delta \tilde{f}_2 \tilde{f}_1) \end{pmatrix}, \end{aligned} \quad (2.15)$$

which reduces to (2.13) if identically one has

$$\begin{aligned}\delta \tilde{f}_2 \tilde{f}_2 &= q_1 D_x^{-1} \left( \delta \tilde{f}_1 \tilde{f}_2 + \delta \tilde{f}_2 \tilde{f}_1 \right), \\ \delta \tilde{f}_1 \tilde{f}_1 &= q_2 D_x^{-1} \left( \delta \tilde{f}_1 \tilde{f}_2 + \delta \tilde{f}_2 \tilde{f}_1 \right).\end{aligned}\tag{2.16}$$

As a simple consequence of (2.16) one obtains that there exists some function  $\alpha \in C^\infty(\mathbb{R}; \mathbb{C})$  for which

$$\delta \tilde{f}_2 = \frac{\alpha}{\tilde{f}_2} q_1, \quad \delta \tilde{f}_1 = \frac{\alpha}{\tilde{f}_1} q_2.\tag{2.17}$$

Having substituted (2.17) into (2.16) one obtains that

$$\frac{d\alpha}{dx} = \alpha \left( q_1 \frac{\tilde{f}_1}{\tilde{f}_2} + q_2 \frac{\tilde{f}_2}{\tilde{f}_1} \right),\tag{2.18}$$

or upon integrating (2.18), one ensues

$$\alpha = \bar{\alpha} \exp \left[ D_x^{-1} \left( q_1 \frac{\tilde{f}_1}{\tilde{f}_2} + q_2 \frac{\tilde{f}_2}{\tilde{f}_1} \right) \right],$$

where  $\bar{\alpha} \in \mathbb{C}$  is an arbitrary constant. Having now summarized the results obtained above one can formulate the following theorem.

**Theorem 2.1.** *Consider two Dirac type equations: the first one (2.1) linear and the second one (2.4) nonlinear, where*

$$\begin{aligned}\delta \tilde{f}_1 &= \frac{\bar{\alpha}}{\tilde{f}_1} q_2 \exp \left[ D_x^{-1} \left( q_1 \frac{\tilde{f}_1}{\tilde{f}_2} + q_2 \frac{\tilde{f}_2}{\tilde{f}_1} \right) \right], \\ \delta \tilde{f}_2 &= \frac{\bar{\alpha}}{\tilde{f}_2} q_1 \exp \left[ D_x^{-1} \left( q_1 \frac{\tilde{f}_1}{\tilde{f}_2} + q_2 \frac{\tilde{f}_2}{\tilde{f}_1} \right) \right]\end{aligned}\tag{2.19}$$

and  $\bar{\alpha} \in \mathbb{C}$  is an arbitrary constant. Then a partial solution to the nonlinear Dirac type equation (2.4) is given by the explicit expression (2.14), represented by means of the fundamental solution  $F(x, x_0) \in \text{End } \mathbb{C}^2$  to the Dirac equation (2.1) and the arbitrary constant matrix  $C \in \text{End } \mathbb{C}^2$ .

Thus, the Dirac equation (2.4), deformed by means of the vector components (2.19), is a nonlinear integro-differential equation depending on the functional element  $q \in M$ , whose  $2\pi$ -periodicity assumed before is not essential, as the main inferences, which are presented above, were based strictly on local reasonings.

**3. Conclusion.** Based on the analogy with the oscillator type equations (2.1) and (2.2) we succeeded in deriving a more general Steen type statement about the relationship between the solution sets to the linear Dirac equation (2.1) and its nonlinear deformation (2.4), specified by the expressions (2.20).

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