UDC 517.9

EXISTENCE AND ATTRACTIVITY RESULTS FOR HILFER FRACTIONAL DIFFERENTIAL EQUATIONS

ІСНУВАННЯ ПРИТЯГУВАЛЬНИХ РОЗВ'ЯЗКІВ ДРОБОВИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ХІЛФЕРА

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We present some results of the existence of attracting solutions of some fractional differential equations of Hilfer type. The results of the existence of solutions are applied to the Schauder fixed point theorem. We prove that all solutions are uniformly locally attracting.

Наведено результати існування притягувальних розв'язків деяких дробових диференціальних рівнянь хілферовського типу. Результати існування розв'язків застосовано для теореми Шаудера про нерухому точку. Доведено, що всі розв'язки однорідно локально притягувальні.

1. Introduction. Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [1, 2]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas et al. [3 – 5], Samko et al. [6], Kilbas et al. [7] and Zhou [8, 9], the papers by Abbas et al. [10 – 13], Benchohra et al. [14 – 16], Lakshmikantham et al. [17 – 19], and the references therein.

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [1, 20 – 25]. In [4, 10 – 13, 26 – 30], Abbas et al. presented some results on the local and global attractivity of solutions for some classes of fractional differential equations involving both the Riemann – Liouville and the Caputo fractional derivatives by employing some fixed point theorems. Motivated by the above papers, in this article we discuss the existence and the attractivity of solutions for the following problem of Hilfer fractional differential equations of the form

$$\begin{cases} \left(D_0^{\alpha,\beta}u\right)(t) = f(t,u(t)), \quad t \in \mathbb{R}_+ := [0,+\infty), \\ \left(I_0^{1-\gamma}u\right)(t)\Big|_{t=0} = \phi, \end{cases} \tag{1}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta,\ \phi \in \mathbb{R},\ f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_0^{1-\gamma}$ is the left-sided mixed Riemann – Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is the generalized Riemann – Liouville derivative operator of order α and type β, introduced by Hilfer in [1]. This paper initiates the concept of local attractivity of solutions of problem (1).

2. Preliminaries. Let C be the Banach space of all continuous functions v from $I := [0,T]$, $T > 0$, into $\mathbb{R}$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} |v(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}$. We denote by $AC^1(I)$ the space defined by

$$AC^1(I) := \left\{ w\colon I \to \mathbb{R}\colon \frac{d}{dt}\, w(t) \in AC(I) \right\}.$$

By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v\colon I \to \mathbb{R}$ with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \left\{ w\colon (0,T] \to \mathbb{R}\colon t^{1-\gamma}w(t) \in C \right\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in I} \left| t^{1-\gamma} w(t) \right|,$$

and

$$C_\gamma^1(I) = \left\{ w \in C\colon \frac{dw}{dt} \in C_\gamma \right\},$$

with the norm

$$\|w\|_{C_\gamma^1} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

Let $BC := BC(\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. By $BC_\gamma := BC_\gamma(\mathbb{R}_+)$ we denote the weighted space of all bounded and continuous functions defined by

$$BC_\gamma = \left\{ w : (0,+\infty) \to \mathbb{R}\colon t^{1-\gamma}w(t) \in BC \right\},$$

with the norm

$$\|w\|_{BC_\gamma} := \sup_{t\in\mathbb{R}_+} \left| t^{1-\gamma} w(t) \right|.$$

In the following we denote $\|w\|_{BC_\gamma}$ by $\|w\|_{BC}$.

Now, we give some results and properties of fractional calculus.

Definition 2.1 [4, 6, 7]. *The left-sided mixed Riemann–Liouville integral of order* $r > 0$ *of a function* $w \in L^1(I)$ *is defined by*

$$(I_\theta^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s) ds \quad \textit{for a.e.} \quad t \in I,$$

where $\Gamma(\cdot)$ *is the (Euler's) Gamma function defined by*

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \quad \xi > 0.$$

Notice that for all $r,\ r_1,\ r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t) \quad \text{for a.e.} \quad t \in I.$$

Definition 2.2 [4, 6, 7]. *The Riemann–Liouville fractional derivative of order* $r \in (0,1]$ *of a function* $w \in L^1(I)$ *is defined by*

$$(D_0^r w)(t) = \left(\frac{d}{dt} I_0^{1-r} w\right)(t) =$$

$$= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s) ds \quad \textit{for a.e.} \quad t \in I.$$

Let $r \in (0,1], \gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t) = w(t) \quad \text{for all} \quad t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C_{1-\gamma}^1(I)$, then the following composition is proved in [6]:

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1} \quad \text{for all} \quad t \in (0,T].$$

Definition 2.3 [4, 6, 7]. *The Caputo fractional derivative of order* $r \in (0,1]$ *of a function* $w \in L^1(I)$ *is defined by*

$$({}^c D_0^r w)(t) = \left(I_0^{1-r} \frac{d}{dt} w\right)(t) =$$

$$= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \frac{d}{ds} w(s) ds \quad \textit{for all} \quad t \in I.$$

In [1], R. Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [22–24].

Definition 2.4 (Hilfer derivative)**.** *Let* $\alpha \in (0,1),\ \beta \in [0,1],\ w \in L^1(I),\ I_0^{(1-\alpha)(1-\beta)} w \in AC^1(I)$. *The Hilfer fractional derivative of order* α *and type* β *of* w *is defined as*

$$\left(D_0^{\alpha,\beta} w\right)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w\right)(t) \quad \textit{for all} \quad t \in I. \tag{2}$$

Properties. Let $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $\left(D_0^{\alpha,\beta} w\right)(t)$ can be written as

$$\left(D_0^{\alpha,\beta} w\right)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w\right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w\right)(t) \quad \text{for all} \quad t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1], \quad \gamma \geq \alpha, \quad \gamma > \beta, \quad 1-\gamma < 1 - \beta(1-\alpha).$$

2. The generalization (2) for $\beta = 0$, coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative,

$$D_0^{\alpha,0} = D_0^{\alpha} \qquad \text{and} \qquad D_0^{\alpha,1} = {}^c D_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and is in $L^1(I)$, then

$$\left(D_0^{\alpha,\beta} I_0^{\alpha} w\right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w\right)(t) \quad \text{for a.e.} \quad t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$\left(D_0^{\alpha,\beta} I_0^{\alpha} w\right)(t) = w(t) \quad \text{for a.e.} \quad t \in I.$$

4. If $D_0^{\gamma} w$ exists and is in $L^1(I)$, then

$$\left(I_0^{\alpha} D_0^{\alpha,\beta} w\right)(t) = \left(I_0^{\gamma} D_0^{\gamma} w\right)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1} \quad \text{for a.e.} \quad t \in I.$$

Corollary 2.1. *Let* $h \in C_\gamma(I)$. *Then the linear problem*

$$\begin{cases} \left(D_0^{\alpha,\beta} u\right)(t) = h(t), & t \in I := [0,T], \\ \left(I_0^{1-\gamma} u\right)(t)\Big|_{t=0} = \phi, \end{cases}$$

has a unique solution given by

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \left(I_0^{\alpha} h\right)(t).$$

From the above corollary, we have the following lemma.

Lemma 2.1. *Let* $f\colon I \times \mathbb{R} \to \mathbb{R}$ *be such that* $f(\cdot, u(\cdot)) \in BC_\gamma$ *for any* $u \in BC_\gamma$. *Then problem (1) is equivalent to the problem of the solutions of the integral equation*

$$u(t) = \frac{\phi}{\Gamma(\gamma)}\, t^{\gamma-1} + \left(I_0^\alpha f(\cdot, u(\cdot))\right)(t).$$

Let $\varnothing \neq \Omega \subset BC$, and let $G\colon \Omega \to \Omega$, and consider the solutions of the equation

$$(Gu)(t) = u(t). \tag{3}$$

We introduce the following concept of attractivity of solutions for equation (3).

Definition 2.5. *Solutions of equation (3) are locally attractive if there exists a ball* $B(u_0, \eta)$ *in the space* BC *such that, for arbitrary solutions* $v = v(t)$ *and* $w = w(t)$ *of equations (3) belonging to* $B(u_0, \eta) \cap \Omega$, *we have*

$$\lim_{t\to\infty} (v(t) - w(t)) = 0. \tag{4}$$

When the limit (4) is uniform with respect to $B(u_0, \eta) \cap \Omega$, *solutions of equation (3) are said to be uniformly locally attractive (or equivalently that solutions of (3) are locally asymptotically stable).*

Lemma 2.2 [31, p. 62]. *Let* $D \subset BC$. *Then* D *is relatively compact in* BC *if the following conditions hold:*

(a) D *is uniformly bounded in* BC;

(b) the functions belonging to D *are almost equicontinuous on* $\mathbb{R}_+$, *i.e., equicontinuous on every compact of* $\mathbb{R}_+$;

(c) the functions from D *are equiconvergent, that is, given* $\epsilon > 0$ *there exists* $T(\epsilon) > 0$ *such that* $|u(t) - \lim_{t\to\infty} u(t)| < \epsilon$ *for any* $t \geq T(\epsilon)$ *and* $u \in D$.

In the sequel we will make use of the following fixed point theorems.

Theorem 2.1 (Schauder fixed point theorem, [32]). *Let* E *be a Banach space and* Q *be a nonempty bounded convex and closed subset of E, and let* $N\colon Q \to Q$ *be a compact and continuous map. Then* N *has at least one fixed point in* Q.

3. Existence and attractivity results. Let us start by defining what we mean by a solution of the problem (1).

Definition 3.1. *By a solution of the problem (1) we mean a measurable function* $u \in BC_\gamma$ *that satisfies the condition* $(I_0^{1-\gamma} u)(0^+) = \phi$, *and the equation* $\left(D_0^{\alpha,\beta} u\right)(t) = f(t, u(t))$ *on* $\mathbb{R}_+$.

The following hypotheses will be used in the sequel.

(H_1) The function $t \mapsto f(t, u)$ is measurable on $\mathbb{R}_+$ for each $u \in BC_\gamma$, and the function $u \mapsto f(t, u)$ is continuous on BC_γ for a.e. $t \in \mathbb{R}_+$,

(H_2) There exists a continuous function $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, u)| \leq \frac{p(t)}{1 + |u|} \quad \text{for a.e.} \quad t \in \mathbb{R}_+ \quad \text{and each} \quad u \in \mathbb{R},$$

and

$$\lim_{t\to\infty} t^{1-\gamma} (I_0^\alpha p)(t) = 0.$$

Set

$$p^* = \sup_{t\in\mathbb{R}_+} t^{1-\gamma}\left(I_0^{\alpha} p\right)(t).$$

Now, we present a theorem concerning the existence and the attractivity of solutions of our problem (1).

Theorem 3.1. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then the problem (1) has at least one solution defined on* $\mathbb{R}_+$. *Moreover, solutions of problem (1) are uniformly locally attractive.*

Proof. Consider the operator N such that, for any $u \in BC_\gamma$,

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)}\, t^{\gamma-1} + \int_0^t (t-s)^{\alpha-1}\frac{f(s,u(s))}{\Gamma(\alpha)}\, ds.$$

The operator N maps BC_γ into BC_γ. Indeed the map $N(u)$ is continuous on $\mathbb{R}_+$ for any $u \in BC_\gamma$, and for each $t \in \mathbb{R}_+$, we have

$$\left|t^{1-\gamma}(Nu)(t)\right| \le \frac{|\phi|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s))|ds \le$$

$$\le \frac{|\phi|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\, ds \le \frac{|\phi|}{\Gamma(\gamma)} + p^*.$$

Thus

$$\|N(u)\|_{BC} \le \frac{|\phi|}{\Gamma(\gamma)} + p^* := R. \tag{5}$$

Hence, $N(u) \in BC_\gamma$. This proves that the operator N maps BC_γ into itself.

By Lemma 2.1, the problem of finding solutions of the problem (1) is reduced to finding the solutions of the operator equation $N(u) = u$. Equation (5) implies that N transforms the ball $B_R := B(0,R) = \{w \in BC_\gamma : \|w\|_{BC} \le R\}$ into itself.

We shall show that the operator N satisfies all the assumptions of Theorem 2.1. The proof will be given in several steps.

Step 1. *N is continuous.*

Let $\{u_n\}_{n\in N}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in \mathbb{R}_+$, we have

$$\left|t^{1-\gamma}(Nu_n)(t) - t^{1-\gamma}(Nu)(t)\right| \le \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u_n(s)) - f(s,u(s))|\, ds. \tag{6}$$

Case 1. If $t \in [0,T]$, $T > 0$, then, since $u_n \to u$ as $n \to \infty$ and f is continuous, by the Lebesgue dominated convergence theorem, equation (6) implies

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as} \quad n \to \infty.$$

Case 2. If $t \in (T, \infty),\ T > 0,$ then from the hypotheses and (6), we get

$$\left|t^{1-\gamma}(Nu_n)(t) - t^{1-\gamma}(Nu)(t)\right| \leq 2\,\frac{t^{1-\gamma}}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} p(s)\,ds. \tag{7}$$

Since $u_n \to u$ as $n \to \infty$ and $t^{1-\gamma}(I_0^\alpha p)(t) \to 0$ as $t \to \infty$, then (7) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. *$N(B_R)$ is uniformly bounded.*

This is clear since $N(B_R) \subset B_R$ and B_R is bounded.

Step 3. *$N(B_R)$ is equicontinuous on every compact subset $[0,T]$ of $\mathbb{R}_+,\ T > 0.$*

Let $t_1, t_2 \in [0,T],\ t_1 < t_2,$ and let $u \in B_R$. Then we have

$$\left|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\right| \leq$$

$$\leq \left| t_2^{1-\gamma} \int\limits_0^{t_2} (t_2-s)^{\alpha-1} \frac{f(s,u(s))}{\Gamma(\alpha)}\,ds - t_1^{1-\gamma} \int\limits_0^{t_1} (t_1-s)^{\alpha-1} \frac{f(s,u(s))}{\Gamma(\alpha)}\,ds \right| \leq$$

$$\leq t_2^{1-\gamma} \int\limits_{t_1}^{t_2} (t_2-s)^{\alpha-1} \frac{|f(s,u(s))|}{\Gamma(\alpha)}\,ds +$$

$$+ \int\limits_0^{t_1} \left| t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1} \right| \frac{|f(s,u(s))|}{\Gamma(\alpha)}\,ds \leq$$

$$\leq t_2^{1-\gamma} \int\limits_{t_1}^{t_2} (t_2-s)^{\alpha-1} \frac{p(s)}{\Gamma(\alpha)}\,ds + \int\limits_0^{t_1} \left| t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1} \right| \frac{p(s)}{\Gamma(\alpha)}\,ds.$$

Thus, from the continuity of the function p and by setting $p_* = \sup_{t\in[0,T]} p(t),$ we get

$$\left|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\right| \leq \frac{p_* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha +$$

$$+ \frac{p_*}{\Gamma(\alpha)} \int\limits_0^{t_1} \left| t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1} \right| ds.$$

As $t_1 \to t_2,$ the right-hand side of the above inequality tends to zero.

Step 4. *$N(B_R)$ is equiconvergent.*

Let $t \in \mathbb{R}_+$ and $u \in B_R$. Then we have

$$\left|t^{1-\gamma}(Nu)(t)\right| \leq \frac{|\phi|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} |f(s,u(s))|\,ds \leq$$

$$\leq \frac{|\phi|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} p(s)\, ds \leq \frac{|\phi|}{\Gamma(\gamma)} + t^{1-\gamma} \left(I_0^\alpha p\right)(t).$$

Since $t^{1-\gamma}(I_0^\alpha p)(t) \to 0,$ as $t \to +\infty,$ we get

$$|(Nu)(t)| \leq \frac{|\phi|}{t^{1-\gamma}\Gamma(\gamma)} + \frac{t^{1-\gamma}(I_0^\alpha p)(t)}{t^{1-\gamma}} \to 0 \quad \text{as} \quad t \to +\infty.$$

Hence,

$$|(Nu)(t) - (Nu)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty.$$

As a consequence of Steps 1 to 4, together with the Lemma 2.2, we can conclude that $N \colon B_R \to$ $\to B_R$ is continuous and compact. From an application of Schauder's theorem (Theorem 2.1), we deduce that N has a fixed point u which is a solution of the problem (1) on $\mathbb{R}_+$.

Step 5. *The uniform local attractivity of solutions.*

Let us assume that u_0 is a solution of problem (1) with the conditions of this theorem. Taking $u \in B(u_0, 2p^*),$ we have

$$\left|t^{1-\gamma}(Nu)(t) - t^{1-\gamma}u_0(t)\right| = \left|t^{1-\gamma}(Nu)(t) - t^{1-\gamma}(Nu_0)(t)\right| \leq$$

$$\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} |f(s,u(s)) - f(s,u_0(s))| ds \leq$$

$$\leq \frac{2t^{1-\gamma}}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} p(s) ds \leq 2p^*.$$

Thus, we get

$$\|N(u) - u_0\|_{BC} \leq 2p^*.$$

Hence, we obtain that N is a continuous function such that

$$N\left(B\left(u_0, 2p^*\right)\right) \subset B\left(u_0, 2p^*\right).$$

Moreover, if u is a solution of problem (1), then

$$|u(t) - u_0(t)| = |(Nu)(t) - (Nu_0)(t)| \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} |f(s,u(s)) - f(s,u_0(s))|\, ds \leq 2\left(I_0^\alpha p\right)(t).$$

Thus

$$|u(t) - u_0(t)| \leq \frac{2t^{1-\gamma}\left(I_0^\alpha p\right)(t)}{t^{1-\gamma}}. \tag{8}$$

By using (8) and the fact that $\lim_{t\to\infty} t^{1-\gamma}\left(I_0^\alpha p\right)(t) = 0,$ we deduce that

$$\lim_{t\to\infty} |u(t) - u_0(t)| = 0.$$

Consequently, all solutions of problem (1) are uniformly locally attractive.

4. An example. As an application of our results we consider the following problem of Hilfer fractional differential equation of the form

$$\begin{cases} \left(D_0^{1/2,1/2}u\right)(t) = f(t,u(t)), \quad t \in \mathbb{R}_+, \\ \left(I_0^{1/4}u\right)(t)|_{t=0} = 1, \end{cases} \tag{9}$$

where

$$\begin{cases} f(t,u) = \dfrac{ct^{-1/4}\sin t}{64(1+\sqrt{t})(1+|u|)}, \quad t \in (0,\infty), \quad u \in \mathbb{R}, \\ f(0,u) = 0, \qquad\qquad\qquad\qquad\qquad u \in \mathbb{R}, \end{cases}$$

and $c = \dfrac{9\sqrt{\pi}}{16}$. Clearly, the function f is continuous.

The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{ct^{-1/4}|\sin t|}{64(1+\sqrt{t})}, \quad t \in (0,\infty), \\ p(0) = 0. \end{cases}$$

Also, we have

$$t^{1-\gamma} I_0^{1/2}\, p(t) = \frac{t^{1/4}}{\Gamma\left(\frac{1}{2}\right)} \int\limits_0^t (t-\tau)^{-1/2} p(\tau)d\tau \le \frac{1}{4}\, t^{-1/4} \to 0 \quad \text{as} \quad t \to \infty.$$

All conditions of Theorem 3.1 are satisfied. Hence, the problem (9) has at least one solution defined on $\mathbb{R}_+$, and solutions of this problem are uniformly locally attractive.

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Received 16.11.16