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REPRESENTATION OF SOLUTIONS OF LINEAR DIFFERENTIAL SYSTEMS OF SECOND-ORDER WITH CONSTANT DELAYS*

ЗОБРАЖЕННЯ РОЗВ'ЯЗКІВ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ СИСТЕМ ДРУГОГО ПОРЯДКУ ІЗ СТАЛИМИ ЗАПІЗНЕННЯМИ

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We derive representations for solutions to initial-value problems for n-dimensional second-order differential equations with delays,

$$x''(t) = 2Ax'(t-\tau) - (A^2 + B^2)x(t-2\tau),$$

and

$$x''(t) = (A+B)x'(t-\tau) - ABx(t-2\tau),$$

by means of special matrix delayed functions. Here A and B are commuting $(n \times n)$-matrices and $\tau > 0$. Moreover, a formula connecting delayed matrix exponential with delayed matrix sine and delayed matrix cosine is derived. We also discuss common features of the two considered equations.

Знайдено зображення розв'язків задач із початковими умовами для диференціальних рівнянь другого порядку розмірності n із запізненнями

$$x''(t) = 2Ax'(t-\tau) - (A^2 + B^2)x(t-2\tau)$$

та

$$x''(t) = (A+B)x'(t-\tau) - ABx(t-2\tau),$$

при цьому використано спеціальні матричні функції із запізненням. Тут A і B — комутативні матриці розмірності $n \times n$ і $\tau > 0$. Також отримано формулу, що зв'язує експоненціальну матрицю з запізненням з sin- *та* cos-*матрицями із запізненням. Також розглянуто загальні властивості обох розглядуваних рівнянь.*

1. Introduction. Recently, much attention was paid to a new formalization of the well-known method of steps in the theory of linear differential equations with constant coefficients and a single delay. Such a formulation was given in [1, 2] utilizing what is called a delayed matrix exponential, which is a matrix polynomial on every interval. After papers [1, 2] were published, this formalization was widely applied, e.g., in boundary-value problems, control problems and stability problems, modification to discrete equations was performed, generalizations to the case of several delays were developed, etc. (see [3–27]). Some of these results are collected in the book [28].

We recall the definition of a delayed matrix exponential. Let $(n \times n)$-matrices Θ, I and A be the zero matrix, the unit matrix, and a general constant matrix, respectively and θ be the

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$(n \times 1)$-zero vector. Let $\tau > 0$. A delayed matrix exponential e_τ^{At}, $t \in \mathbb{R}$, is defined as

$$e_\tau^{At} = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} A^s \frac{(t-(s-1)\tau)^s}{s!}, \tag{1}$$

where $\lfloor \cdot \rfloor$ is the floor function. The delayed matrix exponential equals the unit matrix on $[-\tau, 0]$ and represents a fundamental matrix of a homogeneous linear system with a single delay,

$$\dot{x}(t) = Ax(t-\tau). \tag{2}$$

In [1], a representation of solution of the Cauchy initial problem (2), (3), where

$$x(t) = \varphi(t), \quad -\tau \le t \le 0, \tag{3}$$

and $\varphi\colon [-\tau, 0] \to \mathbb{R}^n$ is continuously differentiable, is given in the integral form

$$x(t) = e_\tau^{At}\varphi(-\tau) + \int\limits_{-\tau}^{0} e_\tau^{A(t-\tau-s)}\varphi'(s)\, ds. \tag{4}$$

The advantage of the representation formula (4), as compared with the well-known representation formulas (e.g. [29–32]), consists in that it uses explicitly the given fundamental matrix (1) and, consequently, provides us with an explicit analytical formula for a solution of problem (2), (3).

The purpose of the present paper is to give representations of solutions to two initial-value problems. The first one is

$$x''(t) - 2Ax'(t-\tau) + (A^2 + B^2)x(t-2\tau) = \theta, \quad t \ge \tau, \tag{5}$$

$$x^{(i)}(t) = \xi^{(i)}(t), \quad i = 0, 1, \quad t \in [-\tau, \tau], \tag{6}$$

where the $(n \times n)$-matrices A, B commute, i.e., $AB = BA$, the matrix B is regular, and function $\xi\colon [-\tau, \tau] \to \mathbb{R}^n$ is assumed to be twice continuously differentiable.

The second one is the problem (6), (7) where

$$x''(t) - (A+B)x'(t-\tau) + ABx(t-2\tau) = \theta, \quad t \ge \tau, \tag{7}$$

with the matrices A and B commuting but regularity of B is not assumed.

The paper is organized as follows. A representation of the solution to the problem (5), (6) is developed in Section 2 while the problem (6), (7) is considered in Section 3. The last Section 4 is devoted to some relations between special matrix functions describing some common features of the considered problems.

2. Representation of the solution to problem (5), (6). Consider a linear system,

$$z'(t) = Cz(t-\tau), \quad t \ge 0, \tag{8}$$

where C is a $(2n \times 2n)$-matrix defined by $(n \times n)$-commuting matrices A and B as

$$C := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{9}$$

and z is a $(2n \times 1)$-vector. Let $z = \begin{pmatrix} x \\ y \end{pmatrix}$ where x, y are $(n \times 1)$-vectors.

We show that, if the vector-valued function $z\colon [-\tau, \infty) \to \mathbb{R}^{2n}$ is a solution to system (8) on the interval $[0, \infty)$, then the vector-valued function $x\colon [-\tau, \infty) \to \mathbb{R}^n$ is a solution to the second-order system (5) on the interval $[\tau, \infty)$. This follows from the following transformations. The system (8) can be written as

$$\begin{aligned} x'(t) &= Ax(t-\tau) + By(t-\tau), \\ y'(t) &= -Bx(t-\tau) + Ay(t-\tau), \end{aligned} \tag{10}$$

where $t \geq 0$ and

$$Ax'(t) - By'(t) = (A^2 + B^2)x(t-\tau). \tag{11}$$

Differentiating (10) and using (11), we derive

$$\begin{aligned} x''(t) &= Ax'(t-\tau) + By'(t-\tau) = 2Ax'(t-r) - Ax'(t-\tau) + By'(t-\tau) = \\ &= 2Ax'(t-\tau) - (A^2 + B^2)x(t-2\tau). \end{aligned} \tag{12}$$

Obviously, (12) is equivalent to (5). Comparing the domains of z and x we see that the above statement holds. The connection between systems (8) and (5) is used to prove the following result.

Theorem 1. *Let $AB = BA$ and the matrix B be invertible. Then the solution of the initial-value problem (5), (6) can be expressed as*

$$\begin{aligned} x(t) = &\left(\operatorname{Re} e_\tau^{(A+iB)t} - \operatorname{Im} e_\tau^{(A+iB)t} B^{-1} A\right) \xi(-\tau) + \left(\operatorname{Im} e_\tau^{(A+iB)t}\right) B^{-1} \xi'(0) + \\ &+ \int\limits_{-\tau}^{0} \left(\left(\operatorname{Re} e_\tau^{(A+iB)(t-\tau-s)}\right) \xi'(s) + \right. \\ &\left. + \left(\operatorname{Im} e_\tau^{(A+iB)(t-\tau-s)}\right) B^{-1}(\xi''(s+\tau) - A\xi'(s))\right) ds \end{aligned} \tag{13}$$

where $t \geq \tau$.

Proof. The strategy of the proof is the following. We will find a solution of a related initial-value problem for the system (8) in a suitable form. Then separating components for the vector x, we will get the representation (13).

First, we compute the powers C^k, $k \in \mathbb{N}$. Let us represent the matrix C as

$$C = A_{2n} I_{2n} + B_{2n} J_{2n},$$

where

$$A_{2n} := \begin{pmatrix} A & \Theta \\ \Theta & A \end{pmatrix}, \quad I_{2n} := \begin{pmatrix} I & \Theta \\ \Theta & I \end{pmatrix}, \quad J_{2n} := \begin{pmatrix} \Theta & I \\ -I & \Theta \end{pmatrix}, \quad B_{2n} := \begin{pmatrix} B & \Theta \\ \Theta & B \end{pmatrix}$$

are $(2n \times 2n)$-matrices (note that the matrix $J_{2n}^2 = J_{2n} \times J_{2n}$ can be viewed as a matrix analogue to the complex unit since $J_{2n}^2 = -I_{2n}$). Then

$$C^k = (A_{2n}I_{2n} + B_{2n}J_{2n})^k = \sum_{s=0}^{k} \binom{k}{s} A_{2n}^s I_{2n}^s B_{2n}^{k-s} J_{2n}^{k-s} =$$

$$= I_{2n}\mathrm{Re}\,(A_{2n} + iB_{2n})^k + J_{2n}\mathrm{Im}\,(A_{2n} + iB_{2n})^k,$$

where i is the imaginary unit. This relation can easily be verified if we show that the general terms on both sides are identical, i.e., if

$$\binom{k}{s} A_{2n}^s I_{2n}^s B_{2n}^{k-s} J_{2n}^{k-s} = I_{2n}\mathrm{Re} \binom{k}{s} A_{2n}^s I_{2n}^s i^{k-s} B_{2n}^{k-s} + J_{2n}\mathrm{Im} \binom{k}{s} A_{2n}^s I_{2n}^s i^{k-s} B_{2n}^{k-s}$$

or

$$A_{2n}^s B_{2n}^{k-s} J_{2n}^{k-s} = \mathrm{Re}\, A_{2n}^s i^{k-s} B_{2n}^{k-s} + J_{2n}\mathrm{Im}\, A_{2n}^s i^{k-s} B_{2n}^{k-s}.$$

In each of the four possible cases, i.e., for

$$J_{2n}^{k-s} = J_{2n} \quad \text{and} \quad i^{k-s} = i,$$

$$J_{2n}^{k-s} = -I_{2n} \quad \text{and} \quad i^{k-s} = -1,$$

$$J_{2n}^{k-s} = -J_{2n} \quad \text{and} \quad i^{k-s} = -i,$$

$$J_{2n}^{k-s} = I_{2n} \quad \text{and} \quad i^{k-s} = 1,$$

we get an equality. From the definition of the delayed matrix exponential (1), we deduce

$$e_\tau^{Ct} = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} C^s \,\frac{(t-(s-1)\tau)^s}{s!} =$$

$$= \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} \left(I_{2n}\mathrm{Re}\,(A_{2n} + iB_{2n})^s + J_{2n}\mathrm{Im}(A_{2n} + iB_{2n})^s\right) \frac{(t-(s-1)\tau)^s}{s!} =$$

$$= \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} \mathrm{Re}\,(A_{2n} + iB_{2n})^s \,\frac{(t-(s-1)\tau)^s}{s!} +$$

$$+ \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} J_{2n}\mathrm{Im}\,(A_{2n} + iB_{2n})^s \,\frac{(t-(s-1)\tau)^s}{s!} =$$

$$= \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} \mathrm{Re} \begin{pmatrix} A + iB & \Theta \\ \Theta & A + iB \end{pmatrix}^s \frac{(t-(s-1)\tau)^s}{s!} +$$

$$+\sum_{s=0}^{\lfloor t/\tau\rfloor+1} J_{2n}\mathrm{Im}\begin{pmatrix} A+iB & \Theta \\ \Theta & A+iB\end{pmatrix}^s \frac{(t-(s-1)\tau)^s}{s!} =$$

$$=\sum_{s=0}^{\lfloor t/\tau\rfloor+1} \mathrm{Re}\begin{pmatrix} (A+iB)^s & \Theta \\ \Theta & (A+iB)^s\end{pmatrix} \frac{(t-(s-1)\tau)^s}{s!}+$$

$$+\sum_{s=0}^{\lfloor t/\tau\rfloor+1} J_{2n}\,\mathrm{Im}\begin{pmatrix} (A+iB)^s & \Theta \\ \Theta & (A+iB)^s\end{pmatrix} \frac{(t-(s-1)\tau)^s}{s!} =$$

$$=\mathrm{Re}\sum_{s=0}^{\lfloor t/\tau\rfloor+1} \begin{pmatrix} (A+iB)^s & \Theta \\ \Theta & (A+iB)^s\end{pmatrix} \frac{(t-(s-1)\tau)^s}{s!}+$$

$$+\,\mathrm{Im}\sum_{s=0}^{\lfloor t/\tau\rfloor+1} \begin{pmatrix} \Theta & (A+iB)^s \\ -(A+iB)^s & \Theta\end{pmatrix} \frac{(t-(s-1)\tau)^s}{s!} =$$

$$=\begin{pmatrix} \mathrm{Re}\, e_\tau^{(A+iB)t} & \mathrm{Im}\, e_\tau^{(A+iB)t} \\ -\mathrm{Im}\, e_\tau^{(A+iB)t} & \mathrm{Re}\, e_\tau^{(A+iB)t}\end{pmatrix}. \tag{14}$$

Now consider the initial-value problem

$$z(t) = \varphi^*(t), \quad -\tau \le t \le 0,$$

for system (8), related to system (5) where the function

$$\varphi^* = \begin{pmatrix} \varphi_x^* \\ \varphi_y^*\end{pmatrix} : [-\tau, 0] \to \mathbb{R}^{2n}$$

is continuously differentiable as specified below. Since $z = \begin{pmatrix} x \\ y\end{pmatrix}$, we set $\varphi_x^*(t) \equiv \xi(t)$, $t \in$ $\in [-\tau, 0]$. Next, we will specify φ_y^*. From (10), due to invertibility of the matrix B, we get

$$y(t-\tau) = B^{-1}(x'(t) - Ax(t-\tau)), \quad t \ge 0,$$

or

$$y(t) = B^{-1}(x'(t+\tau) - Ax(t)), \quad t \ge -\tau.$$

Consequently,

$$\varphi_y^*(t) \equiv B^{-1}(\xi'(t+\tau) - A\xi(t)), \quad t \in [-\tau, 0].$$

Now we utilize the formula (4) where the matrix A is replaced with C, and φ with φ^*. Utilizing

(14), we get

$$z(t) = e_\tau^{Ct}\varphi^*(-\tau) + \int_{-\tau}^{0} e_\tau^{C(t-\tau-s)}\varphi^{*\prime}(s)\,ds =$$

$$= \begin{pmatrix} \operatorname{Re} e_\tau^{(A+iB)t} & \operatorname{Im} e_\tau^{(A+iB)t} \\ -\operatorname{Im} e_\tau^{(A+iB)t} & \operatorname{Re} e_\tau^{(A+iB)t} \end{pmatrix} \begin{pmatrix} \xi(-\tau) \\ B^{-1}(\xi'(0) - A\xi(-\tau)) \end{pmatrix} +$$

$$+ \int_{-\tau}^{0} \begin{pmatrix} \operatorname{Re} e_\tau^{(A+iB)(t-\tau-s)} & \operatorname{Im} e_\tau^{(A+iB)(t-\tau-s)} \\ -\operatorname{Im} e_\tau^{(A+iB)(t-\tau-s)} & \operatorname{Re} e_\tau^{(A+iB)(t-\tau-s)} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \xi'(s) \\ B^{-1}(\xi''(s+\tau) - A\xi'(s)) \end{pmatrix} ds. \tag{15}$$

The solution $x(t)$ of the initial problem (5), (6) is obtained by separating the first n coordinates from (15), i.e., the formula (13) holds.

Theorem 1 is proved.

3. Representation of the solution to problem (7), (6). In this section, we will derive a representation of the solution to the problem (7), (6). Together with equation (7), we consider the linear system (8) where C, in this case, is a $(2n \times 2n)$-matrix defined by

$$C := \begin{pmatrix} A & I \\ \Theta & B \end{pmatrix}. \tag{16}$$

It is easy to see that, for $k \in \mathbb{N}$,

$$C^k = \begin{pmatrix} A^k & \sum_{i=0}^{k-1} A^{k-1-i}B^i \\ \Theta & B^k \end{pmatrix}.$$

For a simple formalization of the delayed exponential e_τ^{Ct}, we define a matrix function $e_\tau^{(A,B)t}$ as

$$e_\tau^{(A,B)t} = \sum_{s=0}^{\lfloor t/\tau \rfloor} \frac{(t-(s-1)\tau)^s}{s!} \sum_{i=0}^{s} A^{s-i}B^i.$$

The following formula can be verified directly by utilizing the definitions of special matrix functions,

$$e_\tau^{Ct} = \begin{pmatrix} e_\tau^{At} & e_\tau^{(A,B)t} \\ \Theta & e_\tau^{Bt} \end{pmatrix}. \tag{17}$$

Let us eliminate $y(t)$ from system (8) with the matrix C given by (16). We get the system

$$x'(t) = Ax(t-\tau) + y(t-\tau), \tag{18}$$

$$y'(t) = By(t-\tau), \tag{19}$$

where $t \geq 0$. Differentiating (18) and, subsequently, using both subsystems (18) and (19), we derive the equation

$$x''(t) = Ax'(t-\tau) + y'(t-\tau) = Ax'(t-\tau) + B(x'(t-\tau) - Ax(t-2\tau))$$

and, after some simplification, we get equation (7).

Theorem 2. *Let $AB = BA$. Then the solution of Cauchy initial problem (7), (6) has the form*

$$x(t) = e_\tau^{At}\xi(-\tau) + e_\tau^{(A,B)t}(\xi'(0) - A\xi(-\tau)) + \\ + \int\limits_{-\tau}^{0} \left(e_\tau^{A(t-\tau-s)}\xi'(s) + e_\tau^{(A,B)(t-\tau-s)}(\xi''(s+\tau) - A\xi'(s)) \right) ds \qquad (20)$$

where $t \geq \tau$.

Proof. Using (18) and (19), we derive

$$y(t) = y(-\tau) + \int\limits_{-\tau}^{t} y'(s)\,ds = y(-\tau) + \int\limits_{-\tau}^{t} B(x'(s) - Ax(s-\tau))\,ds =$$

$$= x'(0) - Ax(-\tau) + \int\limits_{-\tau}^{t} B(x'(s) - Ax(s-\tau))\,ds. \qquad (21)$$

Consider the initial-value problem

$$z(t) = \varphi^*(t), \quad -\tau \leq t \leq 0,$$

for system (8) with the matrix C given by (16), i.e., for the system (18), (19) where

$$\varphi^* = \begin{pmatrix} \varphi_x^* \\ \varphi_y^* \end{pmatrix} : [-\tau, 0] \to \mathbb{R}^{2n}$$

is continuously differentiable as specified below. Since $z = \begin{pmatrix} x \\ y \end{pmatrix}$, we set $\varphi_x^*(t) \equiv \xi(t)$, $t \in$ $\in [-\tau, 0]$. Next, we will specify φ_y^*. From (18), we obtain

$$y(t-\tau) = x'(t) - Ax(t-\tau), \quad t \geq 0,$$

or

$$y(t) = x'(t+\tau) - Ax(t), \quad t \geq \tau.$$

Consequently,

$$\varphi_y^*(t) \equiv \xi'(t+\tau) - A\xi(t), \quad t \in [-\tau, 0],$$

and

$$\varphi^*(t) = \begin{pmatrix} \xi(t) \\ \xi'(t+\tau) - A\xi(t) \end{pmatrix}. \tag{22}$$

Now we utilize formula (4) with the matrix A replaced by C in the form (16) and with φ replaced with φ^* given by (22). Utilizing (17), we get

$$z(t) = e_\tau^{Ct}\varphi^*(-\tau) + \int_{-\tau}^{0} e_\tau^{C(t-\tau-s)}\varphi^{*\prime}(s)ds =$$

$$= \begin{pmatrix} e_\tau^{At} & e_\tau^{(A,B)t} \\ \Theta & e_\tau^{Bt} \end{pmatrix} \begin{pmatrix} \xi(-\tau) \\ \xi'(0) - A\xi(-\tau) \end{pmatrix} +$$

$$+ \int_{-\tau}^{0} \begin{pmatrix} e_\tau^{A(t-\tau-s)} & e_\tau^{(A,B)(t-\tau-s)} \\ \Theta & e_\tau^{B(t-\tau-s)} \end{pmatrix} \begin{pmatrix} \xi'(s) \\ \xi''(s+\tau) - A\xi'(s) \end{pmatrix} ds.$$

By separating the first n components, we get formula (20).

Theorem 2 is proved.

4. Concluding remarks. ***4.1. Relation between special delayed matrix functions.*** In the paper [19], other delayed matrix functions called the delayed matrix sine $\mathrm{Sin}_\tau At$ and delayed matrix cosine $\mathrm{Cos}_\tau At$, where A is an $(n \times n)$-matrix, are defined on $\mathbb{R}$ as

$$\mathrm{Sin}_\tau At = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} (-1)^s A^{2s+1} \frac{(t-(s-1)\tau)^{2s+1}}{(2s+1)!} \tag{23}$$

and

$$\mathrm{Cos}_\tau At = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} (-1)^s A^{2s} \frac{(t-(s-1)\tau)^{2s}}{(2s)!}. \tag{24}$$

The delayed matrix sine and cosine are fundamental matrices of a homogeneous second-order linear system with a single delay,

$$x''(t) = -A^2 x(t-\tau), \tag{25}$$

making it possible to simply express the solutions to initial-value problems. In [19], the solution of the Cauchy initial-value problem (3), (25), assuming that the matrix A is regular, is given in the form

$$x(t) = (\mathrm{Cos}_\tau At)\,\varphi(-\tau) + A^{-1}(\mathrm{Sin}_\tau At)\,\varphi'(-\tau) +$$

$$+ A^{-1} \int_{-\tau}^{0} (\mathrm{Sin}_\tau A(t-\tau-\xi))\,\varphi''(\xi)d\xi. \tag{26}$$

The equation (5) turns into (25) if we set $A = \Theta$ and then replace B with A and τ with $\tau/2$. Therefore, analysing formulas (4) and (26), and the formula

$$e^{Ct}_{\tau/2} = \sum_{s=0}^{\lfloor 2t/\tau \rfloor + 1} C^s \frac{(t-(s-1)\tau/2)^s}{s!} = \begin{pmatrix} \operatorname{Re} e^{iAt}_{\tau/2} & \operatorname{Im} e^{iAt}_{\tau/2} \\ -\operatorname{Im} e^{iAt}_{\tau/2} & \operatorname{Re} e^{iAt}_{\tau/2} \end{pmatrix}, \tag{27}$$

obtained from (14) with the above-mentioned modifications and with

$$C =: \begin{pmatrix} \Theta & A \\ -A & \Theta \end{pmatrix}, \tag{28}$$

we conclude that there exists a relation between the delayed matrix exponential and the delayed sine and cosine matrices. The next theorem provides us with such a relation.

Theorem 3. *The formula*

$$e^{Ct}_{\tau/2} = \begin{pmatrix} \operatorname{Cos}_\tau A(t-\tau/2) & \operatorname{Sin}_\tau A(t-\tau) \\ -\operatorname{Sin}_\tau A(t-\tau) & \operatorname{Cos}_\tau A(t-\tau/2) \end{pmatrix} \tag{29}$$

holds for every $t \in \mathbb{R}$.

Proof. Let us compare the definitions (23) and (24) with the elements of the delayed matrix exponential $e^{Ct}_{\tau/2}$ expressed by (27) where C is given by (28), i.e., with $\operatorname{Re} e^{iAt}_{\tau/2}$ and $\operatorname{Im} e^{iAt}_{\tau/2}$. Next we will use the formula

$$\operatorname{Im}\left(\sum_{s=0}^{m} (iA)^s\right) = \sum_{u=0}^{\lfloor (m-1)/2 \rfloor} (-1)^u A^{2u+1}$$

which holds for an arbitrary integer m. Let k be an integer and $t \in [k\tau, (k+1)\tau)$. Then

$$\left\lfloor \frac{\lfloor 2t/\tau + 1 \rfloor - 1}{2} \right\rfloor = \left\lfloor \frac{\lfloor 2t/\tau \rfloor}{2} \right\rfloor = k$$

and

$$\operatorname{Im} e^{iAt}_{\tau/2} = \operatorname{Im}\left(\sum_{s=0}^{\lfloor 2t/\tau \rfloor + 1} (iA)^s \frac{(t-(s-1)\tau/2)^s}{s!}\right) =$$

$$= \operatorname{Im}\left(\sum_{s=0}^{2k+1} (iA)^s \frac{(t-(s-1)\tau/2)^s}{s!}\right) =$$

$$= \sum_{u=0}^{k} (-1)^u A^{2u+1} \frac{(t-(2u+1-1)\tau/2)^{2u+1}}{(2u+1)!} =$$

$$= \sum_{u=0}^{k} (-1)^u A^{2u+1} \frac{(t-u\tau)^{2u+1}}{(2u+1)!}\,. \tag{30}$$

Moreover, for $t \in [k\tau, (k+1)\tau)$, the definition (23) yields

$$\mathrm{Sin}_\tau A(t-\tau) = \sum_{s=0}^{k}(-1)^s A^{2s+1}\frac{(t-\tau-(s-1)\tau)^{2s+1}}{(2s+1)!} = \sum_{s=0}^{k}(-1)^s A^{2s+1}\frac{(t-s\tau)^{2s+1}}{(2s+1)!}. \qquad (31)$$

Comparing (30) and (31), we get

$$\mathrm{Sin}_\tau A(t-\tau) = \mathrm{Im}\, e_{\tau/2}^{iAt} \qquad (32)$$

for every $t \in \mathbb{R}$.

Next, we use the formula

$$\mathrm{Re}\left(\sum_{s=0}^{m}(iA)^s\right) = \sum_{u=0}^{\lfloor m/2\rfloor}(-1)^u A^{2u}.$$

Let $t \in [(2k-1)\tau/2, (2k+1)\tau/2)$. Then

$$\left\lfloor \frac{\lfloor 2t/\tau+1\rfloor}{2}\right\rfloor = k$$

and

$$\mathrm{Re}\, e_{\tau/2}^{iAt} = \mathrm{Re}\left(\sum_{s=0}^{\lfloor 2t/\tau\rfloor+1}(iA)^s\frac{(t-(s-1)\tau/2)^s}{s!}\right) =$$

$$= \mathrm{Re}\left(\sum_{s=0}^{\lfloor(\lfloor 2t/\tau\rfloor+1)/2\rfloor}(iA)^s\frac{(t-(s-1)\tau/2)^s}{s!}\right) =$$

$$= \mathrm{Re}\left(\sum_{s=0}^{k}(iA)^s\frac{(t-(s-1)\tau/2)^s}{s!}\right) =$$

$$= \sum_{u=0}^{k}(-1)^u A^{2u}\frac{(t-(2u-1)\tau/2)^{2u}}{(2u)!}. \qquad (33)$$

Moreover, for $t \in [(2k-1)\tau/2, (2k+1)\tau/2)$, the definition (24) yields

$$\mathrm{Cos}_\tau A(t-\tau/2) = \sum_{s=0}^{\lfloor(t-\tau/2)/\tau\rfloor+1}(-1)^s A^{2s}\frac{(t-\tau/2-(s-1)\tau)^{2s}}{(2s)!} =$$

$$= \sum_{s=0}^{k}(-1)^s A^{2s}\frac{(t-(2s-1)\tau/2)^{2s}}{(2s)!} \qquad (34)$$

and, comparing (33) and (34), we get

$$\mathrm{Cos}_\tau A(t-\tau/2) = \mathrm{Re}\, e_{\tau/2}^{iAt} \tag{35}$$

for every $t \in \mathbb{R}$. Now it is easy to see that, by (27), (32), and (35), the formula (29) holds.

Theorem 3 is proved.

4.2. On classes of formally solvable equations. The problems (5), (6) and (6), (7), considered in the paper, are special cases of a general problem,

$$\begin{aligned} &x''(t) + Px'(t-\tau) + Qx(t-2\tau) = \theta, \quad t \geq \tau, \\ &x^{(i)}(t) = \xi^{(i)}(t), \quad i = 0,1, \quad t \in [-\tau, \tau], \end{aligned} \tag{36}$$

where P, Q are constant $(n\times n)$-matrices provided that there exists an $(n\times n)$-matrix Λ satisfying the equation

$$\Lambda^2 + P\Lambda\exp(-\tau\Lambda) + Q\exp(-2\tau\Lambda) = \Theta. \tag{37}$$

We assume that a solution of (36) can be found in the form

$$x(t) = \exp(\Lambda t) \tag{38}$$

where Λ is a suitable constant $(n \times n)$-matrix. By substituting (38) into (36), we get

$$\Lambda^2\exp(2\Lambda t) + P\Lambda\exp(\Lambda(t-\tau)) + Q\exp(\Lambda(t-2\tau)) = \Theta$$

and further simplification yields equation (37). Let $Y = \exp(2\Lambda\tau)$ be a new unknown matrix. Then, equation (37) can be written as

$$Y^2 + PY + Q = \Theta. \tag{39}$$

The matrices A and B of the system (5) (i.e., $P = -2A$ and $Q = A^2 + B^2$) generate complex conjugate roots of (39),

$$Y_{1,2} = A \pm iB,$$

and the matrices A and B of the system (7) (i.e., $P = -A - B$ and $Q = AB$) generate real roots of (39),

$$Y_1 = A, \quad Y_2 = B.$$

The systems (5) and (7) are equivalent to system (8) with the matrix C defined by (9) or by (16). Matrices on the right-hand sides of (9) and (16) can be viewed as “Jordan"forms of the matrix C. From this point of view, the last case of the “Jordan"form of the block matrix C is

$$C := \begin{pmatrix} A & \Theta \\ \Theta & B \end{pmatrix}. \tag{40}$$

This matrix defines a pair of independent subsystems. This case is trivial since it is not possible to eliminate n variables to obtain a second-order system. The above analysis leads to a conclusion that the classes of the equations considered formally cover all the possible cases of the roots of the quadratic equation (39) and “Jordan"forms (9), (16) and (40). For the first two of these three cases, we derived a representation of solutions of initial-value problems. Representations of solutions of initial-value problems in the third case was derived, as was mentioned above, in [1].

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