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Group Classification of Systems of Nonlinear Reaction-Diffusion Equations

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Abstract. The completed group classification of systems of two coupled nonlinear reaction-diffusion equations with general diffusion matrix is carried out. A simple and convenient method for the deduction and solution of classifying equations is presented.

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1. Introduction

The group classification of differential equations is one of corner stones of group analysis. This classification specifies the origin of possible applications of powerful group-theoretic tools such as the construction of exact solutions, group generation of solution families starting with known ones, etc. A very important result of group classification consists of an *a priori* description of mathematical models with desired symmetry (e.g., relativistic invariance).

One of the most impressive results in group classification belongs to S. Lie, who had completely classified second-order ordinary differential equations [17]. Lie was also the first who presented the group classification of partial differential equations (PDE), namely, he had classified linear equations including two independent variables [18].

Using the classical Lie approach, whose excellent presentation was given in [25], it is not difficult to derive determining equations for possible symmetries admitted by equations of interest. Moreover, to describe Lie symmetries for a fixed (even if very complicated) equation is a purely technical problem, which is easily solved using special software packages. However, the situation changes dramatically whenever we try to search

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for Lie symmetries for an equation including an arbitrary element that is not *a priori* specified, i.e., when we are interested in group classification of an entire class of differential equations.

The main problem of group classification of a substantially extended class of partial differential equations (PDE) consists of the effective solution of determining equations for the coefficients of generators of a symmetry group. In general, the determining equations are rather complicated systems whose variables are not necessarily separable.

A nice result in group classification of PDE belongs to Dorodnitsyn [26], who had classified nonlinear (but quasilinear) heat equations

$$u_t - u_{xx} = f(u), \tag{1.1}$$

where f is an arbitrary function of the dependent variable u, the subscripts denote derivatives with respect to the corresponding variables, i.e., $u_t = \partial u/\partial t$ and $u_{xx} = \partial^2 u/\partial x^2$. Moreover, in [26], more general equations $u_t - (Ku_x)_x = f(u)$ were classified. The related determining equations appear to be easily integrable, which made it possible to specify all nonequivalent nonlinearities f (which are power, logarithmic, and exponential ones) that correspond to different symmetries of Eq. (1.1). The nonclassical (conditional) symmetries of (1.1) were described by Fushchych and Serov [10] and Clarkson and Mansfield [6].

The results of group classification of Eqs. (1.1) play an important role in the construction of their exact solutions and the qualitative analysis of the nonlinear heat equation (see, e.g., [28]).

In the present paper, we perform the group classification of systems of the nonlinear reaction-diffusion equations

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \Delta(a_{11}u_1 + a_{12}u_2) &= f^1(u_1, u_2), \\ \frac{\partial u_2}{\partial t} - \Delta(a_{21}u_1 + a_{22}u_2) &= f^2(u_1, u_2), \end{aligned} \tag{1.2}$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a function of $t, x_1, x_2, \ldots, x_m$, the symbols $a_{11},\ a_{12}$, $a_{21},\ a_{22}$ denote real constants, and Δ is the Laplace operator in R^m. We also write (1.2) in the matrix form:

$$\frac{\partial u}{\partial t} - A\Delta u = f, \tag{1.3}$$

where A is the matrix whose elements are $a_{11}, \ldots a_{22}$ and $f = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}$.

Mathematical models based on Eqs. (1.2) are widely used in mathematical physics and mathematical biology. Some of these models are discussed in [23] and in Sec. 12 of the present paper, and the entire collection of such models is presented in [20]. Thus, the symmetry analysis of Eqs. (1.2) has a large application value and can be used, e.g., for the construction of exact solutions for a very extended class of physical and biological systems. The comprehensive group analysis of systems (1.2) is also a nice "internal" problem of the Lie theory, which admits an exact general solution for the case of an *arbitrary* number of independent variables $x_1, x_2, \ldots, x_m$.

Symmetries of Eq. (1.3) for the case of a *diagonal* (and invertible) matrix A were investigated by Yu. A. Danilov [7]. Unfortunately, the results presented in [7] and cited in the handbook [14] are neither complete nor correct. We discuss these results in detail in Sec. 12.

Symmetry classification of Eqs. (1.3) with diagonal diffusion matrix was presented in [3], and then some results missing in [3] were added in Addendum [4] and the paper [5]. However, we shall demonstrate that the results given in [3–5] are still incomplete and add the list of nonequivalent equations given in these papers.

Note that symmetries of Eqs. (1.3) with diagonal diffusion matrix were partially described in [15], where symmetries of a more general class of diffusion equations were studied.

Equations (1.3) with *arbitrary* invertible matrix A were investigated in [23], and the related results were announced in [24]. Unfortunately, mainly due to typographical errors made during the publishing procedure, the presentation of classification results in [23] was not satisfactory.[1]

In the present paper, we give the completed group classification of coupled reaction-diffusion equations (1.3) with an arbitrary diffusion matrices A. Moreover, we present a straightforward and easily verified procedure of the solution of determining equations, which guarantees the completeness of the results obtained. We also indicate clearly the equivalence relations used in the classification procedure. In addition, we extend the results obtained in [23] to the case of a *noninvertible* matrix A.

The additional aim of this paper is to present a rather straightforward and conventional algorithm for the investigation of symmetries of a class of partial differential equations that includes (1.3) as a particular case. We show that the classical Lie approach (see, e.g., [11] and [25]) when applied to systems (1.3) admits a rather simple formulation, which can be used even by investigators who are not experts in the group analysis of differential equations. Furthermore, the algorithm can be used for finding conditional symmetries of (1.3) [23] (for the definition of conditional

[1] The tables presenting the results of group classification were deformed and cut off. It is necessary to stress that it was authors' fault, one of whom signed the paper proofs without careful reading.

symmetries, see [12]).

There exist two nonequivalent 2×2 matrices with zero determinant, namely, the diagonal matrix with the only nonzero element and the Jordan cell. We consider the following generalized versions of the related equation (1.2):

$$\begin{aligned} &\partial_t u_1 - \Delta u_1 = f^1(u_1, u_2), \\ &\partial_t u_2 - p_\mu \partial_\mu u_1 = f^2(u_1, u_2) \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\partial_t u_1 - p_\mu \partial_\mu u_2 = f^1(u_1, u_2), \\ &\partial_t u_2 - \Delta u_1 = f^2(u_1, u_2). \end{aligned} \tag{1.5}$$

Here, p_μ are arbitrary constants and the summation is carried out over repeating $\mu = 1, 2, \cdots, m$. Moreover, without loss of generality, we set

$$p_1 = p_2 = \cdots = p_{m-1} = 0, \ p_m = p. \tag{1.6}$$

In the case $p \equiv 0$, Eqs. (1.4) and (1.5) are nothing but particular cases of (1.2), which include such popular models of mathematical biology as the FitzHugh–Nagumo [9] and Rinzel–Keller [27] ones. In addition, (1.5) can serve as a potential equation for the nonlinear d'Alembert equation.

The determining equations for symmetries of Eqs. (1.2) are rather complicated systems of PDE including two arbitrary elements, i.e., unknown functions f^1 and f^2. To handle them we use the approach developed in [29], whose main idea is to make an *a priori* classification of realizations of the related Lie algebras. In fact, this method has roots in works of S. Lie, who used his knowledge of vector-field representations of Lie algebras in a space of two variables to classify second-order ordinary equations [17]. In the case of partial differential equations, we have no hope to classify all related realizations of vector fields . However, for some fixed classes of PDE, it appears to be possible to make this classification, restricting ourselves to realizations that are compatible with equations of interest [29].

Note that an analogous technique was used earlier [13] for the classification of the nonlinear Schrödinger equations with cubic nonlinearity and variable coefficients.

In Sec. 2, we present the general equivalence transformations for Eqs. (1.3) that are valid for arbitrary nonlinearities f^1 and f^2 and give the list of additional equivalence transformations that are valid for some fixed nonlinearities.

In Sec. 3, a simplified algorithm for the investigation of symmetries of systems of reaction-diffusion equations is presented.

In Sec. 4, we deduce determining equations for symmetries admitted by Eqs. (1.3) and specify the general form of the related group generators.

In Sec. 5, we present the kernel of the symmetry group for Eqs. (1.3) and give definitions of main and extended symmetries.

In Secs. 6–8, the results of group classification of Eqs. (1.4) and (1.5) are presented. Equations (1.3) with invertible diffusion matrix are classified in Secs. 9 and 10, and the case of a nilpotent diffusion matrix is studied in Sec. 11.

In Sec. 12, we discuss the results of group classification and present some important model equations that appear to be particular subjects of our analysis. Appendix includes an *a priori* classification of realizations of the low-dimensional Lie algebras that are used in the main text for solving the determining equations.

2. Equivalence Transformations

The problem of group classification of Eqs. (1.2)–(1.5) will be solved up to equivalence transformations.

We say the equation

$$\tilde{u}_t - \tilde{A}\Delta\tilde{u} = \tilde{f}(\tilde{u}) \tag{2.1}$$

is equivalent to (1.3) if there exist an invertible transformation $u \to \tilde{u} = G(u,t,x)$, $t \to \tilde{t} = T(t,x,u)$, $x \to \tilde{x} = X(t,x,u)$, $f \to \tilde{f} = F(u,t,x,f)$ that connects (1.3) with (2.1). In other words, the equivalence transformations should keep the general form of Eq. (1.3) but can change concrete realizations of the matrix A and nonlinear terms f^1 and f^2.

Let us note that there are six *ad hoc* nonequivalent classes of Eqs. (1.3) corresponding to the following forms of matrices A:

$$\begin{gathered} I.\ A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad I^*.\ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad II.\ A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}, \\ III.\ A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad IV.\ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad V.\ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{gathered} \tag{2.2}$$

where a is an arbitrary parameter. Indeed, any 2×2 matrix A can be reduced to one of the forms (2.2) by using linear transformations of dependent variables and scaling independent variables in (1.3). For the matrices I and III, it is possible to restrict ourselves to the cases $a \neq 0, 1$ and $a \neq 0$, respectively, but we prefer to reserve the possibility to treat the version I^* as a particular case of the versions I and III.

The group of equivalence transformations for Eq. (1.3) can be found by using the classical Lie approach and treating f^1 and f^2 as additional dependent variables. In addition to the obvious symmetry transformations

$$t \to t' = t + a, \quad x_\mu \to x'_\mu = R_{\mu\nu} x_\nu + b_\mu, \tag{2.3}$$

where a, b_μ and $R_{\mu\nu}$ are arbitrary parameters satisfying the condition $R_{\mu\nu} R_{\mu\lambda} = \delta_{\mu\lambda}$, this group includes the following transformations:

$$\begin{aligned} &u_a \to K^{ab} u_b + b_a, \quad f^a \to \lambda^2 K^{ab} f^b, \\ &t \to \lambda^{-2} t, \quad x_a \to \lambda^{-1} x_a, \end{aligned} \tag{2.4}$$

where K^{ab} are elements of an invertible constant matrix K commuting with A, $\lambda \neq 0$, and b_a are arbitrary constants.

Let us specify the form of matrices K. By definition, K commutes with A, and so, for the versions I–V presented in (2.2), we have

$$I^* : \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad K_{11} K_{22} - K_{21} K_{12} \neq 0, \tag{2.5}$$

$$I, IV : \quad K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \quad K_1 K_2 \neq 0, \tag{2.6}$$

$$II : \quad K = \begin{pmatrix} K_1 & -K_2 \\ K_2 & K_1 \end{pmatrix}, \quad K_1^2 + K_2^2 \neq 0, \tag{2.7}$$

$$III, V : \quad K = \begin{pmatrix} K_1 & 0 \\ K_2 & K_1 \end{pmatrix}, \quad K_1 \neq 0. \tag{2.8}$$

In addition, for case I there is one more transformation (2.4) with

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda^2 = a. \tag{2.9}$$

Such transformations reduce to the change $a \to \frac{1}{a}$ in the related matrix A, i.e., to scaling the parameter a.

The equivalence transformations (2.4) are also valid for Eqs. (1.4) and (1.5). The related matrices K are given in (2.6) and (2.8).

It is possible to show that there are no more extended equivalence relations valid for arbitrary nonlinearities f^1 and f^2. However, if functions f^1 and f^2 are fixed, the class of equivalence transformations is more extended. In addition to transformations (2.4), it includes symmetry transformations that do not change the form of Eq. (1.3). Moreover, for some classes of functions f^1 and f^2, Eq. (1.3) admits additional equivalence transformations (AET). The corresponding set of equivalence transformations for Eq. (1.3) can be found by using the classical Lie approach and treating f^1 and f^2 as additional dependent variables constrained by the relations specifying the dependence of f^1 and f^2 on u_1 and u_2.

In spite of the fact that we search for AET *after* the description of symmetries of Eqs. (1.3) and the specification of functions f^1 and f^2, for convenience we present the list of additional equivalence transformations in the following formulas:

$$
\begin{array}{rl}
1. & u_1 \to \exp(\omega t)u_1, \quad u_2 \to \exp(\rho t)u_2, \\
2. & u_1 \to u_1 + \omega t + \lambda_a x_a + \mu x^2, \ u_2 \to u_2, \\
3. & u_1 \to u_1, \ u_2 \to u_2 + \rho t + \lambda_a x_a + \mu x^2, \\
4. & u_1 \to u_1 + \rho t, \ u_2 \to u_2 \exp(\rho t), \\
5. & u_1 \to \exp(\omega t)u_1, \ u_2 \to u_2 + \omega t, \\
6. & u_1 \to u_1, \ u_2 \to u_2 + \rho t u_1, \\
7. & u_1 \to \exp(\omega t)u_1, \ u_2 \to u_2 + \omega \frac{t^2}{2}, \\
8. & u_1 \to \exp(\omega t)u_1, \ u_2 \to u_2 + \kappa t u_1 + \rho \frac{t^2}{2}, \\
9. & u_1 \to u_1, \ u_2 \to u_2 - \rho t u_1 + \rho\lambda \frac{t^2}{2}, \\
10. & u_1 \to \exp(\rho t)u_1, \ u_2 \to u_2 - \kappa\rho t, \\
11. & u_1 \to \exp(\rho t)u_1, \ u_2 \to \exp(\rho t)\Big(u_2 + \varepsilon\rho \frac{t^2}{2} u_1\Big), \\
12. & u_1 \to u_1 + \rho t + \nu x^2, \ u_2 \to u_2 - \rho t - \nu x^2, \\
13. & u_1 \to u_1 + \rho t, \ u_2 \to e^{-\frac{\rho}{\nu}t} u_2, \\
14. & u_1 \to u_1 + \rho t, \ u_2 \to u_2 + \rho t u_1 + \rho \frac{t^2}{2}, \\
15. & u_1 \to u_1 \cos\omega t - u_2 \sin\omega t, \ u_2 \to u_2 \cos\omega t + u_1 \sin\omega t, \\
16. & u_1 \to \exp(\omega t)u_1, \ u_2 \to \exp(\omega t)(u_2 - \omega t u_1) \\
17. & \text{Transformations (11.2) valid for equations with matrix } A \\
 & \ \text{of the type } V \text{ only.}
\end{array}
\tag{2.10}
$$

Here, the Greek letters denote parameters that are either arbitrary or specified in the tables presented below. We stress once more that, in contrast to (2.4), the equivalence transformations (2.10) are admitted by some particular equations (1.3), which will be specified in what follows.

3. An Algorithm for the Description of Symmetries for Systems (1.3)–(1.5)

Let us investigate Lie symmetries of systems (1.3)–(1.5), i.e., find all continuous groups of transformations for u, t, x that keep these equations invariant. In contrast to equivalence transformations, symmetry transformations do not change functions f^1 and f^2.

Since any term in (1.3) does not depend on t and x explicitly, this equation *with arbitrary functions* f^1 *and* f^2 admits obvious symmetry with respect to translations of all independent variables and rotations of spatial variables present in (2.3). For Eqs. (1.4) and (1.5) such symmetries also have the form (2.3), where the indices of $R_{\mu\nu}$ run through the values $1, 2, \dots, m-1$.

To find all Lie symmetries we require the form-invariance of the systems of reaction-diffusion equations with respect to one-parameter groups of transformations:

$$t \to t'(t, x, \varepsilon), \qquad x \to x'(t, x, \varepsilon), \qquad u \to u'(t', x', \varepsilon), \tag{3.1}$$

where ε is a group parameter. In other words, we require that $u'(t', x', \varepsilon)$ satisfy the same equation as $u(t, x)$:

$$L'u' = f(u'), \tag{3.2}$$

where L are the linear differential expressions involved into Eqs. (1.3)–(1.5), i.e.,

$$L = \frac{\partial}{\partial t} - A \sum_i \frac{\partial_i^2}{\partial x_i^2} - B \frac{\partial}{\partial x_m}, \quad L' = \frac{\partial}{\partial t'} - A \sum_i \frac{\partial_i^2}{\partial x_i'^2} - B \frac{\partial}{\partial x_m'}.$$

Here, B is the zero matrix for Eq. (1.3), $B = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}$ for Eqs. (1.4), and $B = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$ for Eq. (1.5).

Beginning with the infinitesimal transformations

$$\begin{aligned} &t \to t' = t + \Delta t = t + \varepsilon\eta, \qquad x_a \to x_a' = x_a + \Delta x_a = x_a + \varepsilon\xi^a, \\ &u_a \to u_a' = u_a + \Delta u_a = u_a + \varepsilon\pi_a \end{aligned} \tag{3.3}$$

we obtain the following representation for the operator L':

$$L' = \left[1 + \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] L \left[1 - \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] + O(\varepsilon^2). \tag{3.4}$$

Using the Lie algorithm, one can find find the determining equations for the functions η, ξ_a, and π_a, which specify the generator X of the symmetry group:

$$X = \eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a} - \pi^b \frac{\partial}{\partial u_b}, \tag{3.5}$$

where the summation from 1 to m and from 1 to n is assumed over the repeated indices a and b, respectively. We will obtain these determining equations directly.

First, note that, without loss of generality, we can restrict ourselves to functions η, ξ^a, and π^a that satisfy the conditions

$$\frac{\partial \eta}{\partial u_a} = 0, \quad \frac{\partial \xi^a}{\partial u_b} = 0, \quad \frac{\partial^2 \pi^a}{\partial u_c \partial u_b} = 0. \tag{3.6}$$

This is nothing but a consequence of results of [2], where PDE whose symmetries satisfy (3.6) are classified. These results admit a straightforward generalization to the case of systems (2.2) with invertible matrix A.

Substituting (3.3) and (3.4) into (3.2), using (1.3)–(1.5), and neglecting the terms of order ε^2, we find that

$$[Q, L]\, u - L\omega = \pi f + \frac{\partial f}{\partial u_a} \left(-\pi^{ab} u_b - \omega^a \right), \tag{3.7}$$

where

$$Q = \eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a} + \pi,$$

$[Q, L] = QL - LQ$ is the commutator of the operators Q and L, and π is the matrix whose elements are π^{ab}, so that [23] $\pi^a = \pi^{ab} u_b + \omega^a$, with π^{ab} and ω^a being functions of independent variables t, x.

Equation (3.7) is compatible with (1.3)–(1.5) and does not impose new nontrivial conditions on u if the commutator $[Q, L]$ admits the representation

$$[Q, L] = \Lambda L + \varphi \tag{3.8}$$

where Λ and φ are 2×2 matrices dependent on t, x.

Substituting (3.8) into (3.7), we obtain the following classifying equations for f:

$$\left(\Lambda^{kb} + \pi^{kb} \right) f^b + \varphi^{kb} u^b + (L\omega)^k = (\omega^a + \pi^{ab} u_b) \frac{\partial f^k}{\partial u^a}. \tag{3.9}$$

Thus, to find all nonlinearities f^k generating Lie symmetries for Eqs. (1.3)–(1.5), it is necessary to solve the operator equations (3.8) and find the general form of the matrices Λ, π, and φ and the functions η and

ξ. In the second step, we find the nonlinearities f^a solving system (3.9) with its known coefficients.

We stress that the described procedure of group classification of Eqs. (1.3)–(1.5) is equivalent to the standard Lie algorithm but is more straightforward. In addition, it is rather convenient, and, till an appropriate moment, all equations (1.3) with nonsingular matrices A can be analyzed in a parallel way.

4. Determining Equations

Evaluating the commutator in (3.8) and equating the coefficients for linearly independent differential operators, we obtain the determining equations

$$\left(\frac{\partial \xi^a}{\partial x_b}+\frac{\partial \xi^b}{\partial x_a}\right)A=-\delta_{ab}(\Lambda A+[A,\pi]),\quad \frac{\partial^2\eta}{\partial t\partial x_a}=0,\quad \frac{\partial \eta}{\partial t}=\Lambda, \tag{4.1}$$

$$\frac{\partial \xi^a}{\partial t}-2\frac{\partial}{\partial x_a}A\pi-\Delta A\xi^a=0,\qquad \varphi=\frac{\partial \pi}{\partial t}-\Delta A\pi \tag{4.2}$$

where δ_{ab} is the Kronecker symbol.

The general expressions for the coefficient functions η, ξ^a, and π of symmetry X (3.5) can be obtained by evaluating the determining equations (4.1) and (4.2). We do not reproduce this procedure here but present the general form of the related generator (3.5) found in [23]:

$$\begin{aligned} X=\lambda K+\sigma_\alpha G_\alpha+\omega_\alpha \widehat{G}_\alpha+\mu D-2(C^{ab}u_b+B^a)\frac{\partial}{\partial u_a}\\ +\Psi^{\mu\nu}x_\mu\partial_\nu+\nu\partial_t+\rho_\mu\partial_\mu \end{aligned} \tag{4.3}$$

where the Greek letters denote arbitrary constants, $\Psi^{\mu\nu}=-\Psi^{\nu\mu}$, B^a are functions of t and x, C^{ab} are functions of t such that

$$C^{ab}A^{bk}-A^{ab}C^{bk}=0, \tag{4.4}$$

and

$$\begin{aligned} K&=2t\Big(t\frac{\partial}{\partial t}+x_\mu\frac{\partial}{\partial x_\mu}\Big)-\frac{x^2}{2}(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a}-tm\Big(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}\Big),\\ G_\alpha&=t\partial_\alpha+\frac{1}{2}x_\alpha(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a},\\ \widehat{G}_\alpha&=e^{\gamma t}\Big(\partial_\alpha+\frac{1}{2}\gamma x_\alpha(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a}\Big),\\ D&=2t\frac{\partial}{\partial t}+x_\mu\frac{\partial}{\partial x_\mu}. \end{aligned} \tag{4.5}$$

If $a = 0$, then the related generator X again has the form (4.3), where, however, $\lambda = \sigma_\mu = \omega_\mu = C^2 = 0$ and B^2 is a function of t, x and u.

Formula (4.3) presents a symmetry operator for Eq. (1.2) iff the related classifying equations (3.9) for f^1 and f^2 are satisfied, i.e.,

$$\begin{aligned}
&(\lambda t(m+4)+\mu)f^a + \Big(\frac{1}{2}\lambda x^2 + \sigma_\mu x_\mu + \gamma e^{\gamma t}\omega_\mu x_\mu\Big)(A^{-1})^{ab} f^b \\
&\qquad + C^{ab} f^b + C^{ab}_t u_b + B^a_t - \Delta A^{ab} B^b \\
&\qquad = \Big(B^s + C^{sb} u_b + \lambda t m u_s + \Big(\frac{1}{2}\lambda x^2 + \sigma_\mu x_\mu \\
&\qquad\qquad + \gamma e^{\gamma t}\omega_\mu x_\mu\Big)(A^{-1})^{sk} u_k\Big)\frac{\partial}{\partial u_s} f^a. \qquad (4.6)
\end{aligned}$$

Thus, the group classification of Eqs. (1.3) with nonsingular matrix A reduces to solving Eq. (4.6), where $\lambda, \mu, \sigma_\nu, \omega_\nu$, and γ are arbitrary parameters and B^a and C^{ab} are functions of (t, x) and t, respectively. Moreover, the matrix C with elements C^{ab} should commute with A.

Note that relations (4.3)–(4.6) are valid for the group classification of systems (1.3) of coupled reaction-diffusion equations including an *arbitrary* number n of dependent variables $u = (u_1, u_2, \dots u_n)$, provided that the related $n \times n$ matrix A is invertible [23]. In this case, the indices a, b, s, and k in (4.3)–(4.6) run through the values $1, 2 \dots n$.

Now consider Eq. (1.4) and the related symmetry operator (3.5). The determining equations for η, ξ^μ, and π^a are easily obtained using (3.8) and (3.9) and have the following form:

$$\begin{aligned}
&\eta_{tt} = \eta_{x_\mu} = \frac{\partial \eta}{\partial u_a} = 0, \quad \xi^\mu_t = \frac{\partial \xi^\mu}{\partial u_a} = 0, \\
&\frac{\partial^2 \pi^a}{\partial u_b \partial u_c} = 0, \quad \frac{\partial \pi^a_{x_\mu}}{\partial u_b} = 0, \quad \frac{\partial \pi^1}{\partial u_2} = \frac{\partial \pi^2}{\partial u_1} = 0, \\
&\frac{\partial \pi^1}{\partial u_1} - \frac{\partial \pi^2}{\partial u_2} = \frac{1}{2}\eta_t \ \text{ if } \ p \neq 0, \\
&\xi^\mu_{x_\nu} + \xi^\nu_{x_\mu} = -\delta^{\mu\nu}\eta_t, \ \mu \neq m,
\end{aligned} \qquad (4.7)$$

where the subscripts denote derivatives with respect to the corresponding independent variable, i.e., $\eta_t = \frac{\partial \eta}{\partial t}$, $\xi^\mu_{x_\nu} = \frac{\partial \xi^\mu}{\partial x_\nu}$, etc.

Integrating system (4.7), we obtain the general form of the operator X:

$$X = \nu\partial_t + \rho_\nu\partial_\nu + \Psi^{\mu\nu}\partial_\nu x_\mu + \mu D - 2B^a\frac{\partial}{\partial u_a} - 2Fu_1\frac{\partial}{\partial u_1} - 2Gu_2\frac{\partial}{\partial u_2}, \quad (4.8)$$

$$\mu = 2(F - G) \ \text{ if } p \neq 0, \qquad (4.9)$$

where B^a are functions of (t, x), F and G are functions of t, and the summation over the indices μ and ν is assumed with $\mu, \nu = 1, 2, \cdots, n-1$.

The classifying equations (3.9) reduce to the following ones:

$$\begin{aligned}
&(\mu + F)f^1 + F_t u_1 + (\partial_t - \Delta)B^1 \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + Gu_2\frac{\partial}{\partial u_2}\Big)f^1, \\
&(\mu + G)f^2 + G_t u_2 + B^2_t - pB^2_{x_m} \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + Gu_2\frac{\partial}{\partial u_2}\Big)f^2.
\end{aligned} \tag{4.10}$$

Relations (4.8)–(4.10) are valid for $p \neq 0$ and $p = 0$ as well (in the last case, condition (4.9) should be omitted). Solving (4.10), we specify both the coefficients of the infinitesimal operator (4.8) and the related nonlinearities f^1 and f^2.

For Eqs. (1.5), we establish by analogy that generator (3.5) reduces to

$$X = \mu\left(3t\partial_t + 2x_\nu\partial_\nu - u_2\frac{\partial}{\partial u_2}\right) - F\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right) - B^a\frac{\partial}{\partial u_a}, \tag{4.11}$$

while the classifying equations are

$$\begin{aligned}
&(3\mu + F)f^1 + F_t u_1 + B^1_t - pB^2_{x_m} \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + (F + \mu)u_2\frac{\partial}{\partial u_2}\Big)f^1, \\
&(4\mu + F)f^2 + F_t u_2 + B^2_t - \Delta B^1 \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + (F + \mu)u_2\frac{\partial}{\partial u_2}\Big)f^2,
\end{aligned} \tag{4.12}$$

where F and B^1, B^2 are unknown functions of t and t, x, respectively.

The determining equations for the symmetries of Eq. (1.5) with $p = 0$ are qualitatively different for the cases where the number m of spatial variables $x_1, x_2, \dots x_m$ is $m = 1$, $m = 2$, and $m > 2$. The related generator (3.5) has the form

$$\begin{aligned}
X = \alpha D + &\left(\int (N - M)dt\right)\frac{\partial}{\partial t} + 2mH_a\frac{\partial}{\partial x_a} \\
&- \left(N + (m-2)\frac{\partial H_a}{\partial x_a}\right)u_1\frac{\partial}{\partial u_1} - \left(M + (m+2)\frac{\partial H_a}{\partial x_a}\right)u_2\frac{\partial}{\partial u_2} \\
&- B^1\frac{\partial}{\partial u_1} - B^2\frac{\partial}{\partial u_2} - B^3 u_1\frac{\partial}{\partial u_2},
\end{aligned} \tag{4.13}$$

where the summation from 1 to m is carried out over repeating indices, the Greek letters denote arbitrary parameters, M and N are functions of t, B^1 and B^2 are functions of t, x, B^3 is a function of t, x, u_1, and $H_a = 2\lambda_b x_b x_a - x^2\lambda_a$ for $m > 2$. For $m = 2$, H_a are arbitrary functions satisfying the Cauchy–Riemann conditions $\frac{\partial H_1}{\partial x_1} = \frac{\partial H_2}{\partial x_2}$, $\frac{\partial H_1}{\partial x_2} = -\frac{\partial H_2}{\partial x_1}$; for $m = 1$, H_1 is a function of x, and the sums with respect to a in (4.13) degenerate into single terms.

The corresponding classifying equations have the form

$$
\begin{aligned}
&\Big(\frac{\alpha}{2} + 2N - M + (m-2)\frac{\partial H_a}{\partial x_a}\Big) f^1 + N_t u_1 + B^1_t \\
&\quad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + B^3 u_1\frac{\partial}{\partial u_2} + \Big(N + (m-2)\frac{\partial H_a}{\partial x_a}\Big)u_1\frac{\partial}{\partial u_1} \\
&\qquad\qquad + \Big(M + (m+2)\frac{\partial H_a}{\partial x_a}\Big)u_2\frac{\partial}{\partial u_2}\Big) f^1, \\
&\Big(\frac{\alpha}{2} + N + (m+2)\frac{\partial H_a}{\partial x_a}\Big) f^2 + B^3 f^1 + M_t u_2 + B^3_t u_1 + B^2_t \\
&\qquad - \Delta B_1 + (2-m)\Big(\Delta\frac{\partial H^a}{\partial x^a}\Big)u_1 = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} \\
&\qquad + B^3 u_1\frac{\partial}{\partial u_2} + \Big(N + (m-2)\frac{\partial H_a}{\partial x_a}\Big)u_1\frac{\partial}{\partial u_1} \\
&\qquad\qquad + \Big(M + (m+2)\frac{\partial H_a}{\partial x_a}\Big)u_2\frac{\partial}{\partial u_2}\Big) f^2.
\end{aligned}
\tag{4.14}
$$

Note that, in this case, the symmetry classification appears to be rather complicated and cumbersome. Nevertheless, the classifying equations can be effectively solved using the approach outlined in the subsequent sections.

Thus, the group classification of Eqs. (1.3), (1.4), and (1.5) reduces to finding general solutions of Eqs. (4.6), (4.10), (4.12), and (4.14). To solve these equations it is necessary to make an effective separation of independent variables. To do this we will use an approach that includes the *a priori* specification and simplification of possible forms of generators X (4.3), (4.8), (4.11), and (4.13) using the condition that X belong to an n-dimensional Lie algebra with $n = 1, 2, \ldots$. This specification will be based on the classification of algebras of 3×3 matrices of special form.

5. Basic, Main, and Extended Symmetries

Let us begin with Eq. (1.3). The general form for the related symmetries and the classifying equation for nonlinearities f^1 and f^2 are given by relations (4.3) and (4.6), respectively.

Equation (4.6) does not include the parameters $\Psi^{\mu\nu}$, ν, and ρ_ν present in (4.3), and, thus, for any f^1 and f^2, Eq. (1.3) admits symmetries generated by the following operators:

$$P_0 = \partial_t, \quad P_\lambda = \partial_\lambda, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu. \tag{5.1}$$

For some classes of nonlinearities f^1 and f^2, the invariance algebra of Eq. (1.3) is more extended but includes (5.1) as a subalgebra. We will refer to (5.1) as to *basic symmetries*.

Operators (5.1) generate the maximal local Lie group that is admitted by Eqs. (1.3) for *any* functions f^1 and f^2. In other words, the basic symmetries generate the kernel of the invariance group of Eq. (1.3).

Let us specify *main symmetries* for Eq. (1.3), whose generator $\tilde{X}$ has the form (4.3) with $\Psi^{\mu\nu} = \nu = \rho_\nu = \sigma_\nu = \omega_\nu = 0$, i.e.,

$$\tilde{X} = \mu D + C^{ab} u_b \frac{\partial}{\partial u_a} + B^a \frac{\partial}{\partial u_a}. \tag{5.2}$$

The classifying equation for symmetries (5.2) can be obtained from (4.6) by setting $\mu = \sigma^a = \omega^a = 0$. As a result, we get

$$(\mu\delta^{ab} + C^{ab}) f^b + C^{ab}_t u_b + B^a_t - \Delta A^{ab} B^b = (C^{nb} u_b + B^n) \frac{\partial f^a}{\partial u_n}. \tag{5.3}$$

Operator (5.2) is a particular case of (4.3). Moreover, it is easily verified that operators (5.2) and (5.1) form a Lie algebra that is a subalgebra of symmetries for Eq. (1.3). On the other hand, if Eq. (1.3) admits a more general symmetry (4.3) with $\sigma_a \neq 0$ or (and) $\lambda \neq 0$, $\omega^\mu \neq 0$, then it has to admit symmetry (5.2) as well. To prove this we will calculate multiple commutators of (4.3) with the basic symmetries (5.1) and use the fact that such commutators have to belong to symmetries of Eq. (1.3).

Let Eq. (1.3) admit the extended symmetry (4.3) with $\sigma_\nu \neq 0$ and $\Psi^{\mu\nu} = \rho_\mu = \nu = \lambda = \omega^k = 0$, i.e.,

$$X = \sigma_\alpha G_\alpha + \mu D + (C^{ab} u_b + B^a) \frac{\partial}{\partial u_b}. \tag{5.4}$$

Commuting Y with P_α, we obtain one more symmetry:

$$Y_\alpha = -\frac{\sigma_\alpha}{2} (A^{-1})^{ab} u_b \frac{\partial}{\partial u_a} + B^a_{x_\alpha} \frac{\partial}{\partial u_a} + \mu P_{x_\alpha}. \tag{5.5}$$

The last term belongs to the basic symmetry algebra (5.1) and so can be omitted. The remaining terms are of the type (5.2).

Thus, supposing that the extended symmetry (5.4) is admissible, we conclude that Eq. (1.3) has to admit the main symmetry as well.

Commuting (5.5) with P_0 and P_λ, we arrive at the following symmetries:

$$Y_{\mu\nu} = B^a_{x_\mu x_\nu}\frac{\partial}{\partial u_a}, \quad Y_{\mu t} = B^a_{x_\mu t}\frac{\partial}{\partial u_a}. \tag{5.6}$$

Any symmetry (5.4)–(5.6) generates its own system (4.6) of classifying equations. After straightforward, but rather cumbersome, calculations, we conclude that all these systems are compatible, provided that the following condition is satisfied:

$$(A^{-1})^{ab} f^b = (A^{-1})^{nb} u_b \frac{\partial f^a}{\partial u_n}. \tag{5.7}$$

If (5.7) is satisfied, then Eq. (1.3) admits symmetry (5.4) with $\mu = C^{ab} = B^a = 0$, i.e., the Galilei generators G_α of (4.5).

By analogy, supposing that Eq. (1.3) admits the extended symmetry (4.3) with $\lambda \neq 0$ and $\omega^a = 0$, we prove that it should also admit symmetry (5.4) with $\mu \neq 0$ and $\sigma_\nu \neq 0$. The related functions f^1 and f^2 must satisfy relations (5.7) and (5.3). Moreover, analyzing a possible dependence of C^{ab} and B^a on t in the corresponding relations (4.6), we conclude that they should be either scalars or linear in t, i.e., $C^{ab} = \mu^{ab} t + \nu^{ab}$. Moreover, up to the equivalence transformations (2.4), we can choose $B^a = 0$, and reduce (5.3) to the following system:

$$\begin{aligned} &(m+4) f^a + \mu^{ab} f^b = (\mu^{kb} u_b + m u_k)\frac{\partial f^a}{\partial u_k}, \\ &\nu^{ab} f^b + \mu^{ab} u_b = \nu^{kb} u_b \frac{\partial f^a}{\partial u_k}, \end{aligned} \tag{5.8}$$

where the parameters ν^{ab} and μ^{ab} are different from zero only in the case of the diagonal matrix A.

Finally, for the general symmetry (4.3) it is not difficult to show that the condition $\omega_\nu \neq 0$ leads to the following equation for f^a:

$$(A^{-1})^{kb}(f^b + \gamma u^b) = (A^{-1})^{ab} u_b \frac{\partial f^k}{\partial u_a}. \tag{5.9}$$

Note that relations (5.7) and (5.9) are particular cases of (5.3) for $\mu = 0$, $C^{ab} = (A^{-1})^{ab}$ and $\mu = 0$, $C^{ab} = e^{\gamma t}(A^{-1})^{ab}$, respectively. Thus, if relation (5.7) is valid, then, in addition to G_α (4.5), Eq. (1.3) admits the symmetry

$$X = (A^{-1})^{ab} u_b \frac{\partial}{\partial u_a}. \tag{5.10}$$

Alternatively, if (5.9) is satisfied, then Eq. (1.2) admits the symmetry $\widehat{G}_\alpha$ (2.6) and also the following one:

$$X = e^{\gamma t}(A^{-1})^{ab} u_b \frac{\partial}{\partial u_a}, \quad \gamma \neq 0. \tag{5.11}$$

Thus, it is reasonable first to classify Eqs. (1.3) that admit the main symmetries (5.2) and then to specify all cases where these symmetries can be extended.

Conditions under which system (1.3) admits extended symmetries are given by relations (5.7)–(5.9).

As for Eqs. (1.4) and (1.5), we note that, in accordance with (4.8) and (4.11), they admit only basic symmetries.

We are now ready to seek solutions of the classifying equations (4.10), (4.12), and (5.3). To present clearly the main details of our approach, we begin with the group classification of systems (1.4) because this problem appears to be much simpler than the other ones considered here.

6. Symmetry Algebras of Equations (1.4)

Consider Eqs. (1.4) and assume that the parameter p is nonzero. Then, scaling dependent and independent variables, we can reduce its value to $p = 1$.

To solve the rather complicated classifying equations (4.10), (4.12), and (5.3) we use the main algebraic property of the related symmetries, i.e., the fact that they should form a Lie algebra. In other words, instead of going through all nonequivalent possibilities arising via the separation of variables in the classifying equations, we first specify all nonequivalent realizations of the invariance algebra for our equations whose elements are defined by relations (5.2), (4.8), and (4.11) up to arbitrary constants and arbitrary functions. Then, using the one-to-one correspondence between these algebras and the classifying equations (4.10), (4.12), and (5.3), we easily solve the group classification problems for Eqs. (1.3)–(1.5).

Let us begin with the classifying equations (4.10) and the related symmetries (4.8). For any functions f^1 and f^2, Eqs. (1.4) admit symmetries (5.1), where the indices μ, ν and λ run through the values $1, 2, \dots m-1$ and $1, 2, \dots m$, respectively.

In accordance with (4.8), any symmetry generator extending algebra (5.1) has the following form:

$$X = \mu D - 2B^a \frac{\partial}{\partial u_a} - 2F u_1 \frac{\partial}{\partial u_1} + (\mu - 2F) u_2 \frac{\partial}{\partial u_2}. \tag{6.1}$$

Let X_1 and X_2 be operators of the form (6.1). Then the commutator $[X_1, X_2]$ is also a symmetry whose general form is given by (6.1). Thus, operators (6.1) form a Lie algebra, which we denote by $\mathcal{A}$.

Let us specify algebras $\mathcal{A}$ that can appear in our classification procedure. First, consider one-dimensional $\mathcal{A}$, i.e., suppose that Eq. (1.4) admits the only symmetry of the form (6.1). Then any commutator of operator (5.1) with (6.1) should be reducible to a linear combination of operators (5.1) and (6.1). This condition gives us only the following possibilities:

$$\begin{aligned} X &= X_1 = \mu D - 2\alpha_a \frac{\partial}{\partial u_a} - 2\beta u_1 \frac{\partial}{\partial u_1} - (2\beta - \mu) u_2 \frac{\partial}{\partial u_2}, \\ X &= X_2 = e^{\nu t}\Big(\alpha_a \frac{\partial}{\partial u_a} + \beta u_1 \frac{\partial}{\partial u_1} + \beta u_2 \frac{\partial}{\partial u_2}\Big), \\ X &= X_3 = e^{\nu t + \rho\cdot x}\alpha^a \frac{\partial}{\partial u_a}, \end{aligned} \tag{6.2}$$

where the Greek letters again denote arbitrary parameters and $\rho \cdot x = \rho_\mu x_\mu$.

The other choices of arbitrary functions F and B^a in (6.1) correspond to algebras $\mathcal{A}$ of dimension larger than one.

The next step is to specify all nonequivalent sets of arbitrary constants in (6.2), using the equivalence transformations (2.4).

If the coefficient of $u_a \frac{\partial}{\partial u_a}$ (a is fixed) is nonzero, then, translating u_a, we reduce the related coefficient α_a in X_1 and X_2 to zero; then, scaling u_a, we can reduce all nonzero α_a in (6.2) to ± 1. In addition, all operators (6.2) are defined up to constant multipliers. Using these simple arguments, we come to the following nonequivalent versions of operators (6.2) belonging to one-dimensional algebras $\mathcal{A}$:

$$\begin{aligned} X_1^{(1)} &= \mu D - u_1 \frac{\partial}{\partial u_1} + (\mu - 1) u_2 \frac{\partial}{\partial u_2}, \\ X_1^{(2)} &= D + u_2 \frac{\partial}{\partial u_2} + \nu \frac{\partial}{\partial u_1}, \\ X_1^{(3)} &= D - u_1 \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}, \\ X_2^{(\nu)} &= e^{\nu t + \rho_2\cdot x}\Big(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\Big), \\ X_3^{(1)} &= e^{\sigma_1 t + \rho_1\cdot x}\Big(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\Big), \\ X_3^{(2)} &= e^{\sigma_2 t + \rho_2\cdot x}\frac{\partial}{\partial u_1},\ X_3^{(3)} = e^{\sigma_3 t + \rho_3\cdot x}\frac{\partial}{\partial u_2}. \end{aligned} \tag{6.3}$$

To describe *two-dimensional* algebras $\mathcal{A}$ we represent one of the related basis elements X in the general form (6.1) and calculate the commutators

$$Y = [P_0, X] - 2\mu P_0, \quad Z = [P_0, Y], \quad W = [X, Y],$$

where P_0 is the operator given in (5.1) and Y, Z, and W denote the terms on the right-hand side. After simple calculations, we obtain

$$\begin{aligned} Y &= F_t\Big(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\Big) + B_t^a\frac{\partial}{\partial u_a}, \\ Z &= F_{tt}\Big(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\Big) + B_{tt}^a\frac{\partial}{\partial u_a}, \\ W &= 2\mu t Z + \mu x_b B_{tx_b}^a\frac{\partial}{\partial u_a}. \end{aligned} \tag{6.4}$$

By definition, Y, Z, and W belong to $\mathcal{A}$. Let $F_t \neq 0$. Then it follows from (6.4) that

$$\mu \neq 0: \ B_{tt}^a = F_{tt} = B_{tb}^a = 0, \tag{6.5}$$

$$\mu = 0: \ F_{tt} = \alpha F_t + \gamma^a B_t^a, \quad B_{tt}^a = \gamma^a F_t + \beta^{ab} B_t^b; \tag{6.6}$$

otherwise the dimension of $\mathcal{A}$ is larger than 2. The Greece letters in (6.5) and (6.6) denote arbitrary parameters.

Starting from (6.5), we conclude that, up to translations of t, the coefficients F and B_a have the following form:

$$F = \sigma t \text{ or } F = \beta; \ B^a = \nu^a t + \alpha^a \text{ if } \mu \neq 0.$$

If $F = \sigma t$, then the change

$$u_a \to u_a e^{-\sigma t} - \frac{\nu_a}{\mu} t \tag{6.7}$$

reduces the related operator (4.8) to the following form:

$$X = \mu\left(D + u_2\frac{\partial}{\partial u_2}\right) - 2\alpha_a\frac{\partial}{\partial u_a}, \tag{6.8}$$

i.e., X coincides with X_1 of (6.2) for $\beta = 0$. Moreover, it is possible to show that (6.7) gives an equivalence transformation for the related equations (1.4) (i.e., for Eqs. (1.4) that admit symmetry (6.8)).

The choice $F = \beta$ corresponds to the following operator (6.1):

$$X = X_4 = X_1 - 2t\alpha^a \frac{\partial}{\partial u_a}, \tag{6.9}$$

where X_1 is given in (6.2).

Thus, if one of the basis elements of a two-dimensional algebra $\mathcal{A}$ is of the general form (6.1) with $\mu \neq 0$, then it can be reduced to (6.8) or (6.9). Denote such a basis element by e_1. Without loss of generality, the second basis element e_2 of $\mathcal{A}$ is a linear combination of the operators $X_2^{(\nu)}$ and $X_3^{(a)}$ (6.3). Going over possible pairs (e_1, e_2) and requiring that $[e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2$, we come to the following two-dimensional algebras:

$$\begin{aligned}
&A_1 = \left\langle D + u_2\frac{\partial}{\partial u_2}, X_2^{(0)}\right\rangle, \quad A_2 = \langle X_1^{(2)}, X_3^{(3)}\rangle,\\
&A_3 = \langle X_1^{(3)}, X_3^{(3)}\rangle, \quad A_4 = \langle X_1^{(1)}, X_3^{(3)}\rangle, \quad A_5 = \langle X_1^{(1)}, X_3^{(3)}\rangle,\\
&A_6 = \left\langle D + 2u_2\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_1} + \nu t\frac{\partial}{\partial u_2}, X_3^{(2)}\right\rangle,\\
&A_7 = \left\langle D + 2u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2} + 3\nu t\frac{\partial}{\partial u_1}, \quad X_3^{(1)}\right\rangle.
\end{aligned} \tag{6.10}$$

The form of the basis elements in (6.10) is defined up to transformations (6.7) and (2.4).

If $\mathcal{A}$ does not include operators (6.1) with nontrivial parameters μ, then, in accordance with (6.7), its elements are of the following form:

$$e_a = F_{(a)}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right) + B^b_{(a)}\frac{\partial}{\partial u_b}, \; a = 1, 2, \tag{6.11}$$

where $F_{(\alpha)}$ and $B^b_{(a)}$ are solutions of (6.6).

Formulas (6.10), (6.11) define all nonequivalent two-dimensional algebras $\mathcal{A}$, which have to be considered as possible symmetries of Eqs. (1.4). We will see that, asking for the invariance of (1.4) with respect to these algebras, the related arbitrary functions f^a are defined up to arbitrary constants, and it is impossible to make further specification of these functions by extending the algebra $\mathcal{A}$.

7. Group Classification of Equations (1.4)

To classify Eqs. (1.4) that admit one- and two-dimensional extensions of the basis invariance algebra (5.1) it is sufficient to solve the classifying equations (4.10) for f^a with *known* coefficient functions B^a and F of symmetries (6.1). These functions are easily found by comparing (4.8) with (6.3), (6.10), and (6.11).

Let us present an example of such calculation that corresponds to the algebra A_1 whose basis elements are $X_1 = 2t\partial_t + x_a\partial_a + u_2\frac{\partial}{\partial u_2}$ and $X_2^{(0)} = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}$ (see (6.10)). The operator $X_2^{(0)}$ generates the following form of Eq. (4.10):

$$f^a = \left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right) f^a, \quad a = 1, 2;$$

its general solution is

$$f^1 = u_1 F_1\left(\frac{u_2}{u_1}\right), \quad f^2 = u_1 F_2\left(\frac{u_2}{u_1}\right). \tag{7.1}$$

Here, F_1 and F_2 are arbitrary functions of $\frac{u_2}{u_1}$.

Equations (1.4) with nonlinearities (7.1) admit the symmetry $X_2^{(0)}$. In order that this equation be also invariant with respect to X_1, the functions f^1 and f^2 must satisfy Eq. (4.10) with $F = 0$, i.e.,

$$f^1 = -u_1\frac{\partial f^1}{\partial u_1}; \quad f^2 = -\frac{1}{2}u_1\frac{\partial f^2}{\partial u_1}. \tag{7.2}$$

It follows from (7.1) and (7.2) that

$$f^1 = \alpha u_1^3 u_2^{-2}, \quad f^2 = \lambda u_1^2 u_2^{-1}. \tag{7.3}$$

Thus, Eq. (1.4) admits the symmetries $X_0^{(2)}$ and X_1, which form the algebra A_1 (6.10), provided that f^1 and f^2 are functions given in (7.3). These symmetries are defined up to arbitrary constants α and λ. If one of them is nonzero, then it can be reduced to $+1$ or -1 by scaling independent variables.

By analogy, we solve Eqs. (4.10) corresponding to the other symmetries indicated in (6.3) and (6.10). For the one-dimensional algebras (6.3), the related nonlinearities f^1 and f^2 are defined up to arbitrary functions F_1 and F_2, while, for the two-dimensional algebras (6.10), the functions f^1 and f^2 are defined up to two integration constants. We do not reproduce the related rather routine calculations but present their results in Table 1.

Table 1. Nonlinearities and Symmetries for Eq. (1.4) with p=1

No	Nonlinearities	Arguments of F_1 F_2	Symmetries
1.	$f^1 = u_1^{2\mu+1}F_1$, $f^2 = u_1^{\mu+1}F_2$	$u_2 u_1^{\mu-1}$	$\mu D - u_1\frac{\partial}{\partial u_1} + (\mu-1)u_2\frac{\partial}{\partial u_2}$
2.	$f^1 = F_1 u_2^{-2}$, $f^2 = F_2 u_2^{-1}$	$u_1 - \nu \ln u_2$	$D + u_2\frac{\partial}{\partial u_2} + \nu\frac{\partial}{\partial u_1}$
3.	$f^1 = u_1(F_1 + \lambda \ln u_1)$, $f^2 = u_2(F_2 + \lambda \ln u_1)$	$\frac{u_2}{u_1}$	$e^{\lambda t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right)$
4.	$f^1 = u_1^3 F_1$, $f^2 = u_1^2 F_2$	$u_2 - \ln u_1$	$D - u_1\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}$
5.	$f^1 = F_1$, $f^2 = F_2 + \nu u_2$	u_2	$e^{\nu t}\Psi(x)\frac{\partial}{\partial u_2}$
6.	$f^1 = \alpha u_1 + F_1$, $f^2 = \lambda u + F_2$	u_1	$e^{\lambda t + \nu x_m}\tilde{\Psi}_\mu(\tilde{x})\frac{\partial}{\partial u_1}$, $\mu = \lambda - \nu^2 - \alpha$
7.	$f^1 = \sigma u + F_1$, $f^2 = \lambda v + F_2$	$u - v$	$e^{\lambda t}\ e^{\frac{x_m+t}{2}}\Psi_\mu(\tilde{x}, x_m + t)\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right)$, $\mu = \lambda - \sigma + \frac{1}{4}$
8.	$f^1 = \alpha u_1^3 u_2^{-2}$, $f^2 = \beta u_1^2 u_2^{-1}$		$D + u_2\frac{\partial}{\partial u_2}$, $\quad u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}$
9.	$f^1 = \alpha e^{-2u_1}$, $f^2 = \lambda e^{-u_1}$		$D + u_2\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_1}$, $\quad \Psi(x)\frac{\partial}{\partial u_2}$
10.	$f^1 = \lambda e^{3u_2}$, $f^2 = \alpha e^{2u_2}$		$D - u_1\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}$, $\tilde{\Phi}_0(t, \tilde{x})\frac{\partial}{\partial u_1}$
11.	$f^1 = \alpha u_1^{2\mu+1}$, $f^2 = \lambda u_1^{\mu+1}$		$\mu D - u_1\frac{\partial}{\partial u_1} + (\mu-1)u_2\frac{\partial}{\partial u_2}$, $\Psi(x)\frac{\partial}{\partial u_2}$
12.	$f^1 = \lambda u_2^{3\mu-2}$, $f^2 = \alpha u_2^{2\mu-1}$		$(\mu-1)D - \mu u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$, $\tilde{\Phi}_0(t, \tilde{x})\frac{\partial}{\partial u_1}$
13.	$f^1 = \frac{\alpha}{u_1}$, $\quad f^2 = \ln u_1$		$D + 2u_2\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_1} + t\frac{\partial}{\partial u_2}$, $\Psi(x)\frac{\partial}{\partial u_2}$
14.	$f^1 = \ln u_2$, $f^2 = \alpha u_2^{\frac{1}{3}}$		$D + 2u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2} + 3t\frac{\partial}{\partial u_1}$, $\tilde{\Phi}_0(t, \tilde{x})\frac{\partial}{\partial u_1}$

Here, D is the dilatation operator given in (4.5), $\tilde{x}=(x_1, x_2, \ldots, x_{m-1})$, $\Psi(x)$ is an arbitrary function of spatial variables, and $\tilde{\Psi}_\mu(\tilde{x})$, $\Psi_\mu(\tilde{x}, x_m + t)$, and $\tilde{\Phi}_\mu(t, \tilde{x})$ are solutions of the Laplace and linear heat equations

$$\begin{aligned}
&\tilde{\Delta}\tilde{\Psi}_\mu = \mu\tilde{\Psi}_\mu, \quad \Delta\Psi_\mu = \mu\Psi_\mu, \quad (\frac{\partial}{\partial t} - \tilde{\Delta})\tilde{\Phi}_0 = 0, \\
&\tilde{\Delta} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{m-1}^2}, \quad \Delta = \tilde{\Delta} + \frac{\partial^2}{\partial x_m^2}.
\end{aligned} \tag{7.4}$$

Note that Eqs. (1.4) with nonlinearities 5, 6, 9–14 of Table 1 admit infinite-dimensional algebras $\mathcal{A}$ because the related symmetries are defined up to arbitrary functions $\Psi(x)$ or arbitrary solutions of Eqs. (7.4). Nevertheless, the form of these nonlinearities has been fixed by requiring the invariance with respect to the one- and two-dimensional algebras enumerated in (6.3) and (6.10).

The second note is that Eqs. (1.4) with nonlinearities given in case 8 of Table 1 admit additional equivalence transformations $u_\alpha \to e^{\sigma t}u_\alpha$, while for cases 9, 11, 13 and 10, 12, 14 we have at our disposal transformations 3 and 2, respectively, from list (2.10).

8. Group Classification of Equations (1.5)

Similarly to (1.4), Eqs. (1.5) with arbitrary functions f^1 and f^2 admit the basic symmetries (5.1), where $\mu, \nu = 1, 2, \ldots, m-1$. To classify equations admitting other symmetries it is sufficient to find the general solution for Eqs. (4.12).

We solve (4.12) using the technique applied in Secs. 5 and 6. Comparing (4.11) and (6.1), we conclude that the generators of extended symmetry for Eqs. (1.4) and (1.5) are rather similar, and so we can essentially exploit the algebra classification scheme used in Sec. 5. As a result, we easily come to the following list of one-dimensional algebras A (cf. (6.3)):

$$\begin{aligned}
&\tilde{X}_1^{(1)} = \mu\tilde{D} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}, \\
&\tilde{X}_1^{(2)} = \tilde{D} - \nu\frac{\partial}{\partial u_1}, \quad \tilde{X}_2^{(\nu)} = e^{\nu t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right), \\
&\tilde{X}_1^{(3)} = \tilde{D} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + \nu\frac{\partial}{\partial u_2}, \\
&\tilde{X}_3^{(3)} = e^{\sigma_3 t + \rho_3 x}\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right), \\
&\tilde{X}_3^{(j)} = e^{\sigma_i t + \rho_i \cdot x}\frac{\partial}{\partial u_j}, \quad j = 1, 2,
\end{aligned} \tag{8.1}$$

where $\tilde{D} = 3t\partial_t + 2x_\nu\partial_\nu - u_2\frac{\partial}{\partial u_2}$. The two-dimensional algebras are given by the following relations (cf. (6.10)):

$$
\begin{aligned}
&\tilde{A}_1 = \langle \tilde{D}, \tilde{X}_2^{(0)}\rangle, \quad \tilde{A}_2 = \langle \tilde{X}_1^{(2)}, X_3^{(3)}\rangle,\\
&\tilde{A}_3 = \langle \tilde{X}_1^{(3)}, \tilde{X}_3^{(1)}\rangle, \quad \tilde{A}_4 = \langle \tilde{X}_1^{(1)}, \tilde{X}_3^{(2)}\rangle, \quad \tilde{A}_5 = \langle \tilde{X}_1^{(1)}, \tilde{X}_3^{(1)}\rangle,\\
&\tilde{A}_6 = \Big\langle \tilde{D} + 4\Big(u_2\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_1} + t\frac{\partial}{\partial u_2}\Big), X_3^{(2)}\Big\rangle,\\
&\tilde{A}_7 = \Big\langle \tilde{D} + 3\Big(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + t\frac{\partial}{\partial u_1}\Big), X_3^{(1)}\Big\rangle.
\end{aligned}
\tag{8.2}
$$

Using (8.1) and (8.2) and solving the related classifying equations (4.12), we find the nonlinearities f^1 and f^2 that are given in Table 2. In six cases enumerated in the table, the corresponding equations (1.5) admit infinite-dimensional symmetry algebras whose generators are defined up to arbitrary functions (see cases 5–7 and 9–14 there).

Table 2. Nonlinearities and Symmetries for Eq. (1.5) with $p = 1$

No	Nonlinearities	Arguments of F_1 F_2	Symmetries
1.	$f^1 = u_1^{1+3\mu}F_1$, $f^2 = u_1^{1+4\mu}F_2$	$u_2u_1^{-\mu-1}$	$\mu\tilde{D} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$
2.	$f^1 = u_2^3F_1$, $f^2 = u_2^4F_2$	$u_1 - \nu\ln u_2$	$\tilde{D} - \nu\frac{\partial}{\partial u_1}$
3.	$f^1 = u_1(F_1 + \nu\ln u_1)$, $f^2 = u_2(F_2 + \nu\ln u_1)$	$\frac{u_2}{u_1}$	$e^{\nu t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right)$
4.	$f^1 = u_1^{-2}F_1$, $f^2 = u_1^{-3}F_2$	$u_2 - \nu\ln u_1$	$\tilde{D} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + \nu\frac{\partial}{\partial u_2}$
5.	$f^1 = \lambda u_1 + F_1$, $f^2 = -\mu u_1 + F_2$	u_2	$e^{\lambda t}\Psi_\mu(x)\frac{\partial}{\partial u_1}$
6.	$f^1 = \nu u_2 + F_1$, $f^2 = \lambda u_2 + F_2$	u_1	$e^{\lambda t - \nu x_m}\Psi(\tilde{x})\frac{\partial}{\partial u_2}$
7.	$f^1 = \alpha u_1 + F_1$, $f^2 = \sigma u_2 + F_2$	$u_1 - u_2$	$e^{\lambda t}\, e^{\frac{x_m - t}{2}}\Psi_\mu(\tilde{x}, x_m + t)\Big(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\Big)$, $\mu = \lambda - \sigma + \frac{1}{4}$

8.	$f^1 = \alpha u_1^{-2} u_2^3$, $f^2 = \nu u_1^{-3} u_2^4$		$\tilde{D}, \quad u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}$
9.	$f^1 = \alpha e^{3u_1}$, $f^2 = \nu e^{4u_1}$		$\tilde{D} - \frac{\partial}{\partial u_1}, \ \Psi(\tilde{x}) \frac{\partial}{\partial u_2}$
10.	$f^1 = \alpha e^{-2u_2}$, $f^2 = \nu e^{-3u_2}$		$\tilde{D} + u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_2}$, $\Psi_0(x) \frac{\partial}{\partial u_1}$
11.	$f^1 = \alpha u_1^{3\mu+1}$, $f^2 = \nu u_1^{4\mu+1}$		$\mu\tilde{D} - u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2}$, $\Psi(\tilde{x}) \frac{\partial}{\partial u_2}$
12.	$f^1 = \alpha u_2^{2\nu+1}$, $f^2 = \nu u_2^{3\nu+1}$		$\nu\tilde{D} - u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2}$, $\Psi_0(x) \frac{\partial}{\partial u_1}$
13.	$f^1 = \alpha u_1^{\frac{1}{4}}$, $f^2 = \nu \ln u_1$		$\tilde{D} + 3u_2 \frac{\partial}{\partial u_2} + 4u_1 \frac{\partial}{\partial u_1} + 4\nu t \frac{\partial}{\partial u_2}$, $\Psi(\tilde{x}) \frac{\partial}{\partial u_2}$
14.	$f^1 = \nu \ln u_2$, $f^2 = \frac{\alpha}{\sqrt{u_2}}$		$\tilde{D} + 3u_1 \frac{\partial}{\partial u_1} + 2u_2 \frac{\partial}{\partial u_2} + 2\nu t \frac{\partial}{\partial u_1}$, $\Psi_0(x) \frac{\partial}{\partial u_1}$

Here, $\Psi_\mu(x)$ and $\Psi_\mu(\tilde{x}, x_m + t)$ are arbitrary solutions of the Laplace equation $\Delta\Psi_\mu = \mu\Psi_\mu$, and μ, ν, and λ are arbitrary parameters satisfying $\nu\lambda \neq 0$.

Equations (1.5) with the nonlinearities given in case 8 of Table 2 admit the additional equivalence transformation $u_\alpha \to e^{\sigma t} u_\alpha$. Furthermore, for cases 9,11,13 and 10,12, 14 we have transformations 3 and 2, respectively, from list (2.10).

9. Group Classification of Equations (1.3) with Invertible Diffusion Matrices

In this section, we present the group classification of systems of coupled reaction-diffusion equations (1.3) with invertible matrix A. In accordance with the scheme outlined in Sec. 4, we first describe the main symmetries generated by operators (5.2) and then indicate extensions of these symmetries.

As in Secs. 5 and 7, the first step of our analysis consists of the description of realizations of Lie algebras $\mathcal{A}$ generating basic symmetries of Eq. (1.3). However, the basis elements of $\mathcal{A}$ are now of the general form (5.2), while in Secs. 5 and 7 we have been restricted to representations (6.1) and (4.11), respectively, which are particular cases of (5.1).

Thus, the first step of our analysis is to describe nonequivalent realizations of finite-dimensional algebras $\mathcal{A}$ whose basis elements have the form (5.2).

Let us specify all nonequivalent "tails" of operators (5.2), i.e., the terms

$$\pi = C^{ab}u_b\frac{\partial}{\partial u_a} + B^a\frac{\partial}{\partial u_a}. \tag{9.1}$$

These terms can either be a constituent part of a more general symmetry (5.2) or represent a particular case of (5.2) corresponding to $\mu = 0$.

If Eq. (1.3) admits a one-dimensional invariance algebra $\mathcal{A}$, then the commutators of π with the basic symmetries P_0 and P_a should be equal to a linear combination of π and operators (5.1). In other words, there are three possibilities:

$$1.\ C^{ab} = \mu^{ab}, \quad B^a = \mu^a, \tag{9.2}$$

$$2.\ C^{ab} = e^{\lambda t}\mu^{ab}, \quad B^a = e^{\lambda t}\mu^a, \tag{9.3}$$

$$3.\ C^{ab} = 0, \quad B^a = e^{\lambda t+\omega\cdot x}\mu^a, \tag{9.4}$$

where μ^{ab}, μ^a, λ, and ω are constants.

In any case, the problem of classification of one-dimensional algebras $\mathcal{A}$ includes the subproblem of classification of nonequivalent linear combinations (9.1) with *constant* coefficients μ^{ab} and μ^a. To describe such linear combinations we use the isomorphism of (9.1) with 3×3 matrices of the following form:

$$g = \begin{pmatrix} 0 & 0 & 0 \\ B^1 & C^{11} & C^{12} \\ B^2 & C^{21} & C^{12} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^{11} & \mu^{12} \\ \mu^2 & \mu^{21} & \mu^{12} \end{pmatrix}. \tag{9.5}$$

Equations (1.3) admit the equivalence transformations (2.4), which change the term π (9.1) and can be used to simplify it. The corresponding transformation for matrix (9.5) can be represented as

$$g \to g' = UgU^{-1}, \tag{9.6}$$

where U is a 3×3 matrix of the following special form:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & K^{11} & K^{12} \\ b^2 & K^{21} & K^{22} \end{pmatrix}. \tag{9.7}$$

We will use relations (9.2)–(9.4) and the equivalence transformations (9.6) to construct basis elements of basic symmetry algebras. For different forms of the matrix A specified in (2.2), the transformation matrix (9.7) needs further specification in accordance with (2.5)–(2.8).

The obtained nonequivalent realizations of low-dimensional algebras $\mathcal{A}$ are presented in Appendix. Starting from these realizations, one easily solves the related determining equations (4.6) for the nonlinearities f^1 and f^2 and specify all cases where the main symmetries can be extended (i.e., where relations (5.7)–(5.9) are satisfied). In addition, we have to control all cases where the basis elements of $\mathcal{A}$ depend on arbitrary solutions Ψ of the linear heat equation. Such algebras (whose basis elements can be obtained from (A.1.10), (A.1.11), and (A.1.15)–(A.1.18) by replacing g_5 and g_3 by Ψg_5 and Ψg_3) are infinite-dimensional, but they generate the same number of determining equations as the low-dimensional algebras.

10. Classification Results

We do not reproduce the related exact calculations but present the results of group classification in Tables 3–9. In addition to equations with invertible diffusion matrix, we present here the results of classification related to a diffusion matrix of the type IV, while the type V is considered separately (see (2.2) for a classification of diffusion matrices).

Tables 3–9 present classification results for different types of Eqs. (1.3) corresponding to the nonequivalent diffusion matrices enumerated in (2.2). The type of a diffusion matrix is indicated in the fourth columns of Tables 3 and 4 and the third columns of Tables 5 and 6. In Tables 7–9, the results of the symmetry classification of special equations are presented; these equations are indicated in the table titles. In the last columns of Tables 3, 5, and 6, the additional equivalence transformations (AET) are specified that are possible for the related class of nonlinearities. Finally, D, G_α, and $\widehat{G}_\alpha$ denote generators (4.5), ψ_μ denotes an arbitrary solution of the linear heat equation $\frac{\partial}{\partial t}\psi_\mu - \Delta\psi_\mu = \mu\psi_\mu$,

$$\tilde{\psi}_\nu = \begin{cases} \psi_\nu & \text{for Class } III, \\ e^{\nu t}\Psi(x) & \text{for Class } IV, \end{cases}$$

and $\Psi(x)$ and $\Psi_\nu(x)$ have the same meaning as in Tables 1 and 2.

The results of group classification are briefly discussed in Sec. 12.

Table 3. Nonlinearities with Arbitrary Functions and Extendible Symmetries for Eqs. (1.3) and (2.2)

No	Nonlinear terms	Arguments of F_1 F_2	Type of matrix A	Main symmetries	Additional symmetries	AET (2.10)
1.	$f^1 = u_1^{\nu+1} F_1$, $f^2 = u_1^{\nu+\mu} F_2$	$\frac{u_2}{u_1^\mu}$	$I,\ IV$, $\mu \neq 1$; $I-IV$, $\mu = 1$	$\frac{\nu}{2}D$ $-u_1\frac{\partial}{\partial u_1}$ $-\mu u_2\frac{\partial}{\partial u_2}$	For I: G_α if $\nu = 0$, $a\mu = 1$	$1, \rho$ $= \mu\omega$ if $\nu = 0$
2.	$f^1 = u_1(F_1$ $+\varepsilon \ln u_1)$, $f^2 = u_2(F_2$ $+\varepsilon\mu \ln u_1)$	$\frac{u_2}{u_1^\mu}$	$I,\ IV$, $\mu \neq 1$; $I-IV$, $\mu = 1$	$e^{\varepsilon t}\big(u_1\frac{\partial}{\partial u_1}$ $+\mu u_2\frac{\partial}{\partial u_2}\big)$	For I: $\widehat{G}_\alpha$ if $a\mu = 1$	
3.	$f^1 = u_1F_1 + \nu u_2$, $f^2 = \nu\frac{u_2}{u_1}(u_1$ $+u_2) + u_1F_2$ $+u_2F_1$, $\nu \neq 0$	$u_1 e^{-\frac{u_2}{u_1}}$	I^*, III	$e^{\nu t}\big(u_1\frac{\partial}{\partial u_2}$ $+u_1\frac{\partial}{\partial u_1}$ $+u_2\frac{\partial}{\partial u_2}\big)$	For III: $\widehat{G}_\alpha$ if $a = -1$	
4.	$f^1 = u_1^{\nu+1}F_1$, $f^2 = u_1^\nu(F_1u_2$ $+F_2u_1)$	$u_1 e^{-\frac{u_2}{u_1}}$	I^*, III	$\frac{\nu}{2}D$ $-u_1\frac{\partial}{\partial u_2}$ $-u_1\frac{\partial}{\partial u_1}$ $-u_2\frac{\partial}{\partial u_2}$	For III: G_α if $\nu = 0$, $a = -1$	
5.	$f^1 = e^{\nu\frac{u_2}{u_1}}F_1u_1$, $f^2 = e^{\nu\frac{u_2}{u_1}}(F_1u_2$ $+F_2)$	u_1	I^*, III	$\frac{\nu}{2}D$ $-u_1\frac{\partial}{\partial u_2}$	For I^*: $u_2\frac{\partial}{\partial u_2}$ if $\nu = 0$, $F_2 = 0$	6 if $\nu = 0$
					For I^*: $\psi_0\frac{\partial}{\partial u_2}$, D $+2u_2\frac{\partial}{\partial u_2}$ if $F_1 = 0$, $\nu = 0$	3,6
6.	$f^1 = u_1(F_1 - \nu)$, $f^2 = F_1u_2 + F_2$, $\nu \neq 0$	u_1	I^*, III	$e^{\nu t}u_1\frac{\partial}{\partial u_2}$	$\psi_\mu\frac{\partial}{\partial u_2}$ if $F_1 = \mu$	3 if $F_1 = 0$
7.	$f^1 = u_1F_1 + u_2F_2$ $-\nu z(\mu u_1 + u_2)$, $f^2 = u_2F_1 - u_1F_2$ $+\nu z(u_1 - \mu u_2)$; $R = (u_1^2 + u_2^2)^{\frac{1}{2}}$, $z = \tan^{-1}(\frac{u_2}{u_1})$	$Re^{\mu z}$	I^*, II	$e^{\nu t}\big(\mu R\frac{\partial}{\partial R}$ $-\frac{\partial}{\partial z}\big)$	For II: $\widehat{G}_\alpha$ if $\mu = a$, $\nu \neq 0$; G_α if $\mu = a$, $\nu = 0$	15 if $\mu = 0$

Table 4. Nonlinearities with Arbitrary Functions and Nonextendible Symmetries for Eqs. (1.3) and (2.2)

No	Nonlinear terms	Arguments of F_a	Type of matrix A	Symmetries and AET (2.10) [in square brackets]
1.	$f^1 = u_2^{\nu} F_1$, $f^2 = u_2^{\nu+1} F_2$	$u_2 e^{u_1}$	I, IV	$\nu D - 2u_2 \frac{\partial}{\partial u_2} + 2\frac{\partial}{\partial u_1}$ [4 if $\nu = 0$]
2.	$f^1 = F_1 + \varepsilon u_1$, $f^2 = F_2 u_2 + \varepsilon u_1 u_2$	$u_2 e^{u_1}$	I, IV	$e^{\varepsilon t}\left(u_2 \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right)$,
3.	$f^1 = e^{\nu u_1} F_1$, $f^2 = e^{\nu u_1}(F_2 + F_1 u_1)$	$2u_2 - u_1^2$	I^*, III	$\nu D - 2u_1 \frac{\partial}{\partial u_2} - 2\frac{\partial}{\partial u_1}$
4.	$f^1 = \nu u_1 + F_1$, $f_2 = \nu u_1^2 + F_1 u_1 + F_2$	$2u_2 - u_1^2$	I^*, III	$\tilde{\psi}_\nu \left(u_1 \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_1}\right)$
5.	$f^1 = \nu u_1 + F_1$, $f_2 = -\mu u_1 + F_2$	u_2	II, III	For $II: e^{(\nu - a\mu)t}\Psi_\mu \frac{\partial}{\partial u_1}$, For $III: e^{(\nu+\sigma a)t}\Psi_\sigma$, $\mu = \sigma a$
6.	$f^1 = e^{\nu z}(F_1 u_2 + F_2 u_1)$, $f^2 = e^{\nu z}(F_2 u_2 - F_1 u_1)$	$R e^{-\mu z}$	I^*, II	$\nu D - 2\mu\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\right)$ $-2\left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1}\right)$
7.	$f^1 = 0, \quad f^2 = F$	u_2	I, IV	$\psi_0 \frac{\partial}{\partial u_1}, \quad u_1 \frac{\partial}{\partial u_1}$, [2; 1, $\rho = 0$]
8.	$f^1 = 0, \quad f^2 = F$	u_1	$I, a \neq 1$, IV	$D + 2u_2 \frac{\partial}{\partial u_2}, \ \tilde{\psi}_0 \frac{\partial}{\partial u_2}$, [3, 6]
9.	$f^1 = F_1$, $f^2 = F_2 + \nu u_2$	u_1	I, III, IV	$\tilde{\psi}_\nu \frac{\partial}{\partial u_2}$
10.	$f^1 = F_1 + (\nu - \mu)u_1$, $f^2 = F_2 + (\nu - a\mu)u_2$	$u_2 - u_1$	$I, a \neq 1$ IV	$e^{\nu t}\Psi_\mu(x)\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right)$
11.	$f^1 = \alpha u_1 + \mu$, $f_2 = \nu u_2 + F$, $\alpha\mu = 0$	u_1	I^*, III	$\tilde{\psi}_\nu \frac{\partial}{\partial u_2}$, $e^{(\nu-\alpha)t}(u_1 - \mu t)\frac{\partial}{\partial u_2}$
12.	$f^1 = u_1^2$, $f_2 = u_1 u_2 + \nu u_2 + F$,	u_1	I^*, III	$e^{\nu t} u_1 \frac{\partial}{\partial u_2}$, $e^{\nu t}\left(\frac{\partial}{\partial u_2} + t u_1 \frac{\partial}{\partial u_2}\right)$
13.	$f^1 = \left(u_1^2 - 1\right)$, $f_2 = (u_1 + \nu)u_2 + F$	u_1	I^*, III	$e^{(\nu+1)t}\left(u_1 \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_2}\right)$, $e^{(\nu-1)t}\left(u_1 \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_2}\right)$
14.	$f^1 = \left(u_1^2 + 1\right)$, $f_2 = (u_1 + \nu)u_2 + F$	u_1	I^*, III	$e^{\nu t}(\cos t u_1 \frac{\partial}{\partial u_2} - \sin t \frac{\partial}{\partial u_2})$, $e^{\nu t}(\sin t u_1 \frac{\partial}{\partial u_2} + \cos t \frac{\partial}{\partial u_2})$
15.	$f^1 = e^{\nu u_2} F_1$, $f^2 = e^{\nu u_2} F_2$	$\mu u_2 - u_1$	I, IV $\mu \neq 0$; II, III $\mu = 0$	$\nu D - 2\left(\mu \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right)$
16.	$f^1 = e^{\nu u_1} F_1$, $f^2 = e^{\nu u_1} F_2$	u_2	III	$\nu D - 2\frac{\partial}{\partial u_1}$

Table 5. Nonlinearities with Arbitrary Parameters and Extendible Symmetries for Eqs. (1.3) and (2.2)

No	Nonlinear terms	Type of matrix A	Main symmetries	Additional symmetries	AET (2.10)
1.	$f^1=\lambda u_1^{\nu+1}u_2^{\mu}$, $f^2=\sigma u_1^{\nu}u_2^{\mu+1}$	I, IV	$\mu D-2u_2\frac{\partial}{\partial u_2}$, $\nu D-2u_1\frac{\partial}{\partial u_1}$	G_α if $a\nu=-\mu$ $\neq 0$ & K if $\nu=\frac{4}{m(1-a)}, a\neq 1$;	$1, \nu\omega$ $+\mu\rho$ $=0$
				$\psi_0\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=-1$ & G_α if $\mu=a\neq 0$ & K if $a=1+\frac{m}{4}$;	$2; 1$, $\nu\omega$ $+\mu\rho$ $=0$
				$\psi_0\frac{\partial}{\partial u_1}, u_2\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=-1, a=1$;	$2; 1$, $\nu\omega$ $+\mu\rho$ $=0$
				$G_\alpha, \mathcal{K}$ if $\sigma=1, \lambda\neq 1$ $\mu=-\nu=a=1$;	$1, \nu\omega$ $+\mu\rho$ $=0$
				$u_1\frac{\partial}{\partial u_2}$ if $\mu=0$, $\lambda=\sigma, a=1$ & $\frac{\partial}{\partial u_2}+tu_1\frac{\partial}{\partial u_2}$ if $\nu=1$	$1, \nu\omega$ $+\mu\rho$ $=0$
2.	$f^1=\lambda u_1^{\nu+1}$, $f^2=\sigma u_1^{\nu+\mu}$, $\lambda\sigma\neq 0$	I, IV	$\nu D-2u_1\frac{\partial}{\partial u_1}$ $-2\mu u_2\frac{\partial}{\partial u_2}$, $\tilde{\psi}_0\frac{\partial}{\partial u_2}$	$(u_1-\lambda t)\frac{\partial}{\partial u_2}$ if $-\nu=a=1$;	3
				G_α if $\nu=0,\ a\mu=1$;	3
				$e^{-\lambda t}u_1\frac{\partial}{\partial u_2}$ if $\nu=0,\ a=1,$ & $e^{\lambda t}\left(\sigma u_1\frac{\partial}{\partial u_2}+\lambda\frac{\partial}{\partial u_1}\right)$ if $\mu=2$	3; $1,\ \rho$ $=\mu\omega$
3.	$f^1=\lambda u_1$, $f^2=\sigma u_1^{\mu}$	III	$\psi_0\frac{\partial}{\partial u_2}$, $e^{-\lambda t}u_1\frac{\partial}{\partial u_2}$	$e^{\lambda t}\left(u_1\frac{\partial}{\partial u_2}+\lambda\frac{\partial}{\partial u_1}\right)$ if $\mu=2$	3; 6 if $\lambda=0$
4.	$f^1=\lambda e^{\nu u_1}$, $f^2=\sigma e^{(\nu+1)u_1}$, $\lambda\neq 0$	I, IV	$\nu D-2u_2\frac{\partial}{\partial u_2}$ $-2\frac{\partial}{\partial u_1}$, $\tilde{\psi}_0\frac{\partial}{\partial u_2}$	$u_2\frac{\partial}{\partial u_2}$ if $\sigma=0$ & $u_2\frac{\partial}{\partial u_1}$ if $a=1$;	3; 1, $\omega=0$
				$(u_1-\lambda t)\frac{\partial}{\partial u_2}$ if $\nu=0, a=1$	3,4
5.	$f^1=\lambda e^{u_1}$, $f^2=\sigma e^{u_1}$	III	$D-2\frac{\partial}{\partial u_1}$, $\tilde{\psi}_0\frac{\partial}{\partial u_2}$	$u_1\frac{\partial}{\partial u_2}$ if $\lambda=0$	3; 6 if $\lambda=0$
6.	$f^1=\lambda e^{u_2}$, $f^2=\sigma e^{u_2}$, $\lambda\neq 0$	I, IV	$D-2\frac{\partial}{\partial u_2}$, $\psi_0\frac{\partial}{\partial u_1}$	$u_1\frac{\partial}{\partial u_1}+\frac{\partial}{\partial u_2}$ if $\sigma=0$ & for I $u_2\frac{\partial}{\partial u_1}$ if $a=1$	$2; 5$ if $\sigma=0$

7.	$f^1 = \lambda u_1^{\nu+1} e^{\mu \frac{u_2}{u_1}}$,	I^*, III	$\mu D - 2u_1 \frac{\partial}{\partial u_2}$,	For I^* : G_α if $\nu = 0$;	1, $\rho = \omega$; 6 if $\mu = 0$
	$f^2 = e^{\mu \frac{u_2}{u_1}} (\lambda u_2$ $+\sigma u_1) u_1^\nu$		$\nu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2u_2 \frac{\partial}{\partial u_2}$	For III : G_α if $\mu = a\nu$ & K if $\nu = \frac{4}{m}$	1, $\rho = \omega$ if $\nu = 0$; 6 if $\mu = 0$
8.	$f^1 = e^{\mu z} R^\nu (\lambda u_1$ $-\sigma u_2)$,	I^*, II	$\nu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2u_2 \frac{\partial}{\partial u_2}$,	For I^* : G_α if $\nu = 0$;	1, $\rho = \omega$
	$f^2 = e^{\mu z} R^\nu (\lambda u_2$ $+\sigma u_1)$		$\mu D - 2u_1 \frac{\partial}{\partial u_2}$ $+2u_2 \frac{\partial}{\partial u_1}$	For II : G_α if $\mu = a\nu$ & K if $\nu = \frac{4}{m}$	1, $\rho = \omega$ if $\nu = 0$
9.	$f^1 = \varepsilon u_1^{\mu+1}$, $f^2 = \varepsilon u_1^\mu (u_2$ $- \ln u_1)$, $\mu \neq 0$,	I^*	$\mu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2 \frac{\partial}{\partial u_2}$, $u_1 \frac{\partial}{\partial u_2}$	$\frac{\partial}{\partial u_2} + tu_1 \frac{\partial}{\partial u_2}$ if $\mu = 1$	6
10.	$f^1 = \lambda$, $f^2 = \varepsilon \ln u_1$	$I - IV$	$\frac{1}{2} D + u_1 \frac{\partial}{\partial u_1}$ $+u_2 \frac{\partial}{\partial u_2}$ $+\varepsilon t \frac{\partial}{\partial u_2}$,	For $I, a \neq 1, IV$: $u_1 \frac{\partial}{\partial u_1} + \varepsilon t \frac{\partial}{\partial u_2}$ if $\lambda = 0$;	3, 7, 9 (for II : $3, 7$)
			$\tilde{\psi}_0 \frac{\partial}{\partial u_2}$	For I^*, III $(u_1 - \lambda t) \frac{\partial}{\partial u_2}$; & (for I^*) $u_1 \frac{\partial}{\partial u_1} + \varepsilon t \frac{\partial}{\partial u_2}$ if $\lambda = 0$	$3, 9$; & $6, 7$ if $\lambda = 0$ (7 for I^* only)
11.	$f^1 = 0$, $f^2 = \varepsilon u_2 + \ln u_1$	I, IV	$\mu u_1 \frac{\partial}{\partial u_1}$ $-\varepsilon \frac{\partial}{\partial u_2}$, $\tilde{\psi}_\varepsilon \frac{\partial}{\partial u_2}$	$e^{\varepsilon t} u_1 \frac{\partial}{\partial u_2}$ if $a = 1$	10, $\kappa = \varepsilon$
12.	$f^1 = \lambda u_1 \ln u_1$, $f^2 = \nu u_2 + \ln u_1$	I, IV	$\tilde{\psi}_\nu \frac{\partial}{\partial u_2}$	$e^{\nu t}\Big(u_1 \frac{\partial}{\partial u_1} + t \frac{\partial}{\partial u_2}\Big)$ if $\nu = \lambda$;	10, $\kappa = \nu$
				$e^{\lambda t}\Big((\lambda - \nu) u_1 \frac{\partial}{\partial u_1}$ $+ \frac{\partial}{\partial u_2}\Big)$ if $\nu \neq \lambda$	10, $\kappa = \nu$
13.	$f^1 = \lambda u_1^{\mu+1}$, $f^2 = \sigma u_1^{\mu+1}$, $\lambda\sigma = 0$	III	$\mu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2u_2 \frac{\partial}{\partial u_2}$, $\tilde{\psi}_0 \frac{\partial}{\partial u_2}$	$u_1 \frac{\partial}{\partial u_2}$ if $\lambda = 0$	3; 6 if $\lambda = 0$

Here and in what follows, $\varepsilon = \pm 1$, K is the generator defined in (4.6), and $\mathcal{K} = K + \frac{2}{\lambda - 1}\left[t\left(\lambda u_1 \frac{\partial}{\partial u_1} + (2-\lambda) u_2 \frac{\partial}{\partial u_2}\right) + u_1 \frac{\partial}{\partial u_2}\right]$. In the subsequent table, $Q = 2\left((\mu - a\nu)t - \frac{\nu}{2m} x^2\right)$ for the version II and $Q = 2\left((\mu - \nu)t - \frac{\nu}{2am} x^2\right)$, $a \neq 0$ for the version III.

Table 6. Nonlinearities with Arbitrary Parameters and Nonextendible Symmetries for Eqs. (1.3) and (2.2)

No	Nonlinear terms	Type of matrix A	Symmetries	AET (2.10)
1.	$f^1=\lambda u_2^{\nu+1}$, $f^2=\mu u_2^{\nu+1}$	II, III	$\nu D-2u_1\frac{\partial}{\partial u_1}$ $-2u_2\frac{\partial}{\partial u_2}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$	2
2.	$f^1=\lambda\left(u_1+u_2\right)^{\nu+1}$, $f^1=\mu\left(u_1+u_2\right)^{\nu+1}$	$I, a\neq 1$ IV	$\nu D-2u_1\frac{\partial}{\partial u_1}$ $-2u_2\frac{\partial}{\partial u_2}$, $\Psi_0(x)\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right)$	12
3.	$f^1=\lambda u_1^{\nu+1}$, $f^2=u_1^{\nu}\left(\lambda u_2+\mu u_1^{\sigma}\right)$, $\nu+\sigma\neq 0,1,\mu\neq 0$	I^*	$\nu D-2u_1\frac{\partial}{\partial u_1}$ $-2\sigma u_2\frac{\partial}{\partial u_2}$, $u_1\frac{\partial}{\partial u_2}$	6
4.	$f^1=\lambda e^{u_2}$, $f^2=\sigma e^{u_2}$	II, III	$D-2\frac{\partial}{\partial u_2}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$	2
5.	$f^1=\lambda e^{(u_1+u_2)}$, $f^2=\sigma e^{(u_1+u_2)}$	$I, a\neq 1$, IV	$D-2\frac{\partial}{\partial u_2}$, $\Psi_0(x)\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right)$	12
6.	$f^1=\lambda u_2^{\nu}e^{u_1}$, $f^2=\sigma u_2^{\nu+1}e^{u_1}$, $\nu^2+(a-1)^2\neq 0$	I, IV	$D-2\frac{\partial}{\partial u_1}$, $u_2\frac{\partial}{\partial u_2}-\nu\frac{\partial}{\partial u_1}$	13 if $\sigma=0$
7.	$f^1=\lambda e^{u_1}$, $f^2=\sigma u_1 e^{u_1}$	I^*, III	$D-2\frac{\partial}{\partial u_1}-2u_1\frac{\partial}{\partial u_2}$, $\psi_0\frac{\partial}{\partial u_2}$ (& $u_2\frac{\partial}{\partial u_2}$ for I^*)	3; 6 if $\lambda=0$
8.	$f^1=\varepsilon e^{u_1}$, $\varepsilon=\pm 1$, $f^2=\lambda u_1$	I, IV	$D+2u_2\frac{\partial}{\partial u_2}-2\frac{\partial}{\partial u_1}$ $-2\lambda t\frac{\partial}{\partial u_2}$, $\tilde{\psi}_0\frac{\partial}{\partial u_2}$	3
9.	$f^1=\nu e^{\lambda\left(2u_2-u_1^2\right)}$, $f^2=\left(\nu u_1+\mu\right)e^{\lambda\left(2u_2-u_1^2\right)}$	I^*, III	$\lambda D-\frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_1}+u_1\frac{\partial}{\partial u_2}$	14
10.	$f^1=\lambda\ln(2u_2-u_1^2)$, $f^2=\sigma(2u_2-u_1^2)$ $+\lambda u_1\ln(2u_2-u_1^2)$	I^*	$D+2u_1\frac{\partial}{\partial u_1}+4u_2\frac{\partial}{\partial u_2}$ $+4\lambda t\left(\frac{\partial}{\partial u_1}+u_1\frac{\partial}{\partial u_2}\right)$, $\frac{\partial}{\partial u_1}+u_1\frac{\partial}{\partial u_2}$	14
11.	$f^1=\mu\ln u_2$, $f^2=\nu\ln u_2$	II, III	$\Psi_0(x)\frac{\partial}{\partial u_1}$, $D+2u_1\frac{\partial}{\partial u_1}$ $+2u_2\frac{\partial}{\partial u_2}+Q\frac{\partial}{\partial u_1}$	2
12.	$f^1=\varepsilon\ln\left(u_1+u_2\right)$, $f^2=\nu\ln\left(u_1+u_2\right)$	$I, a\neq 1$, IV, $a=0$	$\Psi_0(x)\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right)$, $(a-1)\Big(D+2u_1\frac{\partial}{\partial u_1}$ $+2u_2\frac{\partial}{\partial u_2}\Big)+(2(a\varepsilon+\nu)t$ $+\frac{\varepsilon+\nu}{m}x^2)\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right)$	12

13.	$f^1 = \lambda u_1^{\nu+1}$, $f^2 = \ln u_1$, $\lambda(\nu+1) \neq 0$	I, IV	$\nu \left(D + 2u_2 \frac{\partial}{\partial u_2} \right)$ $-2u_1 \frac{\partial}{\partial u_1} - 2t \frac{\partial}{\partial u_2}$, $\tilde{\psi}_0 \frac{\partial}{\partial u_2}$	3
14.	$f^1 = \lambda u_1^{\nu+1}$, $f^2 = \lambda u_1^{\nu+1} \ln u_1$	I^*, III	$\nu D - 2\Big(u_1 \frac{\partial}{\partial u_1}$ $+u_2 \frac{\partial}{\partial u_2} + u_1 \frac{\partial}{\partial u_2} \Big)$, $\psi_0 \frac{\partial}{\partial u_2}$	3
15.	$f^1 = \lambda u_1^{\nu+1}$, $f^2 = \lambda u_1^{\nu} u_2 + u_1 \ln u_1$, $\lambda(\nu-1) \neq 0$	I^*	$\nu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2tu_1 \frac{\partial}{\partial u_2}$ $-2(1-\nu)u_2 \frac{\partial}{\partial u_2}, \quad u_1 \frac{\partial}{\partial u_2}$	6
16.	$f^1 = \lambda \left(2u_2 - u_1^2\right)^{\nu+\frac{1}{2}}$, $f^2 = \lambda u_1 \big(2u_2 - u_1^2\big)^{\nu+\frac{1}{2}}$ $+\mu \left(2u_2 - u_1^2\right)^{\nu+1}$	I^*	$\nu D - u_1 \frac{\partial}{\partial u_1} - 2u_2 \frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}$ $\Big(\&\ 2\lambda t \left(\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2} \right)$ $+ \frac{\partial}{\partial u_2} \ \text{ if } \mu = 0,\ \nu = \frac{1}{2} \Big)$	$14; 1$, $\rho = 2\omega$ if $\nu = 0$
17.	$f^1 = 2\nu u_1 \ln u_1 + u_1 u_2$, $f^2 = -(\nu-\mu)^2 \ln u_1$ $+2\mu u_2$	I, IV	$X = e^{(\mu+\nu)t} \Big(u_1 \frac{\partial}{\partial u_1}$ $+(\mu-\nu) \frac{\partial}{\partial u_2} \Big)$, $tX + e^{(\mu+\nu)t} \frac{\partial}{\partial u_2}$	10, $\kappa = 2\nu$ if $\mu + \nu$ $= 0$
18.	$f^1 = 2\nu u_1 \ln u_1 + u_1 u_2$, $f^2 = 2\mu u_2$ $+ \left(1 - (\nu-\mu)^2\right) \ln u_1$	I, IV	$X^{\pm} = e^{\lambda_{\pm} t} \Big(u_1 \frac{\partial}{\partial u_1}$ $+(\lambda_{\pm} - 2\nu) \frac{\partial}{\partial u_2} \Big)$, $\lambda_{\pm} = \mu + \nu \pm 1$	10, $\kappa = 2\nu$ if $\mu + \nu$ $= \pm 1$
19.	$f^1 = 2\nu u_1 \ln u_1 + u_1 u_2$, $f^2 = 2\mu u_2$ $- \left(1 + (\nu-\mu)^2\right) \ln u_1$	I, IV	$e^{(\mu+\nu)t} \Big[\cos t u_1 \frac{\partial}{\partial u_1}$ $-(\sin t + (\nu$ $-\mu) \cos t) \frac{\partial}{\partial u_2} \Big]$, $e^{(\mu+\nu)t} \Big[\sin t u_1 \frac{\partial}{\partial u_1}$ $+(\cos t + (\mu$ $-\nu) \sin t) \frac{\partial}{\partial u_2} \Big]$	
20.	$f^1 = \varepsilon(2u_2 - u_1^2)$, $f^2 = (\mu + \varepsilon u_1) \left(2u_2 - u_1^2\right)$ $-\frac{\mu^2}{2} \varepsilon u_1, \quad \mu \neq 0$	I^*, III	$X_1 = e^{\mu t} \Big(2 \frac{\partial}{\partial u_1}$ $+2u_1 \frac{\partial}{\partial u_2} + \varepsilon\mu \frac{\partial}{\partial u_2} \Big)$, $tX_1 + \varepsilon e^{\mu t} \frac{\partial}{\partial u_2}$	
21.	$f^1 = \varepsilon(2u_2 - u_1^2)$, $f^2 = (\mu + \varepsilon u_1) \left(2u_2 - u_1^2\right)$ $+\frac{1-\mu^2}{2} \varepsilon u_1$	I^*, III	$X^{\pm} = e^{\mu \pm 1} \Big(2 \frac{\partial}{\partial u_1}$ $+2u_1 \frac{\partial}{\partial u_2}$ $+\varepsilon(\mu \pm 1) \frac{\partial}{\partial u_2} \Big)$	14 if $\mu^2 = 1$
22.	$f^1 = \varepsilon(2u_2 - u_1^2)$, $f^2 = -\frac{1+\mu^2}{2} \varepsilon u_1$ $+(\mu + \varepsilon u_1) \left(2u_2 - u_1^2\right)$	I^*, III	$e^{\mu t} \Big(2\varepsilon \cos t \Big(\frac{\partial}{\partial u_1}$ $+u_1 \frac{\partial}{\partial u_2} \Big) + (\mu \cos t$ $- \sin t) \frac{\partial}{\partial u_2} \Big)$, $e^{\mu t} \Big(2\varepsilon \sin t \Big(\frac{\partial}{\partial u_1}$ $+u_1 \frac{\partial}{\partial u_2} \Big) + (\mu \sin t$ $+ \cos t) \frac{\partial}{\partial u_2} \Big)$	

Table 7. Symmetries of Eqs. (1.3) with a Diagonal Matrix A and the Nonlinearities $f^1 = u_1\,(\mu\ln u_1 + \lambda\ln u_2)$ and $f^2 = u_2\,(\nu\ln u_2 + \sigma\ln u_1)$

No	Conditions for coefficients and notation	Main symmetries	Additional symmetries
1.	$\lambda = 0,$ $\mu = \nu$	$e^{\mu t}u_2\frac{\partial}{\partial u_2},$ $e^{\mu t}\left(u_1\frac{\partial}{\partial u_1} + \sigma t u_2\frac{\partial}{\partial u_2}\right)$	$\widehat{G}_\alpha$ if $a \neq 0,$ $\sigma = 0,\ \mu \neq 0$
2.	$\lambda = 0,$ $\mu \neq \nu$	$e^{\nu t}u_2\frac{\partial}{\partial u_2}$, $e^{\mu t}\left((\mu-\nu)\,u_1\frac{\partial}{\partial u_1} + \sigma u_2\frac{\partial}{\partial u_2}\right)$	$\widehat{G}_\alpha$ if $\mu \neq 0,$ $\mu - \nu = a\sigma$
			G_α if $a\sigma = -\nu,$ $\mu = 0;$
			$\psi_0\frac{\partial}{\partial u_2}$ if $\sigma = \nu = 0;$
			$\psi_0\frac{\partial}{\partial u_1}$ if $\sigma = \mu = 0;$
			$u_1\frac{\partial}{\partial u_2},\ \widehat{G}_\alpha$ if $a = 1,\ \nu = 0$ $\mu = \sigma \neq 0$
3.	$\delta = \frac{1}{4}(\mu-\nu)^2 + \lambda\sigma = 0,$ $\mu + \nu = 2\omega_0$	$X_2 = e^{\omega_0 t}\left(2\lambda u_1\frac{\partial}{\partial u_1} + (\nu-\mu)\,u_2\frac{\partial}{\partial u_2}\right),$	$\widehat{G}_\alpha$ if $\nu \neq -\mu,$ $2\lambda = a(\nu-\mu)$
	$\lambda\sigma \neq 0$	$e^{\omega_0 t}2u_2\frac{\partial}{\partial u_2} + tX_2$	G_α if $\lambda = a\nu$ $\mu = -\nu \neq 0$
4.	$\lambda\sigma \neq 0,$ $\delta = 1,$ $\omega_\pm = \omega_0 \pm 1$	$e^{\omega_+ t}\left(\lambda u_1\frac{\partial}{\partial u_1} + (\omega_+ - \mu)\,u_2\frac{\partial}{\partial u_2}\right),$ $e^{\omega_- t}\left(\lambda u_1\frac{\partial}{\partial u_1} + (\omega_- - \mu)\,u_2\frac{\partial}{\partial u_2}\right)$	$\widehat{G}_\alpha$ if $\mu\nu \neq \lambda\sigma,$ $\lambda = a(\nu - \mu + a\sigma)$
			G_α if $\nu\mu = \lambda\sigma,$ $\lambda = -a\mu$
5.	$\delta = -1$	$e^{\omega_0 t}\left(2\lambda\cos t u_1\frac{\partial}{\partial u_1} + ((\nu-\mu)\cos t - 2\sin t)\,u_2\frac{\partial}{\partial u_2}\right),$ $e^{\omega_0 t}\left(2\lambda\sin t u_1\frac{\partial}{\partial u_1} + ((\nu-\mu)\sin t + 2\cos t)\,u_2\frac{\partial}{\partial u_2}\right)$	none

Equations (1.3) with the nonlinearities presented in Table 7 admit the equivalence transformation 1 from list (2.10), provided that $\mu\nu = \lambda\sigma$. The related parameters ρ and ω should satisfy the condition $\mu\omega + \lambda\rho = 0$. In addition, the equations corresponding to the last version in case 2 admit the additional equivalence transformation 6 given by formula (2.10).

Table 8. Symmetries of Eqs. (1.3) with a Matrix A of the Type I^*, II and the Nonlinearities $f^1 = (\mu u_1 - \sigma u_2)\ln R + z(\lambda u_1 - \nu u_2)$ and $f^2 = (\mu u_2 + \sigma u_1)\ln R + z(\lambda u_2 + \nu u_1)$

No	Conditions for coefficients	Main symmetries	Additional symmetries
1.	$\lambda = 0$, $\mu = \nu$	$e^{\mu t}\frac{\partial}{\partial z}$,	For II : $\hat{G}_\alpha$ if $a\sigma = 0,\ \mu \neq 0$
		$e^{\mu t}\left(R\frac{\partial}{\partial R} + \sigma t\frac{\partial}{\partial z}\right)$	For II : G_α if $a = \nu = 0,\ \sigma \neq 0$
			For I^* : $\widehat{G}_\alpha$ if $\sigma = 0,\ \mu \neq 0$
2.	$\lambda = 0$, $\mu \neq \nu$,	$e^{\nu t}\frac{\partial}{\partial z}$, $e^{\mu t}\left(\sigma\frac{\partial}{\partial z} + (\mu - \nu)R\frac{\partial}{\partial R}\right)$	For II : $\hat{G}_\alpha$ if $a\sigma = \nu - \mu$, $\mu \neq 0$ or $a = 0,\ \mu \neq 0$
			For II : G_α if $a\sigma = \nu,\ \mu = 0$
			For I^* : $\widehat{G}_\alpha$ if $\mu \neq 0,\ \sigma = 0$
			For I^* : G_α if $\mu = 0,\ \sigma = 0$
3.	$\delta = 0$, $\lambda \neq 0$	$X_3 = e^{\omega_0 t}\left(2\lambda R\frac{\partial}{\partial R} + (\nu - \mu)\frac{\partial}{\partial z}\right)$	For II : $\widehat{G}_\alpha$ if $\mu \neq -\nu$ $a(\mu - \nu) = 2\lambda$
			For II : G_α if $a\nu = -\lambda,\ \omega_0 = 0$
			For I^* : $\widehat{G}_\alpha$ if $\mu = \nu \neq 0$
			For I^* : G_α if $\mu = \nu = 0$
4.	$\lambda \neq 0$, $\delta = 1$	$e^{\omega_+ t}\left(\lambda R\frac{\partial}{\partial R} + (\omega_+ - \mu)\frac{\partial}{\partial z}\right)$, $e^{\omega_- t}\left(\lambda R\frac{\partial}{\partial R} + (\omega_- - \mu)\frac{\partial}{\partial z}\right)$	$\widehat{G}_\alpha$ if $\mu\nu \neq \lambda\sigma$, $\lambda = a(\nu - \mu + a\sigma)$
			G_α if $\nu\mu = \lambda\sigma,\ \lambda = -a\mu$
			For I^* : $\widehat{G}_\alpha$ if $\sigma = 0,\ \mu \neq 0$
			For I^* : G_α if $\sigma = \mu = 0$
5.	$\delta = -1$	$\exp(\omega_0 t)\left[2\lambda\cos tR\frac{\partial}{\partial R} + ((\nu - \mu)\cos t - 2\sin t)\frac{\partial}{\partial z}\right]$, $\exp(\omega_0 t)\left[2\lambda\sin tR\frac{\partial}{\partial R} + ((\nu - \mu)\sin t + 2\cos t)\frac{\partial}{\partial z}\right]$	none

All equations enumerated in Table 8 admit the additional equivalence transformations 15 from list (2.10).

Table 9. Symmetries of Eqs. (1.3) with Nonlinearities $f^1 = \lambda u_2 + \mu u_1 \ln u_1$ and $f^2 = \lambda \frac{u_2^2}{u_1} + (\sigma u_1 + \mu u_2) \ln u_1 + \nu u_2$ and matrices A of the type III (and I^* if $a = 0$)

No	Conditions for coefficients	Main symmetries	Additional symmetries
1.	$\lambda = 0,$ $\mu \neq \nu$	$e^{\nu t} u_1 \frac{\partial}{\partial u_2},$	$\psi_\nu \frac{\partial}{\partial u_2}$ if $\mu = 0,$ & G_α if $a\nu = \sigma \neq 0$
		$e^{\mu t}\left((\mu - \nu) R \frac{\partial}{\partial R} + \sigma u_1 \frac{\partial}{\partial u_2}\right)$	$\widehat{G}_a,$ if $\mu \neq 0,$ $\sigma = a(\nu - \mu) \neq 0$
2.	$\lambda = 0, \mu = \nu$	$e^{\mu t} u_1 \frac{\partial}{\partial u_2},$ $e^{\mu t}\left(R \frac{\partial}{\partial R} + \sigma t u_1 \frac{\partial}{\partial u_2}\right)$	$\psi_0 \frac{\partial}{\partial u_2}$ if $\mu = 0,$ $\sigma \neq 0$ & $D + u_2 \frac{\partial}{\partial u_2}$ if $a = 0$
			$\widehat{G}_a$ if $\sigma = 0,\ \mu \neq 0$
3.	$\sigma = 0,$ $\mu\lambda \neq 0,$	$e^{\nu t}\left(\lambda R \partial_R + (\mu - \nu) u \partial_v\right)$	G_a if $\nu = 0,$ $\mu = -\lambda$
	$\mu \neq \nu, a = 1$	$e^{\mu t} R \partial_R$	$\hat{G}_a$ if $\nu - \mu = \lambda$
4.	$\delta = 0$ $\mu + \nu = 2\omega_0,$ $\lambda \neq 0$	$X_4 = e^{\omega_0 t}\left(2\lambda R \frac{\partial}{\partial R} + (\nu - \mu) u_1 \frac{\partial}{\partial u_2}\right),$	G_a if $\omega_0 = 0,$ $\nu = -a\lambda$ & $D + 2u_1 \frac{\partial}{\partial u_1}$ if $a = 0$
		$2e^{\omega_0 t} u_1 \frac{\partial}{\partial u_2} + tX_4$	$\widehat{G}_a$ if $\omega_0 \neq 0,$ $2a\lambda = \mu - \nu$
5.	$\lambda \neq 0,$ $\delta = 1,$	$e^{\omega_+ t}\left(\lambda R \frac{\partial}{\partial R} + (\omega_+ - \mu) u_1 \frac{\partial}{\partial u_2}\right),$	G_a if $\mu = a\lambda,$ $\mu\nu = \lambda\sigma$
	$\omega_\pm = \omega_0 \pm 1$	$e^{\omega_- t}\left(\lambda R \frac{\partial}{\partial R} + (\omega_- - \mu) u_1 \frac{\partial}{\partial u_2}\right)$	$\widehat{G}_\alpha$ if $\mu\nu \neq \lambda\sigma,$ $\mu - \nu = \lambda - \sigma,$ $a = 1$ or $\sigma = a = 0,\ \mu \neq 0$
6.	$\delta = -1,$	$e^{\omega_0 t}[2\lambda \cos t R \frac{\partial}{\partial R} + ((\nu - \mu)\cos t - 2\sin t) u_1 \frac{\partial}{\partial u_2}],$ $e^{\omega_0 t}[2\lambda \sin t R \frac{\partial}{\partial R} + ((\nu - \mu)\sin t + 2\cos t) u_1 \frac{\partial}{\partial u_2}]$	none

If $\lambda = \mu = 0$ or $\lambda = \nu = 0$, then the related equation (1.3) admits the additional equivalence transformations 16 or 6, respectively, from list (2.10).

Tables 3–9 present the results of the group classification of Eqs. (1.3) with invertible diffusion matrix A. The results presented in Tables 3–7 are also valid for equations with a singular matrix A of the type IV but do not exhaust all nonequivalent nonlinearities for such equations. Moreover, the equations with singular diffusion matrix admit strong equivalence transformations $u_1 \to u_1$ and $u_2 \to \varepsilon(u_2)$, where $\varepsilon(u_2)$ is an arbitrary function of u_2, that reduce the number of nonequivalent symmetries in Tables 3–9 for $a = 0$.

The completed group classification of Eqs. (1.3) with a matrix A of the type IV is given in [21].

11. Classification of Reaction-Diffusion Equations with Nilpotent Diffusion Matrix

To complete the classification of systems (1.3) we need to consider the remaining class of these equations where the matrix A belongs to the type V, i.e., is nilpotent. The procedure of classification of such equations appears to be more complicated than in the case of invertible or diagonalizable diffusion matrices. The general form of a symmetry admitted by this equation is given by Eq. (4.13), whereas the classifying equations take the form (4.14).

A specific feature of symmetries (4.13) is that the coefficient B^3 can be a function of u_1. One more specific point in the classification of equations with a matrix A of the type V is that they admit powerful equivalence relations

$$u_1 \to u_1, \; u_2 \to u_2 + \Phi(u_1) \tag{11.1}$$

and

$$u_1 \to u_1, \; u_2 \to u_2 + \hat{\Phi}(u_1, t, x), \tag{11.2}$$

which have not appeared in our analysis presented in the previous sections.

Transformation (11.1) (where $\Phi(u_1)$ is an arbitrary function of u_1) are admitted by any Eq. (1.3) with a matrix A of the type V. Transformations (11.2) are valid for the cases where f^1 does not depend on u_2 and, at the same time, f^2 is linear in u_2. Moreover, the related functions $\hat{\Phi}(u_1, t, x)$ should satisfy the following system of equations:

$$\begin{aligned} &f^2_{u_2}\hat{\Phi}_t - \hat{\Phi}_{tt} - f^1\hat{\Phi}_{tu_1} = 0, \\ &f^2_{u_2}\hat{\Phi}_{x_\nu} - \hat{\Phi}_{tx_\nu} - f^1\hat{\Phi}_{u_1x_\nu} = 0 \end{aligned} \tag{11.3}$$

Thus, the group classification of Eq. (1.3) with nilpotent diffusion matrix reduces to solving the classifying equations (4.14) with applying the equivalence transformations discussed in Sec. 2 and transformations

(11.1) and (11.2) as well. To do this we again use the analysis of low-dimensional algebras $\mathcal{A}$, whose results are given in Appendix. We do not reproduce the related routine calculations but present the classification results in Tables 8–10.

In Tables 8–10, we use without explanations the notation applied in Tables 1–9. In addition, a number of classified equations exhibit a specific symmetry $W\partial_{u_2}$, where W is a function of t, x and u_1 that solves the following equation:

$$f^2_{u_2} - W_t - W_{u_1} f^1 = 0.$$

Table 10. Nonlinearities with Arbitrary Functions for Eqs. (1.3) with Nilpotent Diffusion Matrix

No	Nonlinear terms	Arguments of F_α	Symmetries
1.	$f^1 = F_1 u_1^{\mu-\nu}$, $f^2 = F_2 u_1^{\mu}$	$\frac{u_1^{\nu+1}}{u_2}$	$Q_1 = (\mu-1)D - \nu t\frac{\partial}{\partial t}$ $-u_1\frac{\partial}{\partial u_1} - (\nu+1)u_2\frac{\partial}{\partial u_2}$ & $(m-2)x^2\frac{\partial}{\partial x_a} - x_a Q_1$ if $\nu(m-2)=4$, $\mu(m-2)=m+2,\ m\neq 2$
2.	$f^1 = F_1 u_1 u_2^{\mu-1}$, $f^2 = F_2 u_2^{\mu}$, $F_2 \neq 0$	u_1	$\mu D - t\frac{\partial}{\partial t} - u_2\frac{\partial}{\partial u_2}$ & $e^W\frac{\partial}{\partial u_2}$ if $\mu=1$ & $H^a\frac{\partial}{\partial x_a} - H^b_{x_b}u_2\frac{\partial}{\partial u_2}$ if $m=2$
3.	$f^1 = F_1 u_2^{-1}$, $f^2 = F_2 + \nu u_2$	u_1	$e^{\nu t}\left(\frac{\partial}{\partial t} + \nu u_2\frac{\partial}{\partial u_2}\right)$ & $e^W\frac{\partial}{\partial u_2}$ if $F_1 = 0$
4.	$f^1 = F_1 u_2^{\mu-1}$, $f^2 = F_2 u_2^{\mu}$	$u_2 e^{u_1}$	$\mu D - t\frac{\partial}{\partial t} - u_2\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_1}$
5.	$f^1 = \frac{F_1}{u_2} + \nu$, $f^2 = F_2 + \nu u_2$	$u_2 e^{u_1}$	$e^{\nu t}\left(\frac{\partial}{\partial t} + \nu u_2\frac{\partial}{\partial u_2} - \nu\frac{\partial}{\partial u_1}\right)$
6.	$f^1 = 0,\ f^2 = F_2$	u_2	$\Psi_0(x)\frac{\partial}{\partial u_1}$, $x_a\frac{\partial}{\partial x_a} + 2u_1\frac{\partial}{\partial u_1}$
7.	$f^1 = F_1,\ f^2 = 0$	u_1	$e^W\frac{\partial}{\partial u_2}$, $x_a\frac{\partial}{\partial x_a} - 2u_2\frac{\partial}{\partial u_2}$
8.	$f^1 = \frac{\nu}{\mu-1}u_1 + F_1 u_1^{2-\mu}$, $f^2 = \frac{\mu\nu}{\mu-1}u_2 + F_2 u_1$, $\mu \neq 1$	$u_2 u_1^{-\mu}$	$e^{\nu t}\Big((1-\mu)t\frac{\partial}{\partial t} - \nu u_1\frac{\partial}{\partial u_1}$ $-\nu\mu u_2\frac{\partial}{\partial u_2}\Big)$

9.	$f^1 = u_1F_1,\ m=1$ $f^2 = u_2F_2 + u_1$	u_2u^3	$Q_2 = \cos(2x)\left(u_1\frac{\partial}{\partial u_1} - 3u_2\frac{\partial}{\partial u_2}\right)$ $+\sin(2x)x\frac{\partial}{\partial x},\quad Q_3 = (Q_2)_x$
10.	$f^1 = u_1F_1,\ m=1$ $f^2 = u_2F_2 - u_1$	$u_2u_1^3$	$Q_4 = e^{2x}\left(\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_1} - 3u_2\frac{\partial}{\partial u_2}\right),$ $Q_5 = e^{-2x}\left(\frac{\partial}{\partial x} - u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2}\right)$
11.	$f^1 = F_1,$ $f^2 = u_2F_2,$ $m=2$	$u_2e^{u_1}$	$H^a\frac{\partial}{\partial x_a} - H^b_{x_b}\left(u_2\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right)$
12.	$f^1 = \nu e^{\frac{u_2}{u_1}},$ $f^2 = e^{\frac{u_2}{u_1}}F$	u_1	$D - u_1\frac{\partial}{\partial u_2}$
13.	$f^1 = F_1,$ $F_2 = u_2F_2 + F_3$	u_1	$e^W\frac{\partial}{\partial u_2}$
14.	$f^1 = e^{\nu u_2}F_1,$ $f^2 = e^{\nu u_2}F_2$	u_1	$\nu D - \frac{\partial}{\partial u_2}$
15.	$f^1 = e^{\nu u_1}F_1,$ $f^2 = e^{\nu u_1}F_2$	u_2	$\nu D - \frac{\partial}{\partial u_1}$
16.	$f^1 = \nu u_1 + F_1,$ $f_2 = \mu u_1 + F_2$	u_2	$e^{(\nu - a\mu)t}\Psi_\mu(x)\frac{\partial}{\partial u_1}$
17.	$f^1 = u_1(F_1 + \nu\ln u_1),$ $f^2 = u_2(F_2 + \nu\ln u_1),$ $\nu \neq 0$	$\frac{u_1}{u_2}$	$e^{\nu t}(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2})$
18.	$f^1 = u_1F_1 - \nu u_2,$ $f^2 = \nu\frac{u_2}{u_1}(u_2 - u_1)$ $+u_1F_2 - u_2F_1$	$u_1e^{\frac{u_2}{u_1}}$	$e^{\nu t}\left(u_1\frac{\partial}{\partial u_2} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}\right)$ & $\widehat{G}_\alpha$ if $a=1$
19.	$f^1 = u_1^{\nu+1}F_1,$ $f^2 = u_1^\nu(F_2u_1 - F_1u_2)$	$u_1e^{\frac{u_2}{u_1}}$	$\nu D + u_1\frac{\partial}{\partial u_2} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$ & G_α if $\nu=0,\ a=1$
20.	$f^1 = u_1^{\mu+1}F_1,$ $f^2 = u_1^{\mu+1}F_2$	$\frac{u_2}{u_1}$	$\mu D - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$

For the nonlinearities enumerated in cases 2 (for $\mu = 1$), 3 (for $F_1 = 0$), 4, and 8 of Table 8, the related equation (1.3) admits the additional equivalence transformations (11.2). In addition, transformations (2.4) and (11.1) and some equivalence transformations from list (2.10) are admissible, namely, transformations 9 for the nonlinearities given in cases 1 (for $\nu = -1$ and $\mu = 0$), 6, and 18–20, and transformations 1 with $\rho = \omega$ for the nonlinearities from cases 1 (for $\nu = 1$ and $\mu = 0$), 20 (for $\mu = 0$), 18, and 19. Finally, for f^1 and f^2 presented in case 7, transformation 3 of (2.10) is admissible.

Table 11. Nonlinearities with Arbitrary Parameters and Extendible Symmetries for Eqs. (1.3) with Nilpotent Diffusion Matrix

No	Nonlinearities	Main symmetries	Additional symmetries	AET (2.10)
1.	$f^1=\lambda u_1^{\nu+1}u_2^{\mu}$, $f^2=\sigma u_1^{\nu}u_2^{\mu+1}$	$(\mu+\nu)t\frac{\partial}{\partial t}$ $-(\mu+1)u_1\frac{\partial}{\partial u_1}$ $+(\nu-1)u_2\frac{\partial}{\partial u_2}$ $Q_6=2\mu u_1\frac{\partial}{\partial u_1}$ $+(\mu+\nu)x_a\frac{\partial}{\partial x_a}$ $-2\nu u_2\frac{\partial}{\partial u_2}$	$x_aQ_6-2\kappa x^2\frac{\partial}{\partial x_a}$ if $\kappa(m+2)=\nu$, $\kappa(2-m)=\mu$	
			$e^W\frac{\partial}{\partial u_2}$ if $\lambda=0$, $\mu=-1$ & $2x^2\frac{\partial}{\partial x_a}$ $-(m-2)x_aQ_6$ if $\nu=\frac{m+2}{m-2}$, $m\neq 2$	17, $3,6$
			$\Psi_0(x)\frac{\partial}{\partial u_1}$ if $\sigma=0,\nu=-1$, & $x_aQ_6+\frac{2}{m+2}x^2\frac{\partial}{\partial x_a}$ if $\mu=\frac{m-2}{m+2}$, $m\neq 2$	17, 9
			$\Psi_0(x)\frac{\partial}{\partial u_1}$ if $\lambda=\nu=0$	17, 9
			$e^W\frac{\partial}{\partial u_2}$ if $\mu=0$ & $H^a\frac{\partial}{\partial x_a}-\frac{\partial H^a}{\partial x_a}u_2\frac{\partial}{\partial u_2}$ if $m=2$	17 & 6 if $\lambda=\sigma$
2.	$f^1=\lambda u_1^{\nu+1}u_2^{-1}$, $f^2=\sigma u_1^{\nu}+\varepsilon u_2$, $\lambda\neq 0$	$e^{\varepsilon t}\left(\frac{\partial}{\partial t}+\varepsilon u_2\frac{\partial}{\partial u_2}\right)$, ${Q'}_6=Q_6\vert_{\mu=-1}$	$x_a{Q'}_6-\frac{2}{m-2}x^2\frac{\partial}{\partial x_a}$ if $\nu=\frac{m+2}{m-2}$, $m\neq 2$	
			$\Psi_0(x)\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=-1$	9
			$u_2\frac{\partial}{\partial u_2}+u_1\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=1$	1
3.	$f^1=\lambda e^{\nu u_1}$, $f^2=\sigma e^{(\nu+1)u_1}$, $\lambda\sigma=0$	$(\nu+1)D-u_2\frac{\partial}{\partial u_2}$ $-t\frac{\partial}{\partial t}-\frac{\partial}{\partial u_1}$, $e^W\frac{\partial}{\partial u_2}$	$\nu\left(u_2\frac{\partial}{\partial u_2}+t\frac{\partial}{\partial t}\right)$ $-\frac{\partial}{\partial u_1}$ if $\sigma=0$	17, 3
			$u_2\frac{\partial}{\partial u_2}+t\frac{\partial}{\partial t}$ if $\lambda=0$	10, $3,6$
4.	$f^1=\lambda e^{(\nu+1)u_2}$, $f^2=\sigma e^{\nu u_2}$	$(\nu-1)D-u_1\frac{\partial}{\partial u_1}$ $+t\partial_t-\frac{\partial}{\partial u_2}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$	$\nu\left(u_1\frac{\partial}{\partial u_1}-t\frac{\partial}{\partial t}\right)$ $+\frac{\partial}{\partial u_2}$ if $\lambda=0$	9 for any λ
5.	$f^1=\lambda u_2^{\mu-1}e^{u_1}$, $f^2=\sigma u_2^{\mu}e_1^{u}$	$D-\frac{\partial}{\partial u_1}$, $t\partial_t+u_2\frac{\partial}{\partial u_2}-\mu\frac{\partial}{\partial u_1}$	e^W if $\mu=1$ & $H^a\frac{\partial}{\partial x_a}-\frac{\partial H^a}{\partial x_a}u_2\frac{\partial}{\partial u_2}$ if $m=2$	17 if $\mu=1$
6.	$f^1=\lambda\ln u_2$, $f^2=\sigma u_2^{\frac{\mu+1}{2}}$	$\mu D-\frac{\mu+1}{2}t\frac{\partial}{\partial t}$ $-\frac{1-\mu}{2}u_1\frac{\partial}{\partial u_1}$ $-u_2\frac{\partial}{\partial u_2}-\lambda t\frac{\partial}{\partial u_1}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$	$x_a\frac{\partial}{\partial x_a}+2u_1\frac{\partial}{\partial u_1}$, $u_1\frac{\partial}{\partial u_1}+2u_2\frac{\partial}{\partial u_2}$ $+t\partial_t+2\lambda t\frac{\partial}{\partial u_1}$ if $\sigma=0$	9

Table 12. Nonlinearities with Arbitrary Parameters and Nonextendible Symmetries for Eqs. (1.3) with $a = 0$

No	Nonlinearities	Conditions	Symmetries	AET (2.10)
1.	$f^1 = \lambda u_1^{3\mu+1}u_2^{\mu}$, $f^2 = \sigma u_1^{3\mu}u_2^{\mu+1} - \alpha u_1$,	$\mu \neq 0$, $m = 1$, $\alpha = -1$	$Q_7 = 4\mu t\frac{\partial}{\partial t}$ $-(\mu+1)u_1\frac{\partial}{\partial u_1}$ $+(3\mu-1)u_2\frac{\partial}{\partial u_2}$, Q_2, Q_3	
		$\mu \neq 0$, $m = 1$, $\alpha = 1$	Q_4, Q_5, Q_7	
2.	$f^1 = \lambda u_1^{-2}u_2^{-1}$,	$m = 1$, $\alpha = -1$	$e^{\varepsilon t}\left(\frac{\partial}{\partial t} + \varepsilon u_2\frac{\partial}{\partial u_2}\right)$, $Q_2,\ Q_3$	17 if
	$f^2 = \sigma u_1^{-3} + \varepsilon u_2 - \alpha u_1$	$m = 1$, $\alpha = 1$	$e^{\varepsilon t}\left(\frac{\partial}{\partial t} + \varepsilon u_2\frac{\partial}{\partial u_2}\right)$, $Q_4,\ Q_5$	$\lambda = 0$
3.	$f^1 = \lambda u_2^{\mu+1}$, $f^2 = \sigma u_2^{\mu-\nu+1}$	$\mu \neq -1$	$(\mu-2\nu)\,D + \nu t\frac{\partial}{\partial t}$ $-u_2\frac{\partial}{\partial u_2} - (\nu+1)u_1\frac{\partial}{\partial u_1}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$	9
4.	$f^1 = \lambda u_2,\ f^2 = e^{-u_2}$	$\lambda \neq 0$	$2D - t\partial_t + u_1\frac{\partial}{\partial u_1}$ $+\frac{\partial}{\partial u_2} + \lambda t\frac{\partial}{\partial u_1}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$	9
5.	$f^1 = \lambda e^{u_2}$, $f^2 = \sigma e^{u_2}$	$\lambda\sigma \neq 0$	$D - \frac{\partial}{\partial u_2},\ \Psi_0(x)\frac{\partial}{\partial u_1}$	9
6.	$f^1 = \lambda u_1^{\nu+1}e^{\mu\frac{u_2}{u_1}}$, $f^2 = e^{\mu\frac{u_2}{u_1}}(\lambda u_2 + \sigma u_1)u_1^{\nu}$	$\mu\lambda \neq 0$	$\mu D - u_1\frac{\partial}{\partial u_2}$, $\nu D - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$	
7.	$f^1 = \mu \ln u_2$, $f^2 = \nu \ln u_2$	$\nu \neq 0$	$\Psi_0(x)\frac{\partial}{\partial u_1}$, $D + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}$ $+\left(\mu t - \frac{\nu}{2m}x^2\right)\frac{\partial}{\partial u_1}$	9
8.	$f^1 = 0,\ f^2 = \varepsilon \ln u_1$	$\varepsilon = \pm 1$	$D - t\partial_t + u_1\frac{\partial}{\partial u_1}$ $+\varepsilon t\frac{\partial}{\partial u_2},\ t\partial_t + u_2\frac{\partial}{\partial u_2}$, $\Phi(u_1, x)\frac{\partial}{\partial u_2}$	3, 6, 17
9.	$f^1 = \varepsilon(\ln u_2 - \kappa \ln u_1)\,u_1$, $f^2 = \varepsilon(\ln u_2 - \kappa \ln u_1)\,u_2$	$m \neq 2$, $\kappa \neq \frac{m+2}{m-2}$	$(1-\kappa)x_a\frac{\partial}{\partial x_a}$ $+2\kappa u_2\frac{\partial}{\partial u_2} + 2u_1\frac{\partial}{\partial u_1}$, $e^{(1-\kappa)\varepsilon t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right)$	1, $\rho=\omega$ if $\kappa=1$
10.	$f^1 = \varepsilon u_1\,((m+2)\ln u_1$ $+(2-m)\ln u_2)$,	$m \neq 1, 2$ $\alpha = 0$	$Q_1,\ x_aQ_1 - x^2\partial x_a$, $e^{4\varepsilon t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right)$	
	$f^2 = \varepsilon u_2\,((m+2)\ln u_1$ $+(2-m)\ln u_2) - \alpha u_1$	$m = 2$, $\alpha = 0$	$H^a\frac{\partial}{\partial x_a} - H^a_{x_a}u_2\frac{\partial}{\partial u_2}$, $e^{4\varepsilon t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right)$	

		$m=1$, $\alpha=1$, $\varepsilon=1$	$Q_2,\ Q_3$, $e^{4t}\left(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}\right)$	
		$m=1$, $\alpha=1$, $\varepsilon=-1$	$Q_4,\ Q_5$, $e^{-4t}\left(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}\right)$	
11.	$f^1=\mu u_1\ln u_1$, $f^2=\mu u_2\ln u_1+\nu u_2$	$\mu\neq 0$	$e^W\frac{\partial}{\partial u_2}$, $e^{\mu t}(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$	17
12.	$f^1=\varepsilon u_2$, $f^2=\lambda\frac{u_2^2}{u_1}+2\nu u_2$ $+\sigma u_1\ln u_1$	$\lambda=\pm 1$, $\sigma=\mp\nu^2$	$Q_8=e^{\nu t}(\lambda(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$ $+\nu u_1\frac{\partial}{\partial u_2})$, $e^{\nu t}u_1\frac{\partial}{\partial u_2}+tQ_8$	
		$\lambda\neq 0$, $\nu^2+\lambda\sigma=1$	$X_\pm=e^{\nu\pm 1}(\lambda(u_1\frac{\partial}{\partial u_1}$ $+u_2\frac{\partial}{\partial u_2})+(\nu\pm 1)u_1\frac{\partial}{\partial u_2})$	1, $\rho=\omega$ if $\sigma=0$
		$\lambda\neq 0$, $\nu^2+\lambda\sigma$ $=-1$	$e^{\nu t}(\lambda\cos t(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$ $+(\nu\cos t-\sin t)u_1\frac{\partial}{\partial u_2})$, $e^{\nu t}(\lambda\sin t(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$ $+(\nu\sin t+\cos t)u_1\frac{\partial}{\partial u_2})$	

12. Discussion

In this paper, we present a completed group classification of systems of two coupled reaction-diffusion equations with general diffusion matrix. In other words, we specify essentially different equations of this type defined up to equivalence transformations and describe their symmetries.

We consider only nonlinear equations, i.e., we exclude the cases where f^1 and f^2 on the right-hand side of (1.3) are linear in u_1 and u_2. Such cases are presented in [23].

The analyzed class of equations includes six nonequivalent subclasses corresponding to the different canonical forms of the diffusion matrix A enumerated in (2.2). In the particular case where the matrix A has the forms I and I^* from (2.2), our results can be compared with those of [7] and also of [3–5].

The paper [7] was apparently the first work where the problem of the group classification of Eqs. (1.3) with diagonal diffusion matrix was formulated and partially solved. Unfortunately, the classification results presented in [7] are incomplete and, in many points, incorrect. Thus, all cases enumerated above in Table 7, cases 1 and 2 of Table 3, cases 1, 2, 7–10, and 15 of Table 4, and cases 2, 12, 16, and 17 of Table 6 were overlooked, the symmetries of equations with nonlinearities given in cases 1 and 2 of Table 5 were presented incompletely, etc.

In [3–5], the Lie symmetries of the same equations and also of systems of diffusion equations with unit diffusion matrix were classified. The results obtained in [3–5] are much more advanced than the pioneer Davidov ones; nevertheless, they are still incomplete. In particular, the cases indicated above in cases 5 and 6 of Table 3, cases 12–14 of Table 4, the last line of case 1, case 9, and case 11 for a=1 of Table 5, cases 15 and 22 of Table 6, and case 1 for $\sigma = 0$ and $\mu \neq 0$ of Table 7 were not indicated in [5], which is in conflict with the statement of Theorem 1 formulated here. Moreover, many of equations presented in [5] as nonequivalent ones are, in fact, equivalent to one another even within the framework of the equivalence relations (7) of [3]. The related examples are not enumerated here because we believe that all nonequivalent equations (1.3) with different symmetries are present in Tables 1–9.

Except the points mentioned in the previous paragraph, our results concerning equations with diagonal diffusion matrix are in accordance with those obtained in [3–5].

Consider examples of well-known reaction-diffusion equations that appear to be particular subjects of our analysis.

- The Jackiw–Teitelboim model of two-dimensional gravity with nonrelativistic gauge [19]

$$\begin{aligned} &\frac{\partial}{\partial t}u_1 - \frac{\partial^2 u_1}{\partial x^2} - 2ku_1 + 2u_1^2u_2 = 0, \\ &\frac{\partial}{\partial t}u_2 + \frac{\partial^2 u_2}{\partial x^2} + 2ku_2 - 2u_1u_2^2 = 0 \end{aligned} \tag{12.1}$$

  admits the equivalence transformation 1 (2.10) for $\rho = -\omega$. Choosing $\rho = 2k$, we transform Eq. (12.1) to the form (1.2), where $a = -1$, $f^1 = -2u_1^2u_2$, and $f^2 = 2u_2^2u_1$. The symmetries corresponding to these nonlinearities are given in the first line of Table 5. Symmetries of Eqs. (12.1) were investigated in [16]. In accordance with our analysis, the generalized equation (12.1) with two spatial variables admits the additional conformal symmetry generated by operator K (4.5).

- The primitive predator–prey system can be defined by [20]

$$\dot{u}_1 - D\frac{\partial^2 u_1}{\partial x_2} = -u_1u_2, \qquad \dot{u}_2 - \lambda D\frac{\partial^2 u_2}{\partial x^2} = u_1u_2,$$

  and this is again a particular case of Eq. (1.2) with the nonlinearities given in the first line of Table 3, where, however, $\mu = \nu = 1$ and $F_1 = -F_2 = \frac{u_1}{u_2}$. In addition to the basic symmetries $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \rangle$, this equation admits the (main) symmetry

$$X = \left(D - 2u_1\frac{\partial}{\partial u_1} - 2u_2\frac{\partial}{\partial u_2}\right).$$

- The $\lambda - \omega$ reaction-diffusion system

$$\dot{u}_1 = D\Delta u_1 + \lambda(R)u_1 - \omega(R)u_2, \quad \dot{u}_2 = D\Delta u_2 + \omega(R)u_1 + \lambda(R)u_2, \tag{12.2}$$

where $R^2 = u_1^2 + u_2^2$, has symmetries that were analyzed in [1]. Again we recognize that this system is a particular case of (1.2) with nonlinearities given in case 6 of Table 4 with $\mu = \nu = 0$. Hence, it admits the five-dimensional Lie algebra generated by the main symmetries (2.2) with $\mu, \nu = 1, 2$ and

$$X = \left(u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1}\right), \tag{12.3}$$

which is in accordance with results of [1] for arbitrary functions λ and ω. Moreover, using Table 5 (case 8), we find that, in the case where

$$\lambda(R) = \tilde{\lambda}R^\nu, \quad \omega = \sigma R^\nu, \tag{12.4}$$

Eq. (12.2) admits the additional symmetry with respect to scaling transformations generated by the operator

$$X = \left(u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1}\right) + \nu D. \tag{12.5}$$

The other extensions of the basic symmetries correspond to the case where $\lambda(R) = \mu\ln(R)$ and $\omega(R) = \sigma\ln(R)$; the related additional symmetries are given in Table 8 for $\nu = \lambda = 0$.

- The nonlinear Schrödinger equation (NSE) in an m-dimensional space

$$\left(i\frac{\partial}{\partial t} - \Delta\right)\psi = F(\psi, \psi^*) \tag{12.6}$$

is also a particular case of (1.2). If we denote $\psi = u_1 + iu_2$ and $F = f_1 + if_2$, then (12.6) reduces to the form (1.3) with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In other words, any solution given in Tables 3–6 and 8 with matrices A belonging to Class II gives rise to NSE (9.4) that admits a main or an extended symmetry. Thus, our analysis makes it possible to present the completed group classification of the NSE as a particular case of the general study of systems of reaction-diffusion equations with arbitrary diffusion matrix. Our results are in complete agreement with those obtained in [22], where symmetries of the general NSE were described.

Among the solutions presented in Tables 3–6 and 8, we recognize those corresponding to the well-known nonlinearities [11]

$$F = F(\psi^*\psi)\psi, \quad F = (\psi^*\psi)^k\psi, \quad F = (\psi^*\psi)^{\frac{2}{m}}\psi, \quad F = \ln(\psi^*\psi)\psi.$$

One more interesting particular case of the NSE with extended symmetry can be found using case 1 of Table 6 for $\nu = 2$ and $m = 1$, namely,

$$\left(i\frac{\partial}{\partial t} - \Delta\right)\psi = (\psi - \psi^*)^2,$$

which is a potential equation for the Boussinesq equation for the function $V = \frac{\partial}{\partial t}(\psi - \psi^*)$.

- The generalized complex Ginzburg–Landau (CGL) equation

$$\frac{\partial W}{\partial \tau} - (1 + i\beta)\Delta W = F(W, W^*) \tag{12.7}$$

is a particular case of system (1.3) with a matrix A belonging to Class II with $a \neq 0$ (see (2.2)). Indeed, representing W and F as $W = (u_1 + iu_2)$ and $F = \beta(f^1 + if^2)$ and changing the independent variable $\tau \to t = \beta\tau$, we transform (12.7) to the form (1.3) with $A = \begin{pmatrix} \beta^{-1} & -1 \\ 1 & \beta^{-1} \end{pmatrix}$. All nonequivalent nonlinearities f^1 and f^2 and the corresponding symmetries are given in Table 3 (cases 1 and 3), Table 4 (cases 5, 6, and 15), Table 5 (cases 8 and 10), Table 6 (cases 1, 4, and 11), and Table 8. The ordinary CGL equation corresponds to the case $F = W - (1 + i\alpha)W|W|^2, m = 2$ and admits only the basic symmetries (5.1).

- The nonautonomous dynamical systems in a phase space [8]

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} - A(u_1, u_2) = h_1(t, x), \\ &\frac{\partial u_2}{\partial t} + \alpha\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} - \nu u_1 = h_2(t, x) \end{aligned} \tag{12.8}$$

are also equivalent to a system of the type (1.3) at least in the case of constant h_1 and h_2. The related matrix A belongs to Type III.

Using the results presented in Tables 3–6 and 9, we can specify all cases where the considered system admits main or extended symmetries.

We see that the class of equations that is classified in the present paper includes a number of important particular systems. Moreover, we present an *a priori* description of symmetries of all possible systems of two reaction-diffusion equations with general diffusion matrix.

Appendix

A.1. Algebras $\mathcal{A}$ for Equations (1.3) with Diagonal Diffusion Matrix

Let us consider Eq. (1.3) with diagonal matrix A (version I of (2.2) where $a \neq 0$) and find the related low-dimensional algebras $\mathcal{A}$. In this case, matrix (9.5) and the equivalence transformation matrix (9.7) reduce to the forms

$$g = \begin{pmatrix} 0 & 0 & 0 \\ B^1 & C^{11} & 0 \\ B^2 & 0 & C^{22} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^{11} & 0 \\ \mu^2 & 0 & \mu^{22} \end{pmatrix} \tag{A.1.1}$$

and

$$U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & K^1 & 0 \\ b^2 & 0 & K_2 \end{pmatrix}. \tag{A.1.2}$$

Up to the equivalence transformations (9.6) and (A.1.2), there exist three nonequivalent matrices (A.1.1), namely

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A.1.3}$$

In accordance with (9.1)–(9.4), the related symmetry operator can be represented in one of the following forms:

$$X_1 = \mu D - 2(g_a)_{bc}\tilde{u}_c\frac{\partial}{\partial u_b}, \quad X_2 = e^{\lambda t}(g_a)_{bc}\tilde{u}_c\frac{\partial}{\partial u_b} \tag{A.1.4}$$

or

$$X_3 = e^{\lambda t+\omega\cdot x}\left(\frac{\partial}{\partial u_2} + \mu\frac{\partial}{\partial u_1}\right). \tag{A.1.5}$$

Here, $(g_a)_{bc}$ are elements of matrices (A.1.3), $b, c = 0,\ 1,\ 2$, $\tilde{u} =$ column (u_0, u_1, u_2), and $u_0 = 1$.

Formulas (A.1.4) and (A.1.5) give the principal description of one-dimensional algebras $\mathcal{A}$ for Eq. (1.3) with a matrix A of the type I.

To describe two-dimensional algebras $\mathcal{A}$, we classify the matrices g (A.1.1) forming two-dimensional Lie algebras. Choosing one of the basis elements in the form given in (A.1.3) and the other element in the general form (A.1.1), we find that, up to the equivalence transformations (9.6), there exist six algebras $\langle e_1, e_2\rangle$:

$$A_{2,1} = \{\tilde{g}_1, g_4\}, \quad A_{2,2} = \{\tilde{g}_1, \tilde{g}_3\}, \quad A_{2,3} = \{g_5, \tilde{g}_3\}, \tag{A.1.6}$$

$$A_{2,4} = \{g_1, g_5\}, \quad A_{2,5} = \{g_1', g_3\}, \quad A_{2,6} = \{g_2, \tilde{g}_3\}, \tag{A.1.7}$$

where $\tilde{g}_1 = g_1|_{\lambda=0}$, $g_1' = g_1|_{\lambda=1}$, $\tilde{g}_3 = g_3|_{\lambda=0}$, and

$$g_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.1.8}$$

Algebras (A.1.6) are Abelian, whereas algebras (A.1.7) are characterized by the following commutation relations:

$$[e_1, e_2] = e_2, \tag{A.1.9}$$

where e_1 is the first element given in braces in (A.1.7), i.e., for $A_{2,4}$ $e_1 = g_1$, etc.

Using (A.1.6) and (A.1.7) and applying arguments analogous to those that follow Eqs. (6.2), we easily find pairs of operators (5.2) forming Lie algebras. Denoting

$$\hat{e}_\alpha = (e_\alpha)_{ab}\tilde{u}_b \frac{\partial}{\partial u_a}, \quad \alpha = 1, 2,$$

we represent them as follows:

$$\begin{array}{ll} \langle \mu D + \hat{e}_1 + \nu t\hat{e}_2, \hat{e}_2\rangle, & \langle \mu D + \hat{e}_2 + \nu t\hat{e}_1, \hat{e}_1\rangle, \\ \langle \mu D - \hat{e}_1, \nu D - \hat{e}_2\rangle, & \langle F_1\hat{e}_1 + G_1\hat{e}_2,\ F_2\hat{e}_1 + G_2\hat{e}_2\rangle \end{array} \tag{A.1.10}$$

for e_1 and e_2 belonging to algebras (A.1.6), and

$$\langle \mu D - \hat{e}_1, \hat{e}_2\rangle, \quad \langle \mu D + \hat{e}_1 + \nu t\hat{e}_2, \hat{e}_2\rangle \tag{A.1.11}$$

for e_1 and e_2 belonging to algebras (A.1.7).

Here, μ and ν are parameters that can take any (including zero) finite values, and $\{F_1, G_1\}$ and $\{F_2, G_2\}$ are fundamental solutions of the following system:

$$F_t = \lambda F + \nu G, \quad G_t = \sigma F + \gamma G, \tag{A.1.12}$$

where λ, ν, σ, and γ are arbitrary parameters.

List (A.1.10)–(A.1.11) does not include algebras spanned by the vectors $\langle F\hat{e},\ G\hat{e}\rangle$ (with F and G satisfying (A.1.12)) and $\langle \mu D + \lambda e^{\nu t+\omega\cdot x}\hat{e}, e^{\nu t+\omega\cdot x}\hat{e}\rangle$, which either are incompatible with the classifying equations (4.6) or reduce to one-dimensional algebras. In what follows, we ignore algebras $\mathcal{A}$ that include such subalgebras.

The other two-dimensional algebras $\mathcal{A}$ can be reduced to one of the forms given in (A.1.10) and (A.1.11) by using the equivalence transformations (2.4) and (6.7).

There exists one more type of (m+2)-dimensional algebras $\mathcal{A}$ generated by two-dimensional algebras (A.1.6), namely

$$\langle \mu D + \hat{e}_1 + (\alpha t + \lambda_{\sigma\rho}x_\sigma x_\rho)\hat{e}_2, x_\nu \hat{e}_2, \hat{e}_2\rangle,$$

where ν, σ, and ρ run from 1 to m. The related classifying equations generated by all symmetries $x_1\hat{e}_2, x_2\hat{e}_2, \cdots, x_m\hat{e}_2$, and $\hat{e}_2$ coincide, and we have the same number of constraints for f^1 and f^2 as in the case of two-dimensional algebras $\mathcal{A}$.

Up to equivalence, there exist three realizations of three-dimensional algebras in terms of matrices (A.1.3) and (A.1.8):

$$\begin{aligned} A_{3,1}: &\quad e_1 = \tilde{g}_1,\ e_2 = g_4,\ e_3 = \tilde{g}_3,\\ A_{3,2}: &\quad e_1 = g_5,\ e_2 = g_4,\ e_3 = \tilde{g}_3, \end{aligned} \tag{A.1.13}$$

$$A_{3,3}: \quad e_1 = g_1',\ e_2 = g_5,\ e_3 = \tilde{g}_3. \tag{A.1.14}$$

The nonzero commutators for matrices (A.1.13) and (A.1.14) are $[e_2, e_3] = e_3$ and $[e_1, e_\alpha] = e_\alpha (\alpha = 2, 3)$. The algebras of operators (5.2) corresponding to realizations (A.1.13) and (A.1.14) are of the following general forms:

$$\langle \mu D - 2\hat{e}_1,\ \nu D - 2\hat{e}_2 - 2\lambda t\hat{e}_3, \hat{e}_3\rangle \tag{A.1.15}$$

and

$$\begin{aligned} &\langle \mu D - 2\hat{e}_1 - 2\nu t\hat{e}_2 - 2\sigma t\hat{e}_3,\ \hat{e}_2,\ \hat{e}_3\rangle,\\ &\langle \hat{e}_1,\quad F_1\hat{e}_2 + G_1\hat{e}_3,\ F_2\hat{e}_2 + G_2\hat{e}_3\rangle, \end{aligned} \tag{A.1.16}$$

respectively.

In addition, we have the only four-dimensional algebra

$$\hat{A}_{4,1}: \quad e_1 = \tilde{g}_1,\ e_2 = g_5,\ e_3 = \tilde{g}_3,\ e_4 = g_4, \tag{A.1.17}$$

which generates the following algebras of operators (5.2):

$$\begin{aligned} &\langle \mu D - 2\hat{e}_1 - 2\nu t\hat{e}_2,\ \hat{e}_2, \hat{e}_3,\ \hat{e}_4\rangle,\\ &\langle \mu D - 2\hat{e}_1 - 2\nu t\hat{e}_3,\ \hat{e}_2,\ \hat{e}_3,\ \hat{e}_4\rangle,\\ &\langle \mu D - 2\hat{e}_1,\ \nu D - \hat{e}_4,\ \hat{e}_2,\ \hat{e}_3\rangle. \end{aligned} \tag{A.1.18}$$

Thus, we have specified all low-dimensional algebras $\mathcal{A}$ that can be admitted by Eqs. (1.3) with diagonal (but not unit) matrix A.

A.2. Algebras $\mathcal{A}$ for Equations (1.3) with $A^{12} \neq 0$

Consider Eq. (1.3) with a matrix A of the type II (see (2.3)) and find the corresponding algebras $\mathcal{A}$. The related matrices (9.5) and (9.7) are

$$g = \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^2 & \mu^3 \\ \mu^4 & -\mu^3 & \mu^5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & k^1 & k^2 \\ b^2 & -k^2 & k^3 \end{pmatrix}. \tag{A.2.1}$$

Up to the equivalence transformations (9.6) and (A.2.1), there exist three matrices g, namely

$$g_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; g_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; g_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & -1 \\ 0 & 1 & \mu \end{pmatrix}, \tag{A.2.2}$$

and three two-dimensional algebras of matrices g (A.2.1):

$$A_{2,7} = \{g_1', g_6\}, \;\; A_{2,8} = \{g_5, \tilde{g}_3\}, \tag{A.2.3}$$

$$A_{2,9} = \{g_1', g_5\}, \tag{A.2.4}$$

where $\tilde{g}_3$ is matrix (A.1.3) with $\lambda = 0$.

Algebras (A.2.3) are Abelian, while the basis elements of $A_{2,9}$ satisfy the commutation relations (A.1.9).

As in the previous subsection, we easily find the related basis elements of one-dimensional algebras $\mathcal{A}$ in the form (A.1.4) and (A.1.5) for $\mu = 0$.

The two-dimensional algebras $\mathcal{A}$ generated by (A.2.3) and (A.2.4) are again given by relations (A.1.10) and (A.1.11), respectively, where e_1 and e_2 are the first and second elements of the algebras $A_{2,7} - A_{2,9}$.

In addition, we have two three-dimensional algebras

$$\begin{aligned} &A_{3,3}: \quad e_1 = g_1', \; e_2 = g_5, \; e_3 = \tilde{g}_3, \\ &A_{3,4}: \quad e_1 = g_5, \; e_2 = g_6, \; e_3 = \tilde{g}_3 \end{aligned} \tag{A.2.5}$$

and the only four-dimensional algebra

$$A_{4,2}: \quad e_1 = g_1', \; e_2 = g_6, \; e_3 = \tilde{g}_3, \; e_4 = g_5. \tag{A.2.6}$$

The algebra $A_{3,4}$ generates algebras (A.1.16), while $A_{3,5}$ corresponds to (A.1.15) with $\nu = 0$. Finally, $A_{4,2}$ generates the following algebras $\mathcal{A}$:

$$\begin{aligned} &\langle \mu D - 2\hat{e}_1, \; \nu D - 2\hat{e}_2, \; \hat{e}_3, \; \hat{e}_4 \rangle, \\ &\langle \hat{e}_1, \; \hat{e}_2, \; e^{\mu t + \nu \cdot x}\hat{e}_3, \; e^{\mu t + \nu \cdot x}\hat{e}_4 \rangle. \end{aligned} \tag{A.2.7}$$

A.3. Algebras $\mathcal{A}$ for Equations (1.3) with Triangular Matrix A

If the matrix A belongs to the type III given in (2.3), then the related matrices (9.5) and (9.7) take the form

$$g = \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^2 & 0 \\ \mu^3 & \mu^4 & \mu^5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & k^1 & 0 \\ b^2 & k^2 & k^3 \end{pmatrix}. \tag{A.3.1}$$

There exist six nonequivalent matrices g, i.e., the matrices g_1', g_3, and g_5 given by (A.1.3) and (A.2.2) and the following matrices:

$$g_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_9 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{A.3.2}$$

In addition, we have six two-dimensional algebras

$$\begin{aligned} &A_{2,3} = \{g_5, \tilde{g}_3\}, \; A_{2,10} = \{g_1', g_8\}, \\ &A_{2,11} = \{g_8, \tilde{g}_3\}, \; A_{2,12} = \{g_9, \tilde{g}_3\}, \end{aligned} \tag{A.3.3}$$

$$A_{2,5} = \{g_1', g_3\}, \; A_{2,13} = \{g_1', g_5\}, \tag{A.3.4}$$

four three-dimensional algebras

$$\begin{aligned} A_{3,3} :& \quad e_1 = g_1', \; e_2 = g_5, \; e_3 = \tilde{g}_3, \\ A_{3,5} :& \quad e_1 = g_8, \; e_2 = g_1', \; e_3 = \tilde{g}_3, \\ A_{3,6} :& \quad e_1 = \tilde{g}_3, \; e_2 = g_8, \; e_3 = g_9, \\ A_{3,7} :& \quad e_1 = \tilde{g}_3, \; e_2 = g_5, \; e_3 = g_7 \end{aligned} \tag{A.3.5}$$

and the only four-dimensional algebra

$$A_{4,3} : \quad e_1 = \tilde{g}_3, \; e_2 = g_5, \; e_3 = g_1', \; e_4 = g_8. \tag{A.3.6}$$

Algebras (A.3.3) are Abelian, while (A.3.4) are characterized by the commutation relations (A.1.9). The related two-dimensional algebras $\mathcal{A}$ are given by formulas (A.1.10) and (A.1.11), respectively.

The algebra $A_{3,3}$ generates the three-dimensional algebras $\mathcal{A}$ enumerated in (A.1.16). The algebra $A_{3,5}$ is isomorphic to $A_{3,1}$, and so we come to the related algebras $\mathcal{A}$ given in (A.1.15). The algebras $A_{3,6}$ and $A_{3,7}$ are characterized by the following nonzero commutators:

$$[e_2, e_3] = e_1 \tag{A.3.7}$$

and

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_2 + e_3, \tag{A.3.8}$$

respectively.

Using (A.3.7) and (A.3.8), we come to the following related three-dimensional algebras $\mathcal{A}$:

$$\begin{gathered}\langle \mu D - 2\hat{e}_2,\ \nu D - 2\hat{e}_3,\ \hat{e}_1\rangle,\ \langle \hat{e}_1,\ D + 2e_\alpha + 2\nu t\hat{e}_1,\ \hat{e}_{\alpha'}\rangle,\\ \langle e^{\nu t+\omega\cdot x}\hat{e}_1,\ e^{\nu t+\omega\cdot x}\hat{e}_\alpha,\ \hat{e}_{\alpha'}\rangle,\end{gathered} \tag{A.3.9}$$

where $\alpha,\ \alpha' = 2, 3, \alpha' \neq \alpha$, and

$$\langle \mu D - 2\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3\rangle,\ \langle \hat{e}_1,\ e^{\nu t+\omega\cdot x}\hat{e}_2,\ \ e^{\nu t+\omega\cdot x}\hat{e}_3\rangle. \tag{A.3.10}$$

Finally, the four-dimensional algebras $\mathcal{A}$ corresponding to $A_{4,3}$ have the following general form:

$$\langle \mu D - 2\hat{e}_1,\ \nu D - 2\hat{e}_2,\ \hat{e}_3,\ \hat{e}_4\rangle,\ \ \langle e^{\nu t+\omega\cdot x}\hat{e}_1,\ e^{\nu t+\omega\cdot x}\hat{e}_2,\ \hat{e}_3,\ \hat{e}_4\rangle \tag{A.3.11}$$

A.4. Algebras $\mathcal{A}$ for Equations (1.3) with Unit Matrix A

The group classification of these equations appears to be the most complicated. The related matrices g are of the most general form (9.5) and are defined up to the general equivalence transformation (9.6), (9.7). In other words, there are seven nonequivalent matrices (9.5), namely g_1, g_2 (A.1.3), g_5 , g_6 (A.2.2) and $g_7 - g_9$ (A.3.2). In addition, we have 15 two-dimensional algebras of matrices (9.5)

$$\begin{gathered}A_{2,1} = \{\tilde{g}_1, g_4\},\ A_{2,2} = \{\tilde{g}_1, \tilde{g}_3\},\ A_{2,3} = \{\tilde{g}_3, g_5\},\\ A_{2,10} = \{g_7, g_8\},\ A_{2,11} = \{\tilde{g}_3, g_8\},\\ A_{2,12} = \{\tilde{g}_3, g_9\},\ A_{2,13} = \{g_1', g_6\},\end{gathered} \tag{A.4.1}$$

$$\begin{gathered}A_{2,4} = \{g_1, g_5\},\ A_{2,5} = \{g_1', g_3\},\ A_{2,6} = \{g_2, \tilde{g}_3\},\\ A_{2,14} = \{g_1|_{\lambda\neq 1}, g_8\},\ A_{2,15} = \{g_{11}, -g_8\},\ A_{2,16} = \{g_9, g''{}_1\},\\ A_{2,17} = \{g_4, g_8\},\ A_{2,18} = \{g_7, \tilde{g}_3\},\end{gathered} \tag{A.4.2}$$

where

$$g_{10} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},\ g_1'' = g_1|_{\lambda=2} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}.$$

Algebras (A.4.1) are Abelian, while algebras (A.4.2) are characterized by relations (A.1.9).

The three-dimensional algebras are $A_{3,1} - A_{3,7}$ given by relations (A.1.13), (A.2.5), and (A.3.3) (where tildes should be omitted) and also $A_{3,8} - A_{3,11}$ given below:

$$\begin{aligned}&A_{3,8}:\ e_1 = g_1,\ e_2 = g_8,\ e_3 = \tilde{g}_3,\\ &A_{3,9}:\ e_1 = g_4,\ e_2 = g_8,\ e_3 = \tilde{g}_3,\\ &A_{3,10}:\ e_1 = g_2,\ e_2 = g_8,\ e_3 = -\tilde{g}_3,\\ &A_{3,11}:\ e_1 = \tilde{g}_1,\ e_2 = -g_8,\ e_3 = \tilde{g}_4.\end{aligned}$$

The algebras $(A_{3,8}, A_{3,11})$, $A_{3,9}$, and $A_{3,10}$ are isomorphic to $A_{3,1}$, $A_{3,3}$, and $A_{3,6}$, respectively. The related algebras $\mathcal{A}$ are given by (A.1.15), (A.1.16), and (A.3.9), respectively.

Finally, the four-dimensional algebras of matrices (9.6) are $A_{4,1}$, $A_{4,2}$, and $A_{4,3}$ given by (A.1.17), (A.2.6), and (A.3.6), and also $A_{4,4}$ and $A_{4,5}$ given below:

$$\begin{aligned} A_{4,4}:&\ e_1 = g_1,\ e_2 = g_4,\ e_3 = g_8,\ e_4 = g_3, \\ A_{4,5}:&\ e_1 = g_4,\ e_2 = g_8,\ e_3 = g_5,\ e_4 = g_3. \end{aligned}$$

Using the found algebras and solving the related equations (4.6), we easily make the group classification of Eqs. (1.3).

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