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THE COMPLETE SET OF SYMMETRY OPERATORS OF THE SCHRÖDINGER EQUATION

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UDC 517.9:519.46

The complete set of symmetry operators of an arbitrary order associated with the Schrödinger equation is found. It is shown that this equation is invariant with respect to a 28-dimensional Lie algebra, realized in the class of differential operators of the second order. Higher-order symmetries of the Levi-Leblond equation are investigated.

1. Introduction

A description of symmetry operators of higher orders, associated with basic equations of mathematical physics, has become an increasingly important problem, since it is a necessary step in the investigation of coordinate systems, where equations admit solutions in separated variables [1]. As was shown quite recently [2, 3], the order of symmetry operators, generating such coordinate systems, can be arbitrarily large, and it can exceed the order of an equation. Therefore, the problem of a description of symmetry operators of an arbitrary order is of great importance. Note that in the 1970s integral differential symmetry operators of the Dirac equation were obtained, which can be interpreted as symmetries of an infinite order [4].

In papers [5-7] complete sets of symmetry operators, associated with the scalar wave equation and Diract equation are obtained, and their algebraic properties are studied. The present article continues the investigation of symmetries of higher orders of basic equations of mathematical physics and is devoted to a study of the Schrödinger equation and other equations of nonrelativistic quantum mechanics. Below a complete description of symmetry operators of an arbitrary order n, associated with this equation is given.

Institute of Mathematics, Academy of Sciences of the Ukraine, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 43, No. 11, pp. 1521-1526, November, 1991. Original article submitted June 28, 1991.

It is our pleasure to advise that our interest in symmetry problems related to equations of mathematical physics stems from discussions with O. S. Parayuk, who was a supervisor of our dissertation at Kiev University.

2. Generalized Killing's Tensors and Symmetry Operators of the Schrödinger Equation

Schrödinger equations for a complex scalar function $\Psi(x)$, $x = (x_0, x_1, x_2, x_3)$, $\Psi \supset L_2(R_4)$ can be written in the form

$$L\Psi = 0, \quad L = p_0 - \frac{p^2}{2m}, \tag{1}$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$, $p_0 = i\partial^0 = i(\partial/\partial x_0)$, $p_a = -i\partial^a = -i(\partial/\partial x_a)$.

It is known that Eq. (1) is invariant with respect to a 12-parametric Schrödinger group, whose generators have the form

$$\begin{gathered} P_0 = p_0, \quad P_a = p_a, \\ J_a = \varepsilon_{abc} x_b p_c, \quad G_a = x_0 p_a - m x_a, \\ D = 2x_0 p_0 - x_a p_a + \frac{3}{2} i, \\ A = x_0^2 p_0 - x_0 D - \frac{1}{2} m x^2. \end{gathered} \tag{2}$$

The only invariant symmetry with respect to algebra (2) is the maximal (in the sense of Lie) symmetry of the Schrödinger equation [8].

Definition. A linear differential operator of order n

$$Q^n = \sum_{j=0}^{n} [[\ldots [F^{a_1 a_2 \ldots a_j}, p_{a_1}]_+, p_{a_2}]_+, \cdots]_+, p_{a_j}]_+, \tag{3}$$

where $[A, B]_+ = AB + BA$, $F^{a_1 \ldots}$ are arbitrary functions of x, and $a_\nu = 1, 2, 3$, $\nu = 1, 2, \ldots, j$, is called a symmetry operator of the Schrödinger equation of order n, if

$$[Q; L]\,\Psi = 0 \tag{4}$$

for every Ψ, satisfying (1).

Operator (3) does not include the differentiation with respect to x_0, which can be replaced by differentiation with respect to the spacial coordinates on the solution set of Eq. (1).

Substituting (3) and (1) into (4), and comparing coefficients of linearly independent differentiation operators, we derive the following equations for coefficients of symmetry operators:

$$\partial^{(a_{j+1}} F^{a_1 a_2 \ldots a_j)} = -2m \dot{F}^{a_1 a_2 \ldots a_{j+1}}, \quad j = 0, 1, \ldots, n-1, \tag{5}$$

$$\partial^{(a_{n+1}} F^{a_1 a_2 \ldots a_n)} = 0; \qquad \dot{F} = 0, \quad j = 0, \tag{6}$$

where the dot denotes the differentiation with respect to x_0, and the symmetrization with respect to indices is marked by putting the indices in parentheses.

The system of equations (5), (6) can be integrated immediately for an arbitrary n, using results from [5, 6]. For this purpose, we first consider differential consequences of these equations. Differentiating (5) for $j = n - 1$, with respect to x_{n+1}, and using (6) we obtain

$$\partial^{(a_{n+1}} \partial^{a_n} F^{a_1 a_2 \ldots a_{n-1})} = 0. \tag{7}$$

Then, differentiating (5) for $j = n - 2$, with respect to x_n and x_{n+1}, and using (7), we obtain the equation

$$\partial^{(a_{n+1}}\partial^{a_n}\partial^{a_{n-1}}F^{a_1a_2\cdots a_{n-2})}=0.$$

Repeating this procedure, we obtain an equation for an arbitrary $j = n - s + 1$:

$$\partial^{(a_{j+1}}\partial^{a_{j+2}}\ldots\partial^{a_{j+s}}F^{a_1a_2\cdots a_j)}=0,\quad s=n-j+1. \tag{8}$$

From (5) and (6) it also follows that

$$\frac{\partial^{j+1}}{(\partial x_0)^{j+1}}F^{a_1a_2\cdots a_j}=0. \tag{9}$$

Formula (8) gives a system of overlapping equations whose general solution was obtained in [5, 6]. A symmetric tensor $F^{a_1\cdots}$, satisfying Eqs. (8), has been called a generalized Killing's tensor of range j and order s. This tensor is a polynomial of the order $j + s - 1 = n$ (its explicit form was given in [5]), and involves N^{js} arbitrary parameters, where [6]

$$N^{js}=\frac{s}{12}(j+1)(j+2)(j+1+s)(j+2+s). \tag{10}$$

A general solution of Eqs. (8) and (9) can be represented in the form

$$F^{a_1a_2\cdots a_j}=\sum_{\alpha=0}^{j}F_{s\alpha}^{a_1a_2\cdots a_j}x_0^{\alpha},\quad s=n+1-j, \tag{11}$$

where $F_{s\alpha}^{a_1}$ are arbitrary Killing's tensors of order s. Substituting (11) into the initial equations (5) and (6), we derive relations

$$\alpha F_{s\alpha}^{a_1a_2\cdots a_j}=-2m\partial^{(a_j}F_{s+1\,\alpha-1}^{a_1a_2\cdots a_{j-1})},\quad \alpha\neq 0. \tag{12}$$

Thus, the problem of description of symmetry operators of higher orders, allowed by the Schrödinger equation, reduces to a derivation of explicit forms of generalized Killing's tensors of range j and order $n + 1 - j$, satisfying complementary conditions (12). Using results from [5, 6], Eqs. (12) can be reduced to algebraic equations for coefficients of tensors $F_{s\alpha}^{a_1\cdots}$, which are easily solvable. Here we will confine ourselves to the enumeration of the number of independent solutions and their presentation in an explicit form.

According to (12), tensors $F_{s\alpha}^{a_1a_2\cdots a_j}$ with $\alpha \neq 0$ are uniquely expressed by $F_{s+1\alpha-1}^{a_1a_2\cdots a_{j-1}}$, while $F_{s0}^{a_1a_2\cdots a_j}$ is an arbitrary generalized Killing's tensor, subject to no restriction pertained to Eq. (12) (cf., e.g., [5, p. 38], Lemma 4). This means that the number of linearly independent solutions N_n of system (5), (6) is equal to the number of independent parameters, determining $F_{s0}^{a_1a_2\cdots a_j}$ for all $j \leq n$. According to (10),

$$N_n=\sum_{j=0}^{n}N^{js}=\frac{1}{3!\,4!}(n+1)(n+2)^2(n+3)^2(n+4).$$

Thus, the Schrödinger equation allows N_n linearly independent symmetry operators of order $j \leq n$. Excluding symmetry operators of order $j' \leq n - 1$, we obtain the number $\tilde{N}_n$ of symmetry operators of order n

$$\tilde{N}_n=N_n-N_{n-1}=\frac{1}{4!}(n+1)(n+2)^2(n+3). \tag{13}$$

Explicit expressions for the corresponding symmetry operators can be chosen in the form

$$Q^n=\sum_{c=0}^{n}\sum_{k=0}^{n-c}\lambda^{a_1a_2\cdots a_cb_1b_2\cdots b_{n-c}}P_{a_1}P_{a_2}\ldots P_{a_k}G_{a_{k+1}}G_{a_{k+2}}\ldots G_{a_c}J_{b_1}J_{b_2}\ldots J_{b_{n-c}}, \tag{14}$$

where P_a, G_a, and J_b are generators (2), $\lambda^{a_1\cdots}$ are arbitrary tensors that are symmetric with respect to transpositions of indices $a_i \leftrightarrow a_j$ and $b_k \leftrightarrow b_m$, and satisfy conditions $\lambda^{a_1a_2\cdots a_cb_1b_2\cdots b_{n-c}}\delta_{a_1{}'_1}=0$. Indeed, all parts in (14) are linearly independent, and the number of independent components of all arbitrary tensors $\lambda^{a_1\cdots}$ coincides with $\tilde{N}_n$ (13). We see that all symmetry operators of a finite order, associated with the Schrödinger equation, belong to an algebra spanned by generators (2).

We will formulate the obtained result.

THEOREM. The Schrödinger equation allows $\tilde{N}_n$ symmetry operators of order n. Their explicit form is given in (14), and the form of $\tilde{N}_n$ in (13).

Generalized Killing's tensors have played the key role in the solution of the considered problem.

3. Algebraic Properties of Symmetry Operators

We will investigate the algebraic structure of symmetry operators of the Schrödinger equation. We will confine ourselves to symmetry operators of the second order which, according to (14), are exhausted by the following members:

$$P_{ab} = P_a P_b, \quad G_{ab} = G_a G_b, \quad Q_{ab} = \frac{1}{2}(P_a G_b + P_b G_a), \tag{15}$$

$$\begin{gathered} F_{ab} = J_a J_b + J_b J_a, \quad F_a = \varepsilon_{abc} P_b J_c, \quad \Gamma_a = \varepsilon_{abc} G_b J_c, \\ L_{ab} = P_a J_b + P_b J_a, \quad N_{ab} = G_a J_b + G_b J_a \end{gathered} \tag{16}$$

(tensors L_{ab} and N_{ab} have the null trace).

Operators (15) and (16), as opposed to (2), are not lie algebras, but include subsets that are such algebras. Namely: operators (15) form a 28-dimensional Lie algebra, together with operators P_a, J_a, and G_a (2) and the unit operator I [P_0, A, and D reduce to traces of tensors (15) on the solution set of Eq. (1)]. Indeed, by a direct computation we obtain the following commutative relations:

$$\begin{gathered}
[P_a, P_b] = [P_a, P_0] = [P_0, J_a] = [G_a, G_b] = 0, \\
[P_a, J_b] = i\varepsilon_{abc} P_c, \quad [G_a, J_b] = i\varepsilon_{abc} G_c, \\
[P_0, G_a] = iP_a, \quad [P_a, G_b] = i\delta_{ab} mI, \quad [J_a, J_b] = i\varepsilon_{abc} J_c, \\
[P_{ab}, P_{cd}] = [G_{ab}, G_{cd}] = 0, \\
[P_{ab}, G_{cd}] = im(\delta_{ac} Q_{bd} + \delta_{bd} Q_{ac} + \delta_{ad} Q_{bc} + \delta_{bc} Q_{ad}), \\
[P_{ab}, Q_{cd}] = im(\delta_{ac} P_{bd} + \delta_{bd} P_{ac} + \delta_{ad} P_{bc} + \delta_{bc} P_{ad}), \\
[Q_{ab}, G_{cd}] = -im(\delta_{ac} G_{bd} + \delta_{bd} G_{ac} + \delta_{ad} G_{bc} + \delta_{bc} G_{ad}), \\
[Q_{ab}, Q_{cd}] = im(\delta_{ac} Q_{bd} + \delta_{bd} Q_{ac} - \delta_{ad} Q_{bc} - \delta_{bc} Q_{ad}), \\
[P_a, P_{bd}] = 0, \quad [P_a, Q_{bd}] = im(\delta_{ab} P_d + \delta_{ad} P_a), \\
[P_a, G_{bd}] = im(\delta_{ab} G_d + \delta_{ad} G_b), \\
[J_a, R_{bd}] = i(\varepsilon_{abk} R_{kd} + \varepsilon_{adk} R_{kb}), \quad R_{bd} = (Q_{bd}, G_{bd}, P_{bd}),
\end{gathered}$$

defining a 28-dimensional Lie algebra A_{28}. This algebra includes the subalgebra ASchr(1, 3) (the Lie algebra of the Schrödinger group), which contains operators P_{nn}, I, P_a, G_a, J_a, G_{nn}^2, and Q_{nn}, and also the subalgebra AO(1, 2) $\subset P_{nn}, G_{nn}, Q_{nn}$; AIGL(3) $\subset P_a, J_a, Q_{ab}$, and AP(2, 1) $\subset P_a, J_1, Q_{12}, Q_{13}$.

We see that symmetry operators of the second order have quite nontrivial algebraic structure, which can be used in a construction of groups of implicit symmetries of Eq. (1), in generating nonequivalent sets of symmetry operators, corresponding to coordinate systems, where there exist solutions in separated variables, etc.

4. Symmetry Operators of the Levi-Leblond Equation

The Levi-Leblond equation, describing a nonrelativistic particle with the spin 1/2, has the form [9]

$$L\Psi \equiv \left[\frac{1}{2}(1+\gamma_0) p_0 + (1-\gamma_0) m - \gamma_a p_a\right]\Psi = 0, \tag{17}$$

where γ_0 and γ_a are Dirac matrices, and Ψ is a four-component wave function. Equation (17) is invariant with respect to the Schrödinger group, whose generators have the form

$$P_0 = p_0, \quad P_a = p_a, \quad \hat{J}_a = J_a + S_a, \quad \hat{G}_a = G_a + \eta_a, \tag{18}$$

where J_a and G_a are operators (2),

$$S_a = \frac{i}{4}\varepsilon_{abc}\gamma_b\gamma_c \quad \eta_a = \frac{1}{2}(1-\gamma_0)\gamma_a.$$

In order to describe symmetry operators of higher orders, associated with Eq. (17), we will transform it to the following equivalent form:

$$L'\Psi' = 0, \tag{19}$$

where

$$L' = ULU^{-1}, \quad \Psi' = U\Psi, \tag{20}$$

$$U = 1 - \frac{i}{m}\eta_a p_a, \quad U^{-1} = 1 + \frac{i}{m}\eta_a p_a,$$
$$L' = \frac{1}{2}(1+\gamma_0)\left(p_0 - \frac{p^2}{2m}\right) + (1-\gamma_0)m. \tag{21}$$

Equation (19), as opposed to (17), includes only one γ-matrix, which simplifies the investigation of its symmetries. Choosing a diagonal γ_0, we infer that ψ' has only two non-null components, satisfying the Schrödinger equation.

We can decompose symmetry operators of Eq. (19) with respect to complete set of matrices $\{S_a, I\}$:

$$Q = S_a Q_a + IQ_0, \quad [Q_\mu, \gamma_\nu] = 0, \quad \mu, \nu = 0, 1, 2, 3.$$

Then the operators Q_μ should be symmetry operators of the Schrödinger equation and, hence, have form (14). This means that all symmetry operators of Eq. (19) belong to an algebra spanned by the subalgebra ASchr(1, 3). Indeed, generators (18) under transformation (20) take the form

$$P'_\mu = UP_\mu U^{-1} = P_\mu, \quad \hat{J}'_a = U\hat{J}_a U^{-1} = J_a + S_a, \quad \hat{G}'_a = UG_a U^{-1} = G_a,$$

where P_μ, J_a, and G_a are generators (2), and matrices S_a on the solution set of Eq. (19) are expressible by generators (2)

$$S_a\Psi' = \frac{1}{m}(m\hat{J}'_a - \varepsilon_{abc}P'_b\hat{G}'_c)\Psi'.$$

Since Eqs. (17) and (18) are related by the inverse relation, it follows that all symmetry operators of an arbitrary finite order are polynomials of generators (18).

An analogous assertion can be proved for equations proposed in [10, 11], which describe a Galilean particle with an arbitrary spin s. The methods developed here and in [5-7] allow one to find complete sets of symmetry operators also for the Schrödinger equation with a non-null potential, for example for the harmonic oscillator potential and other potentials, found in [12].

Integral symmetry operators of the Schrödinger equation are considered in [13].

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GEOMETRY IN NONLINEAR QUANTUMLIKE MODELS ON STIEFEL MANIFOLDS AND BIFURCATIONS OF ASSOCIATED AUTONOMOUS SYSTEMS

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UDC 530.145;530.182

Based on the geometric characteristics of Stiefel manifolds $V_{N,k} = SO(N)/SO(N - k)$ that have been previously found, two-loop β functions (a matrix β function, and a pair of scalar functions) of the renormalized group and a dynamic system that together describe the renormalization group evolution of effective interaction in nonlinear σ-models on such manifolds are obtained. It is shown that for definite values of the parameter bifurcations of saddle-node type equilibrium positions are observed in this dynamic system.

1. Analysis of dynamic equations (systems) associated with motion equations in classical non-Abelian gauge fields has demonstrated their nontrivial behavior in phase space. It is known that if static sources of sufficient strength are present, this will lead to a solution that undergoes bifurcation [1]. The motion of non-Abelian fields is stochastic [2, 3]; however, the stochastic feature may be eliminated if a Higgs field is added, i.e., under the condition that the parameter that characterizes the "Yang—Mills—Higgs" system attains its critical value and that there is a transition [4] from random to regular behavior. Whether Yang—Mills—Higgs quantum fields possess corresponding properties remains an open question. (Specific features that are found in a quantum-mechanical analog of the Matinyan—Savvidi system have been considered in [5].)

Two-dimensional (d = 2) nonlinear σ-models have been interpreted as, in some sense, analogs of non-Abelian gauge (d = 4) gauge theories, basically due to the existence of, in certain cases, localized (instanton) classical solutions [6-8] and, in the quantum treatment, due to the presence of the property of asymptotic freedom (a tendency for the effective coupling constant to vanish at short distances) [9-11]. Because of this analogy, we are justified in asking whether any of the nonlinear σ-models exhibit properties that are to some extent analogous to the results found in [1-4]. In the present article it will be shown that the phenomenon of bifurcation of stationary solutions in nonlinear σ-models in fact occurs if we consider, instead of the classical field motion equations, the autonomous dynamic system associated with the renormalization group behavior of effective interaction, more pre-

Institute of Theoretical Physics, Academy of Sciences of the Ukraine, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 43, No. 11, pp. 1527-1537, November, 1991. Original article submitted June 28, 1991.

 0041-5995/91/4311-1418$12.50