Supersymmetries of the Dirac equation for a charged particle interacting with electric field

A.G. Nikitin*

Institute of Mathematics of the Academy of Sciences of Ukraine,

Tereshchenkivska Street 3, 01601 Kyiv-4, Ukraine

Abstract

It is shown that the Dirac equation for a charged particle interacting with an external electric field admits extended supersymmetry provided the related potential has well definite parities. In particular, $N = 4$ and $N = 6$ SUSY for the relativistic Hydrogen atom is indicated.

1. Introduction. First introduced in particle physics [1], SUSY plays more and more essential role in quantum mechanics, refer e.g. to survey [2]. Moreover, some of important quantum mechanical problems (such as an interaction of an electron with constant and homogeneous magnetic or Coulomb fields) admit exact SUSY [3, 4].

It was pointed out long time ago [5], that the Dirac and Schrödinger–Pauli equations for an electron interacting with a time-independent magnetic field are supersymmetric, provided the related vector-potential has a definite parity w.r.t simultaneous reflection of all spatial coordinates.

Recently, generalizing this idea of paper [5], the *extended* $N = 3$, $N = 4$ and $N = 6$ SUSY for an electron in three-dimensional magnetic field was found [6]–[9]. A sufficient condition of existence of such a symmetry is that the vector-potential has definite parities w.r.t. reflection of *any* spatial variable. These results establish deep connections between supersymmetries and discrete involutive symmetries and stimulate systematic search for discrete symmetries of the Dirac and Schrödinger–Pauli equations [7, 9, 11].

In the present paper we prove existence of $N = 3$, $N = 4$ and $N = 6$ SUSY for the Dirac equation for an electron interacting with the electric field. We also indicate symmetries of this equation w.r.t. algebras of discrete transformations which appear to be rather extended. In particular we prove the symmetry of the related Columb problem w.r.t. the algebra $gl(8, R)$.

2. Two forms of the Dirac equation. Consider the stationary Dirac equation for a particle interacting with a time independent electric field

$$L\Psi \equiv (\varepsilon - \gamma_0\gamma_a p_a - \gamma_0 m - eA_0)\,\Psi = 0, \tag{1}$$

*e-mail: nikitin@imath.kiev.ua

where $p_a = -i\frac{\partial}{\partial x_a}$, $a = 1,2,3$, ε is the Hamiltonian eigenvalue, $A_0 = A_0(\mathbf{x})$ is a potential of electric field, γ_μ $(\mu = 0,1,2,3)$ are Dirac matrices (we choose $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$ diagonal).

To search for SUSY of (1) it is convenient to transform this equation to the following equivalent form

$$\left(\varepsilon^2 - 2eA_0\varepsilon + e^2A_0^2 - p_ap_a - m^2 - 2ieS_aE_a\right)\hat{\Phi} = 0, \tag{2}$$

$$(1 + i\gamma_4)\,\hat{\Phi} = 0, \tag{3}$$

where $S_a = \frac{i}{4}\varepsilon_{abc}\gamma_b\gamma_c$, $E_a = -i\frac{\partial A_0}{\partial x_a}$. The corresponding transformation can be represented as [12]

$$\begin{aligned}
&\hat{\Phi} =\!\!\rightarrow V^+\Psi, \quad \Psi = V^-\hat{\Phi}, \quad L \rightarrow V^+\gamma_0 L V^-,\\
&V^\pm = 1 \pm \frac{1}{m}(1 + i\gamma_4)\,(\gamma_0 L - m)\,.
\end{aligned} \tag{4}$$

In accordance with (3) function $\hat{\Phi}$ has only two non-zero components which we denote by Φ. Moreover, equation (2) reduce to the form

$$\left(\varepsilon^2 - m^2\right)\Phi = \left(\mathbf{p}^2 + ie\sigma_aE_a - e^2A_0^2 + 2e\varepsilon A_0\right)\Phi, \tag{5}$$

where σ_a are the Pauli matrices.

The system of two second-order equations (5) is mathematically equivalent to the system of four first order equations given by relations (1). Thus there exist one-to-one correspondence between symmetries of equations (1) and (5). Nevertheless, equation (5) is much more convenient for symmetry analysis then (1) because of reduction of the number and dimension of the involved matrices.

Let us suppose that A_0 depends on some parameters $a = (a_1, a_2, \ldots)$, and is a homogeneous function of $\mathbf{x}$ and a:

$$A_0(k\mathbf{x}, ka) = \frac{1}{k}A_0(\mathbf{x}, a). \tag{6}$$

A familiar example of such a potential is the potential generated by a system of point charges, i.e., $A_0 = \Sigma\frac{g_i e}{\|\mathbf{x} - \mathbf{a_i}\|}$, where $\mathbf{a_i}$ are charges coordinates.

Choosing new variables $\mathbf{r} = \mathbf{x}\varepsilon$ and $b = a\varepsilon$ we reduce (5) to the form

$$\lambda\Phi = H\Phi, \quad H = -p'_ap'_a + ie\sigma_aE'_a + \left(1 - eA_0(\mathbf{r}, b)\right)^2, \tag{7}$$

where $p'_a = -i\frac{\partial}{\partial r_a}$, $E'_a = -\frac{\partial A_0}{\partial r_a}$, $\lambda = \frac{m^2}{\varepsilon^2}$, $\mathbf{r} = (r_1, r_2, r_3)$.

We say equation (7) admits $N = n$ SUSY, if there exist n constants of motion Q_A which commute with "Hamiltonian" H and satisfy the following relations:

$$Q_AQ_B + Q_BQ_A = 2g_{AB}H, \quad [Q_A, H] = 0, \quad A, B = 1, 2, \ldots, n. \tag{8}$$

If $g_{AB} = \delta_{AB}$ (δ_{AB} is the Kronecker symbol), then relations (8) define superalgebra characterizing SUSY quantum mechanics with n supercharges [2]. We will consider also a more general case when the diagonal elements of the tensor g_{AB} are equal either to $+1$ or to -1 (and $g_{AB} = 0$ for $A \neq B$).

Discrete symmetries and supercharges. In addition to (6), we suppose that $A_0(\mathbf{r})$ is an even function w.r.t. reflections of space variables. Let us consider consequently all possible combinations of such parities.

Let

$$A_0(-\mathbf{r}) = A_0(\mathbf{r}), \tag{9}$$

then equation (7) is invariant w.r.t. the space reflection transformation $\Phi(\mathbf{r}) \to R\Phi(\mathbf{r}) = \Phi(-\mathbf{r})$. In addition, we can construct a symmetry operator (supercharge) Q:

$$Q = Rq, \quad q = \sigma_a p'_a - 1 + eA_0 \tag{10}$$

which satisfies the condition $Q^2 = H$ and generates $N = 1$ SUSY for equation (7).

Analogously, equation (7) admits N=1 SUSY provided A_0 is an even function w.r.t. reflection of one of co-ordinate axis, say

$$A_0(\hat{r}_1\mathbf{r}) = A_0(\mathbf{r}), \quad \hat{r}_1\mathbf{r} = (-r_1, r_2, r_3). \tag{11}$$

The corresponding supercharge is $Q = R_1 q$, where operator R_1 is defined as follows: $R_1\Phi(\mathbf{r}) = \sigma_1\Phi(\hat{r}_1\mathbf{r})$.

If A_0 is an even function w.r.t. reflections of two given coordinate axes, say

$$A_0(\hat{r}_1\mathbf{r}) = A_0(\mathbf{r}), \quad A_0(\hat{r}_2\mathbf{r}) = A_0(\mathbf{r}), \quad \hat{r}_2\mathbf{r} = (r_1, -r_2, r_3) \tag{12}$$

then there exist two supercharges for equation (7), namely

$$Q_1 = R_1 q, \quad Q_2 = iR_2 q. \tag{13}$$

Operators (13) satisfy relations (8) for $g_{11} = -g_{22} = 1$.

Finally, if A_0 is an even function w.r.t. reflection of any co-ordinate axis, i.e.,

$$A_0(r_a\mathbf{r}) = A_0(\mathbf{r}), \quad a = 1, 2, 3, \tag{14}$$

then equation (7) admits $N = 3$ SUSY generated by following supercharges

$$Q_1 = R_1 q, \quad Q_2 = R_2 q, \quad Q_3 = R_3 q. \tag{15}$$

We notice that all supercharges introduced in the above are Hermitian w.r.t. the following indefinite metrics

$$(\Phi_1, \Phi_2) = \int d^3x \Phi_1 \hat{R} \Phi_2, \tag{16}$$

where $\hat{R} = R$ for the case when A_0 satisfies (9) and $\hat{R} = R_1$ for the case when parity properties of A_0 are defined by relations (11), (12) and (14).

In all considered cases equation (7) is invariant w.r.t. the following "antiunitary" [13] transformation

$$\Phi(\mathbf{r}) \to C\Phi(\mathbf{r}) = i\sigma_2\Phi^*(\mathbf{r}) \tag{17}$$

where the asterisk denotes the complex conjugation. Using this symmetry and taking into account the relations

$$\{R_a, \sigma_a p'_a\} = 0, \quad [C, \sigma_a p'_a] = 0, \quad \{R_a, C\} = 0, \quad R_1^2 = -C^2 = 1,$$

it is possible to construct additional supercharges and obtain the following bases of superalgebra (8)

$$Q_1 = iRq, \quad Q_2 = CQ_1 \quad (g_{11} = g_{22} = -1), \tag{18}$$

$$Q_1 = R_1 q, \quad Q_2 = CQ_1 \quad (g_{11} = -g_{22} = 1), \tag{19}$$

$$\begin{array}{l} Q_1 = R_1 q, \quad Q_2 = iR_2 q, \quad Q_3 = CR_1 q \\ (g_{11} = -g_{22} = -g_{33} = 1) \end{array} \tag{20}$$

and

$$\begin{array}{l} Q_1 = CR_1 q, \quad Q_2 = iR_2 q, \quad Q_3 = CR_3 q, \quad Q_4 = CRq \\ (g_{11} = -g_{22} = g_{33} = -g_{44} = 1) \end{array} \tag{21}$$

for the cases (9), (11), (12) and (14) correspondingly.

We see that extended SUSY is admitted by a number of problems describing interaction of spin 1/2 particle with an electric field, provided the corresponding potentials have definite parities. Let us present simple examples of such potentials:

$$A_0 = \frac{ge}{\|\mathbf{x}\|}, \tag{22}$$

$$A_0 = \frac{ge}{\|\mathbf{x}+\mathbf{a}\|} - \frac{ge}{\|\mathbf{x}-\mathbf{a}\|}, \tag{23}$$

$$A_0 = \frac{ge}{\|\mathbf{x}+\mathbf{a}\|} - \frac{ge}{\|\mathbf{x}-\mathbf{a}\|} + \frac{ge}{\|\mathbf{x}+\mathbf{b}\|} - \frac{ge}{\|\mathbf{x}-\mathbf{b}\|}, \tag{24}$$

$$\begin{aligned} A_0 = {} & \frac{ge}{\|\mathbf{x}+\mathbf{a}\|} - \frac{ge}{\|\mathbf{x}-\mathbf{a}\|} + \frac{ge}{\|\mathbf{x}+\mathbf{b}\|} - \\ & - \frac{ge}{\|\mathbf{x}+\mathbf{b}\|} + \frac{ge}{\|\mathbf{x}+\mathbf{c}\|} - \frac{ge}{\|\mathbf{x}+\mathbf{c}\|}. \end{aligned} \tag{25}$$

Here $\mathbf{a} = (a, 0, 0)$, $\mathbf{b} = (a, b, 0)$, $\mathbf{c} = (a, b, c)$, $a \neq b$, $b \neq c$, $c \neq a$.

Relations (22), (23), (24) and (25) define potentials of a point charge, of electric dipole, of two and three parallel dipoles correspondingly (the three last examples correspond to elementary units of the crystal of NaCL). These potentials have parities defined by relations (14), (12), (11) and (9) respectively.

Extended SUSY for the hydrogen atom. Here we show that for the case of the Coulomb potential (22) equation (7) admits more extended, $N = 6$ SUSY. This extension is caused by existence of the Johnson–Lippman [14] constant of motion for the Dirac equation and additional symmetry operators

$$J^2 = J_a J_a, \quad D = \sigma_a J_a - 1/2, \tag{26}$$

(where $J_a = \varepsilon_{abc} r_b {p'}_c + \sigma_a/2$) for the corresponding equation (7).

Let us suppose that Φ is an eigenfunction of symmetry operators J^2 and D with eigenvalues $j(j+1)$ and $\pm\kappa = \pm(j+1/2)$ correspondingly, and rewrite equation (7), (22) in the form

$$\mu\Phi = \hat{H}\Phi, \tag{27}$$

where

$$\begin{aligned} \hat{H} &= p'^2 + i\alpha\frac{\sigma_a r_a}{r^3} - \left(\frac{\alpha}{r} - 1\right)^2 + \frac{1}{b^2}\left(\sigma_a J_a - \frac{1}{2}\right)^2, \\ \mu &= \left(\frac{\kappa^2}{b^2} - \frac{m^2}{\varepsilon^2}\right), \quad \alpha = ge^2, \quad b^2 = \kappa^2 - \alpha^2. \end{aligned} \tag{28}$$

Using the relations

$$\{D, \sigma_a p'_a\} = \{D, \sigma_a r_a\} = 0, \quad \left[\sigma_a p'_a, \frac{\sigma_b r_b}{r}\right] = -\frac{2i}{r}D, \tag{29}$$

it is not difficult to verify that the operator

$$Q = \frac{i}{\kappa}D\left(\sigma_a p'_a + \frac{\alpha}{\rho} + \frac{\alpha^2}{b^2}\right) + \frac{\alpha\kappa}{b^2}\frac{\sigma_a r_a}{r} \tag{30}$$

is a supercharge for "Hamiltonian" $\hat{H}$, satisfying the relation $Q^2 = \hat{H}$.

To find additional supercharges we use (29) and the following relations

$$\begin{aligned} &[R_{ab}, Q] = [R_{ab}, \Sigma] = [C, \Sigma] = \{C, Q\} = \{\Sigma, Q\} = 0, \\ &\Sigma^2 = R_{ab}^2 = 1, \end{aligned} \tag{31}$$

where $\Sigma = \frac{1}{b}\left(D - i\alpha\frac{\rho_a\sigma_a}{\rho}\right)$, $R_{ab} = iR_aR_b$.

Products of Q with Σ or R_{ab} are supercharges too, moreover, there exist exactly four of them:

$$Q_1 = R_{23}Q, \quad Q_2 = R_{31}Q, \quad Q_3 = R_{12}Q, \quad Q_4 = i\Sigma Q. \tag{32}$$

Operators (32) commute with $\hat{H}$ and satisfy relations (8) where $g_{11} = g_{22} = g_{33} = -g_{44} = 1$. They are Hermitian w.r.t. the following scalar product

$$(\Phi_1, \Phi_2) = \int d^3x \Phi_1^+ M \Psi_2, \tag{33}$$

where $M = J_aJ_a + \frac{1}{4} + i\alpha\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r}D$ is a positive defined metric operator (we suppose $\alpha \ll 1$).

A more extended set of supercharges can be obtained using antiunitary symmetry (17). It includes six operators

$$\begin{aligned} &Q_1 = Q, \quad Q_2 = i\Sigma Q, \quad Q_3 = CQ, \\ &Q_4 = CR_{12}Q, \quad Q_5 = CR_{31}Q, \quad Q_6 = CR_{12}Q \end{aligned} \tag{34}$$

which satisfy relations (8) with $H = \hat{H}$, $g_{11} = g_{22} = g_{33} = -g_{44} = -q_{55} = -g_{66} = 1$.

Using explicit solutions for the Dirac equation with the Coulomb potential (refer e.g. to ref. [12]), it is possible to show that the found SUSY is exact in as much as the ground state of the system (27) is not degenerated.

With a help of transformations (4) it is possible to find symmetry operators (which correspond to supercharges (34)) for the initial Dirac equation. In this way we obtain the following operators which satisfy superalgebra (8) and are defined on solutions of the Dirac equation:

$$\begin{aligned} &Q_1 = \hat{Q}, \quad Q_2 = i\hat{R}_1\hat{R}_2\hat{R}_3\hat{Q}, \quad Q_3 = \hat{C}\hat{Q}, \\ &Q_4 = i\hat{C}\hat{R}_1\hat{R}_2\hat{Q}, \quad Q_5 = i\hat{C}\hat{R}_2\hat{R}_3\hat{Q}, \quad Q_6 = i\hat{C}\hat{R}_3\hat{R}_1\hat{Q}, \end{aligned} \tag{35}$$

where $\hat{Q}$ is the Johnson-Lippman [14] constant of motion

$$\hat{Q} = m\alpha\frac{\hat{\sigma}_a x_a}{x} + \gamma_0 D\left(\hat{\sigma}_a p_a + i\gamma_4\frac{\alpha}{x}\right),$$

$\hat{\sigma}_a = \frac{i}{2}\varepsilon_{abc}\gamma_b\gamma_c$ and $\hat{C}$, $\hat{R}_a$ are analogues of operators C and R_a defined on the solutions of the initial Dirac equation (1)

$$\hat{C}\psi(\mathbf{x}) = i\gamma_2\psi^*(\mathbf{x}), \quad \hat{R}_a\psi(\mathbf{x}) = \gamma_4\gamma_a\psi(\hat{r}_a\mathbf{x}).$$

It can be verified by direct calculation that operators (35) commute with L of (1) and satisfy relations (8) where $H = (L-\varepsilon)^2$, $g_{11} = g_{22} = g_{33} = -g_{44} = -q_{55} = -g_{66} = 1$.

Discussion. Thus, in addition to known SUSY systems including interactions with a magnetic field [6]–[12] we describe an origin of extended SUSY for interaction of an electron with an electric field. By this we present additional arguments for physical relevance of extended SUSY, and describe a class of quantum mechanical systems which admit it.

It is interesting to search for such quantum mechanical systems which admit extended SUSY and describe an interaction of spinning particles with a superposition of electric and magnetic fields. An example of such a system with time-dependent potentials was proposed in paper [15].

A natural question arises what kind of SUSY degeneracy appears for such well studied system as a relativistic Hydrogen atom (described by equation (27)). Acting by operators (34) on known solutions of this equation (which are present e.g. in book [12]) we recognize, that they change either the signs of the quantum numbers κ and m (eigenvalues of operators D and J_3) or the relative phases of wave functions with different m. In other words, such a degeneracy does exists. Being more or less obvious for the considered system, it can play a non-trivial role if we add a small perturbing interaction. Moreover, the found extended SUSY is preserved for more complicated systems such as a charged particle interacting with superposed Columb and Aharonov–Bohm potentials [17].

In addition to the SUSY context, the above results can be used to construct internal symmetries for the equations under consideration. Thus, starting with supercharges (34) and fixing in (27) $\varepsilon \neq 0$, we can define the operators $\Gamma_0 = \Sigma$, $\Gamma_k = \frac{Q_k}{\mu}$, $k = 1, 2, \ldots, 6$, which form the seven- dimensional Clifford algebra, i.e., satisfy the following relations

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2g_{\mu\nu}, \tag{36}$$

where $\mu, \nu = 0, 1, \ldots, 6$, $g_{00} = g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = -g_{66} = 1$. All linearly independent products of operators Γ_μ include 64 operators which form a basis of algebra

$gl(8, R)$. In accordance with the above, this algebra generates an external symmetry for the Hydrogen atom. This algebra is more extended then known $so(2, 4)$ symmetry (refer, e.g., to [16]) and is isomorphic to the involutive symmetry algebra of the free Dirac equation found in papers [7, 10].

In analogous way it is possible to find internal symmetry algebras for the problems characterized by parities (9), (11), and (12). These algebras are equivalent to the orthogonal Lie algebras $so(1, 2)$, $so(1, 3) \oplus so(1, 3)$ and $so(1, 4)$ correspondingly.

[1] Gol'fand Yu.A., Lichtman E.P. // Sov. Phys. JETP Lett. — 1971. — **13**. — P. 452;
Volkov D.V., Akulov V.P. // Phys. Lett. B. — 1973. — **46**. — P. 109;
Wess J., Zumino B. // Nucl. Phys. B. — 1974. — **70**. — P. 39.

[2] Cooper F., Khare A., Sukhatme U. // Phys. Rep. — 1995. — **211**. — P. 268.

[3] Ravndal F. // Phys. Rev. D. — 1980. — **21**. — P. 2461;
Khare A., Maharana J. // Nucl. Phys. B. — 1984. — **244**. — P. 409.

[4] Sukhumar S.V. // J. Phys. A. — 1985. — **18**. — L697.

[5] Gendenshtein L.E. // Yad. Fiz. — 1985. — **41**. — P. 261; Gendenshtein L.E., Krive N.V. // Usp. Fiz. Nauk. — 1985. — **146**. — P. 583.

[6] Nikitin A.G., On Extended Supersymmetries and Parasupersymmetries / X International Conference "Problems of Quantum Field Theory", JINR E2-96-369. — Dubna, 1996;
Nikitin A.G. On extended supersymmetries in quantum mechanics / GROUP21 Physical applications and mathematical aspects of geometry, groups and algebras, Eds. H.-D. Doebner, W. Sherer, P. Natterman. — Singapoore: World Scientific, 1997. — **1**. — P. 509–514.

[7] Niederle J., Nikitin A.G. // J. Phys. A. — 1997. — **30**. — P. 999.

[8] Tkachuk V.M., Vakarchuk S.I. // Phys. Lett. A. — 1997. — **228**. — P. 141.

[9] Niederle J., Nikitin A.G. // J. Math. Phys. — 1999. — ???

[10] Niederle J., Nikitin A.G. // J. Nonlin. Math. Phys. — 1997. — **4**. — P. 436.

[11] Nikitin A.G. // Int. J. of Mod. Phys. — 1998. — ???

[12] Fushchich W.I., Nikitin A.G. Symmetries of equations of quantum mechanics. — New York: Allerton Press Inc., 1994.

[13] Wigner E.P. in: Lectures Istanbul School of Theoretical Physics. — London: Gordon and Breach, 1964.

[14] Johnson M.P., Lippman B.A. // Phys. Rev. — 1960. — **78**. — P. 369.

[15] Tkachuk V.M. // J. Phys. A. — 1998. — **31**. — P. 1879.

[16] Malkin I.A., Man'ko V.I., Dynamical symmetries and coherent states of quantum systems. — Moscow: Nauka, 1979.

[17] Nikitin A.G., Superalgebras of symmetry operators for Coulomb and Aharonov-Bohm-Coulomb sstems / Photon and Poincaré group, Ed. V. Dvoeglazov. — N.Y.: Nova Science, 1999. — P. 74–80.