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Supermultiplets and relativistic problems: II. The Bhabha equation of arbitrary spin and its properties

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Abstract. In 1945 Bhabha was probably the first to discuss the problem of a free relativistic particle with arbitrary spin in terms of a single linear equation in the four-momentum vector p_ν, but substituting the γ^ν matrices of Dirac by other ones. He determined the latter by requiring that their appropriate Lorentz transformations lead to their formulation in terms of the generators of the O(5) group. His program was later extensively amplified by Krajcik, Nieto and others. We returned to this problem because we had an *ab-initio* procedure for deriving a Lorentz-invariant equation of arbitrary spin and furthermore could express the matrices appearing in them in terms of ordinary and what we called sign spins. Our procedure was similar to that of the ordinary and isotopic spin in nuclear physics that give rise to supermultiplets, hence the appearance of this word in the title. In the ordinary and sign spin formulation it is easy to transform our equation into one linear in both the p_ν and some of the generators of O(5). We can then obtain the matrix representation of our equation for an irrep $(n_1 n_2)$ of O(5) and find, through a similarity transformation, that for the irrep mentioned the particle satisfying our equation will have, in general, several spins and masses determined by a simple algorithm.

1. Introduction

The series title of this paper is related to the fact that the equations for relativistic particles of spin up to $\frac{1}{2}n$ were obtained from the sum of n Dirac equations in which all the momenta and coordinates are taken to be equal [1]. We then noted that the α and β matrices, or equivalently in this paper the γ's, could be expressed as direct products of the ordinary spin and a similar operator that we called sign spin. Thus our equation has two types of spins in analogy with the ordinary and isotopic spin of nuclear physics. As the latter gives rise to supermultiplets, this is the reason for the appearance of this word in the series title [2].

The subtitle includes the name of Bhabha as he was one of the first to consider that free particles of higher spin should be written in Dirac form, i.e. linear in the four-vector momentum p_ν, $\nu = 0, 1, 2, 3$ with coefficients that should be matrices, but only in Dirac form γ_ν if the spin is $\frac{1}{2}$. According to Bhabha, for arbitrary spin, the equation should then have the form [3]

$$(\Gamma^\nu p_\nu + M)\psi = 0 \tag{1.1}$$

where M is a constant and Γ are appropriate matrices for a particle that may have spins up to $\frac{1}{2}n$, that he derived through a sophisticated analysis based on the fact that the Γ^ν, $\nu = 0, 1, 2, 3$, must transform as a four vector under elements of the Lorentz group.

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We shall proceed to show that using the analysis presented in the first paragraph we can prove in an elementary fashion that *our* equation of the Bhabha type is Lorentz invariant and furthermore it can be expressed in terms of the generators of an O(5) group.

Besides, by a similarity transformation closely connected with the one of Foldy–Wouthuysen type, we can determine the spin and mass content of the particle that the equation represents.

The Bhabha equation and its properties have been discussed by many authors and, in particular, by Krajcik and Nieto [4], but we believe that our approach differs from all the others presented in the literature.

2. A linear equation in p_ν for a particle with arbitrary spins that is Lorentz invariant

We start with the well known proof of the Lorentz invariance of the ordinary Dirac equation, so that we can later extend it to the problem we are interested in. Thus we have

$$(\gamma^\nu p_\nu + 1)\psi = 0 \tag{2.1}$$

where the index $\nu = 0, 1, 2, 3$, and when it is repeated it means a sum over the values indicated. Throughout we shall use units in which

$$\hbar = m = c = 1 \tag{2.2}$$

and the 4×4 matrices γ^μ are given by

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad i = 1, 2, 3 \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.3}$$

where $\sigma_i,\ i = 1, 2, 3$, are 2×2 Pauli spin matrices.

As p_ν is a four vector the equation (2.1) will be Lorentz invariant if the $\gamma^\nu,\ \nu = 0, 1, 2, 3$ also transform as a four vector under this operation, which implies the existence of a 4×4 matrix $\mathcal{U}$ such that

$$\gamma^{'\nu} = a^\nu_\mu \gamma^\mu = \mathcal{U}\gamma^\nu\mathcal{U}^{-1} \tag{2.4}$$

where $A \equiv \|a^\nu_\mu\|$ is a Lorentz transformation.

The existence of such a matrix $\mathcal{U}$ is given in many places, but for completness we derive it in the appendix.

We now consider n equations of the type (2.1) distinguished by the fact that we have $\gamma^\nu_r, p_{\nu r},\ r = 1, 2, \ldots, n$ and sum them making all four-momenta equal to get the equation

$$(\Gamma^\nu p_\nu + n)\,\psi = 0 \tag{2.5}$$

where

$$\Gamma^\nu = \sum_{r=1}^{n} \gamma^\nu_r \tag{2.6}$$

with γ^ν_r being the direct product of 4×4 matrices

$$\gamma^\nu_r = I \otimes I \otimes \cdots \otimes I \otimes \gamma^\nu \otimes I \cdots \otimes I \tag{2.7}$$

with γ^ν in the r position where the σ_i in it is replaced by σ_{ir}. Because the $\sigma_{ir},\ r = 1, \ldots, n$, this equation can represent particles with spin going from $\frac{1}{2}n, \frac{1}{2}n - 1, \ldots, \frac{1}{2}$ or 0.

Equation (2.5) is Lorentz invariant, because if we introduce the direct product matrix

$$U \equiv \mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \cdots \otimes \mathcal{U}_r \otimes \cdots \otimes \mathcal{U}_n \tag{2.8}$$

where $\mathcal{U}_r$ is given as in the appendix by replacing σ_i by σ_{ir} we immediately see that

$$\Gamma'^{\nu} = a^{\nu}_{\mu}\Gamma^{\mu} = U\Gamma^{\nu}U^{-1}. \tag{2.9}$$

Thus equation (2.5) has the Lorentz invariance and spin properties of the Bhabha equation and we shall proceed to show, first going through its supermultiplet formulation, that it is also invariant under an O(5) group.

3. The supermultiplet form of the Bhabha equation

To achieve the object indicated in the title of this section we first have to review some results of [2], but now as applied to the γ^{ν}, $\nu = 0, 1, 2, 3$ matrices.

We start by introducing two types of spin vectors, the ordinary one and what we have called the sign spin, which have the same mathematical form, but which will be distinguished here by round and square brackets, respectively, [2]:

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad s_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad s_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

$$\check{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad t_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad t_2 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad t_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.2}$$

From equations (2.3) and (3.1), (3.2) it is immediately clear that the γ^{ν} can be expressed as the direct products [2]

$$\gamma^{j} = i4s_j \otimes t_2 \quad j = 1, 2, 3 \qquad \gamma^0 = 2\hat{I} \otimes t_3. \tag{3.3}$$

We can now add an index $r = 1, 2, \ldots, n$, to all these matrices interpreting them in the direct product form (2.7) and we immediately see that equation (2.5) takes the form

$$\left\{\sum_{r=1}^{n}[4i(s_{jr} \otimes t_{2r})p_j] + \sum_{r=1}^{n}[2(\hat{I} \otimes t_{3r})p_0] + n\right\}\psi = 0 \tag{3.4}$$

where repeated latin indices (i, j, k) are summed over their values $1, 2, 3$.

Now we define

$$S_i = \sum_{r=1}^{n}(s_{ir} \otimes \check{I}) \quad R_{ij} = \sum_{r=1}^{n}(s_{ir} \otimes t_{jr}) \quad T_j = \sum_{r=1}^{n}(\hat{I} \otimes t_{jr}) \qquad i, j = 1, 2, 3 \tag{3.5}$$

and, as we indicated in [2], the 15 operators close under commutation and correspond to the SU(4) Lie algebra.

Using the definitions (3.5) the Bhabha equation (3.4) can be written as

$$\left\{4iR_{j2}p_j + 2T_3p_0 + n\right\}\psi = 0. \tag{3.6}$$

As only R_{i2}, T_3 appear in the equation we may assume that it admits a smaller symmetry group than SU(4). In fact we see from the commutation relations given in [2], that the ten operators

$$S_i \quad R_{i1} \quad R_{i2} \quad T_3 \qquad i = 1, 2, 3 \tag{3.7}$$

close under commutation as

$$\begin{aligned} &[S_i, S_j] = i\epsilon_{ijk}S_k \qquad [S_i, R_{j1}] = i\epsilon_{ijk}R_{j1} \\ &[S_i, R_{j2}] = i\epsilon_{ijk}R_{k2} \qquad [T_3, R_{j1}] = iR_{j2} \qquad [T_3, R_{j2}] = -iR_{j1} \\ &[R_{i1}, R_{j2}] = \tfrac{1}{4}iT_3\delta_{ij}. \end{aligned} \tag{3.8}$$

Thus the ten operators of (3.7) form a Lie algebra which clearly is a subalgebra of SU(4) and in fact is the unitary symplectic algebra Sp(4) whose Casimir operators commute with the operators in (3.6) and thus is its symmetry Lie algebra.

As will be discussed in the next section Sp(4) is isomorphic to O(5) and thus we get the symmetry Lie algebra that Bhabha derived by a very different procedure.

4. The O(5) symmetry algebra of the Bhabha equation

The generators of an orthogonal Lie algebra of dimension d are given by antisymmetric operators $\wedge_{mm'} = -\wedge_{m'm}$ where $m, m' = 1, 2, \ldots, d$, and thus there are $(d/2)(d-1)$ of them satisfying the commutations relations

$$[\wedge_{mm'}, \wedge_{nn'}] = i\,[\delta_{m'n}\wedge_{n'm} + \delta_{mn'}\wedge_{nm'} + \delta_{mn}\wedge_{m'n'} + \delta_{m'n'}\wedge_{mn}]\,. \tag{4.1}$$

Comparing them with the commutation relations (3.8) we easily see that when $d = 5$ the $\wedge_{mm'}$ with $m < m'$ (to avoid the repetition due to the antisymmetry) are correlated with $S_i, R_{i1}, R_{i2}, T_3;\ i = 1, 2, 3$ in the following way:

$$\begin{array}{llll} \wedge_{12} = S_3 & \wedge_{14} = 2R_{11} & \wedge_{15} = 2R_{12} & \\ \wedge_{13} = -S_2 & \wedge_{24} = 2R_{21} & \wedge_{25} = 2R_{22} & \wedge_{45} = T_3 \\ \wedge_{23} = S_1 & \wedge_{34} = 2R_{31} & \wedge_{35} = 2R_{32}. & \end{array} \tag{4.2}$$

The O(5) has the following chains of subgroups O(5) $\supset$ O(4) $\supset$ O(3) $\supset$ O(2) whose generators, in terms of the operators appearing in the commutation rules (3.8), can be selected as

$$\begin{array}{lllll} 10 & S_i, R_{i1}, R_{i2}, T_3 & \text{or} & \wedge_{12}, \wedge_{13}, \wedge_{23}, \wedge_{i4}, \wedge_{i5}, \wedge_{45},\ i = 1, 2, 3 & \mathrm{O}(5) \\ 6 & S_i, R_{i2} & \text{or} & \wedge_{12}, \wedge_{13}, \wedge_{23}, \wedge_{i5},\ i = 1, 2, 3 & \mathrm{O}(4) \\ 3 & S_i & \text{or} & \wedge_{12}, \wedge_{13}, \wedge_{23} & \mathrm{O}(3) \\ 1 & S_3 & \text{or} & \wedge_{12} & \mathrm{O}(2) \end{array} \tag{4.3}$$

where on the left-hand side we give the number of generators and on the right-hand side the group in question, with the generators expressed both in the supermultiplet notation $S_i, R_{ij}, T_3;\ i, j = 1, 2, 3$ and the orthogonal one $\wedge_{mm'},\ m < m',\ m = 1, 2, 3, 4, 5$.

We note now that in the supermultiplet notation the Bhabha equation is given by (3.6), so using the relations (4.2) we can also write it in the form

$$\left[2i \wedge_{i5} p_i + 2 \wedge_{45} p_0 + n\right]\psi = 0. \tag{4.4}$$

It is with this equation we want to deal with but with a small modification that would allow us to make use of a simple form of the matrix representation of the generators of orthogonal groups that we require in (4.4). For this purpose we note that the transposition (4,5) can be represented by the 5×5 orthogonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.5}$$

and thus is an element of the group O(5).

As O(5) is the symmetry group of (4.4), we can apply (4.5) to it and get a completely equivalent equation that now has the form

$$H\psi \equiv \left[2i \sum_{q=-1}^{1} (-1)^q \wedge_{q4} p_{-q} - 2 \wedge_{45} p_0 + n\right]\psi = 0 \tag{4.6}$$

where we also replaced the scalar product in Cartesian coordinates $i = 1, 2, 3$ by the spherical ones $q = 1, 0, -1$, and denote the operator in the square bracket by H.

We shall proceed to discuss this equation by first getting the matrix elements, in an appropriate basis, of $\wedge_{q4},\ q = 1, 0, -1$ and $\wedge_{45}$.

5. Matrix elements of the generators $\wedge_{45}, \wedge_{q4},\ q = 1, 0, -1$ in a basis of irreps in the chain $O(5) \supset O(4) \supset O(3) \supset O(2)$

As is well known [9, 10] the irreps of $O(2k + 1)$ and $O(2k)$ are characterized by partition involving k numbers that can be integer or semi-integer and non-negative, except for the last one in the even case which sometimes can be negative.

Rather than discussing the general theory analysed in [9, 10], we shall restrict our analysis to the chain of orthogonal groups that appear in the title of this section where the irreps will be denoted as follows:

$$\begin{aligned} &O(5);\ n_1, n_2\\ &O(4);\ m_1, m_2\\ &O(3);\ s\\ &O(2);\ \sigma. \end{aligned} \tag{5.1}$$

As O(5) is a symmetry group of the operator (4.6) the n_1, n_2 are integrals of motion of the problem and remain fixed. Turning now our attention to O(4), m_1, m_2 are restricted by the inequalities [9, 10]

$$n_1 \geqslant m_1 \geqslant n_2 \geqslant |m_2|. \tag{5.2}$$

For O(3) we have the single number s restricted by

$$m_1 \geqslant s \geqslant |m_2|. \tag{5.3}$$

Finally σ of O(2) is restricted by $|\sigma| \leqslant s$, which implies that is given by

$$\sigma = s, s - 1, \ldots, -s + 1, -s \tag{5.4}$$

as all the values indicated can only change by one unit at a time within the limits indicated in the inequalities. We note then that the integer or semi-integer character of the representation (n_1, n_2) of O(5) propagates to all of its subgroups.

The kets for the spin part of $O(5) \supset O(4) \supset O(3) \supset O(2)$ chain of groups, can be denoted by

$$\left|\begin{matrix} n_1 n_2\\ m_1 m_2\\ s\\ \sigma \end{matrix}\right\rangle \tag{5.5}$$

and the matrix elements of $\wedge_{45}, \wedge_{q4}$ with respect to them have been calculated in [12, 13]. Before giving them explicitly here, we note that $\wedge_{q4}$ is a Racah tensor of order 1 with respect to the O(3) group and, in particular, $\wedge_{04}$ corresponds to the component 0 of this tensor so by the Wigner–Eckart theorem we have [11] that

$$\left\langle \begin{matrix} n_1n_2 \\ m_1'm_2' \\ s' \\ \sigma' \end{matrix} \right| \wedge_{04} \left| \begin{matrix} n_1n_2 \\ m_1m_2 \\ s \\ \sigma \end{matrix} \right\rangle = \langle s\sigma, 10|s'\sigma'\rangle \left\langle \begin{matrix} n_1n_2 \\ m_1'm_2' \\ s' \end{matrix} \right\| \wedge_4 \left\| \begin{matrix} n_1n_2 \\ m_1m_2 \\ s \end{matrix} \right\rangle \tag{5.6}$$

where $\langle\cdot|\cdot\rangle$ is a standard O(3) Clebsch–Gordan coefficient. Thus for $\wedge_{q4}$ we need only the reduced matrix element on the right hand side of (5.6), and its explicit value, together with that of $\wedge_{45}$, is given below [12, 13]:

$$\begin{aligned}
&\left\langle \begin{matrix} n_1n_2 \\ m_1'm_2' \\ s \end{matrix} \right| \wedge_{45} \left| \begin{matrix} n_1n_2 \\ m_1m_2 \\ s \end{matrix} \right\rangle \\
&= -\frac{i}{2}\sqrt{\frac{(m_1-s+1)(m_1+s+2)(n_1-m_1)(n_1+m_1+3)(m_1-n_2+1)(m_1+n_2+2)}{(m_1+m_2+1)(m_1+m_2+2)(m_1-m_2+1)(m_1-m_2+2)}} \\
&\times \delta_{m_1',m_1+1}\delta_{m_2',m_2} \\
&-\frac{i}{2}\sqrt{\frac{(s-m_2)(s+m_2+1)(n_2-m_2)(n_2+m_2+1)(n_1-m_2+1)(n_1+m_2+2)}{(m_1+m_2+2)(m_1+m_2+1)(m_1-m_2)(m_1-m_2+1)}} \\
&\times \delta_{m_1',m_1}\delta_{m_2',m_{2+1}} \\
&+\frac{i}{2}\sqrt{\frac{(s+m_1+1)(m_1-s)(n_1-m_1+1)(n_1+m_1+2)(m_1-n_2)(m_1+n_2+1)}{(m_1+m_2)(m_1+m_2+1)(m_1-m_2)(m_1-m_2+1)}} \\
&\times \delta_{m_1',m_1-1}\delta_{m_2',m_2} \\
&+\frac{i}{2}\sqrt{\frac{(s-m_2+1)(s+m_2)(n_2-m_2+1)(n_2+m_2)(n_1-m_2+2)(m_2+n_1+1)}{(m_1+m_2)(m_1+m_2+1)(m_1-m_2+2)(m_1-m_2+1)}} \\
&\times \delta_{m_1',m_1}\delta_{m_2',m_2-1}
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\left\langle \begin{matrix} n_1n_2 \\ m_1m_2 \\ s' \end{matrix} \right\| \wedge_4 \left\| \begin{matrix} n_1n_2 \\ m_1m_2 \\ s \end{matrix} \right\rangle &= -i\sqrt{\frac{(m_1-s)(m_1+s+2)(s-m_2+1)(s+m_2+1)}{(2s+3)(s+1)}} \\
&\quad \times \delta_{s',s+1} + \frac{(m_1+1)m_2}{\sqrt{s(s+1)}}\delta_{s',s} \\
&\quad + i\sqrt{\frac{(m_1-s+1)(m_1+s+1)(s-m_2)(s+m_2)}{(2s-1)s}}\delta_{s',s-1}.
\end{aligned} \tag{5.8}$$

We can now return to our equation (4.6) and see that the operator appearing in it commutes with the components $p_q, q = 1, 0, -1$ of the momentum so they are integrals of motion that we can denote by the constants k_q. Furthermore, without loss of generality we can select our coordinate axis so the vector $\boldsymbol{k}$ is along the third of them so $k_0 \equiv k, k_{\pm 1} = 0$. The p_0 is also an integral of motion and we can replace it by a numerical constant we call E as in the units (2.2) it would be the energy. If we now consider the numerical finite and Hermitian matrix

$$\left\| 2i\langle s\sigma, 10|s'\sigma\rangle k \left\langle \begin{matrix} n_1n_2 \\ m_1'm_2' \\ s' \end{matrix} \right\| \wedge_4 \left\| \begin{matrix} n_1n_2 \\ m_1m_2 \\ s \end{matrix} \right\rangle - 2E \left\langle \begin{matrix} n_1n_2 \\ m_1'm_2' \\ s' \end{matrix} \right| \wedge_{45} \left| \begin{matrix} n_1n_2 \\ m_1m_2 \\ s \end{matrix} \right\rangle \delta_{ss'} + n\delta_{m_1'm_1}\delta_{m_2'm_2}\delta_{s's} \right\| \tag{5.9}$$

where the indices $m_1, m_2, s, m_1', m_2', s'$ vary according to the rules (5.2)–(5.4) and σ is diagonal, we see that if we equate its determinant to 0 we will get a secular equation that gives several expressions of E as function k, n_1, n_2, σ and n.

We can then denote the energy as

$$E(knn_1n_2\sigma\mu) \tag{5.10}$$

where μ is a number that differentiates the energies that correspond to the same $kn_1n_2\sigma$.

The eigenstate ψ of (4.6) can be denoted by

$$\begin{aligned} \psi &= |knn_1n_2\sigma\mu\rangle \exp\{i[kx_3 - E(knn_1n_2\sigma\mu)t]\} \\ &= \sum_{m_1m_2s} A^{kn_1n_2\sigma\mu}_{m_1m_2s} \left| \begin{matrix} n_1n_2 \\ m_1m_2 \\ s \\ \sigma \end{matrix} \right\rangle \exp\{i[kx_3 - E(knn_1n_2\sigma\mu)t]\} \end{aligned} \tag{5.11}$$

where the A's are coefficients determined from the process of diagonalization of the matrix (5.9). As we choose $\boldsymbol{k}$ in a fixed direction only its absolute value k appears with x_3, and we also replace x_0 by t.

We shall see, in the following section, that by a similarity transformations, analogous to the Foldy–Wouthuysen (FW) one, we can get the relation of the energy E with the other variables in (5.10) in a much simpler way.

6. A similarity transformation for the operator H in (4.6)

We now wish to find an operator $\exp(i\Delta)$ such that under the similarity transformation

$$e^{-i\Delta}He^{i\Delta} = \mathcal{H} \tag{6.1}$$

we get a new operator $\mathcal{H}$ in which the positive and negative energy parts are separated and, besides, $\mathcal{H}$ becomes proportional to an operator related to the numerical term $(E^2 - k^2)^{1/2}$ associated with Einstein's expression for the mass.

To achieve our purpose we follow a procedure similar to that of FW [14] and propose a Δ of the form

$$\Delta = (\wedge_{i5}p_i/p)\theta \tag{6.2}$$

where p is magnitude of the three vector p_i and θ will be a function of p to be determined later by requiring that

$$\mathcal{H} = \wedge_{45}f + n \tag{6.3}$$

with f also being a function of p which we obtain in the process of determining (6.3). The appearance of n is due to the fact that in H of (4.6) it is a number and it will remain the same after the similarity transformation.

We shall analyse the problem in an inverse fashion by writing

$$H = e^{i\Delta}\mathcal{H}e^{-i\Delta} \tag{6.4}$$

which from the Campbell–Hausdorf formula can be written as

$$H = \mathcal{H} + \sum_{m=1}^{\infty} \frac{i^m}{m!} \left[\Delta, \left[\Delta, \left[\Delta \cdots [\Delta, \mathcal{H}]\right] \cdots\right]\right]_m \tag{6.5}$$

where the last term is a m-tuple commutator of order m.

The first term $m = 0$ is obviously the $\mathcal{H}$ operator, while for $m = 1$ we have

$$\left[\frac{\wedge_{i5} p_i}{p}\theta,\ \wedge_{45} f\right] = \theta f \frac{p_i}{p}[\wedge_{i5}, \wedge_{45}] = if\ \frac{\theta p_i}{p} \wedge_{i4}\,. \tag{6.6}$$

The second, i.e. with $m = 2$, is given from (6.6) by

$$\left[\frac{\wedge_{i5} p_i}{p}\theta,\ if\ \frac{\wedge_{j4} p_j}{p}\theta\right] = \ if\ \frac{p_i p_j \theta^2}{p^2}[\wedge_{i5}, \wedge_{j4}] = \theta^2 f \wedge_{45}\,. \tag{6.7}$$

Now using equations (6.7) and (6.6) we see that for $m = 3$ we obtain

$$\left[\frac{\wedge_{i5} p_i}{p}\theta,\ \theta^2 \wedge_{45} f\right] = i\theta^3 f \frac{\wedge_{i4} p_i}{p} \tag{6.8}$$

and continuing in this way we finally obtain the following expression for the right-hand side of (6.5):

$$f\left[\wedge_{45}\cos\theta - \frac{\wedge_{i4} p_i}{p}\sin\theta\right] + n. \tag{6.9}$$

Comparing it now with H of (4.6), which in Cartesian components is

$$H = 2i \wedge_{i4} p_i - 2 \wedge_{45} p_0 + n \tag{6.10}$$

we get the relations

$$f\frac{\sin\theta}{p} = -2i \qquad f\cos\theta = -2p_0 \tag{6.11}$$

from which we conclude that θ must be imaginary, i.e. $\theta = i\phi$ with ϕ real and we get the relations

$$f\sinh\phi = -2p \qquad f\cosh\phi = -2p_0 \tag{6.12}$$

which imply

$$\phi = \operatorname{arctanh}(p/p_0) \qquad f = -2\sqrt{p_0^2 - p^2}. \tag{6.13}$$

The $-$ sign for the f comes from to the fact that if $p = 0, \phi = 0$ and from (6.12) we conclude that $f = -2p_0$.

The operator $\mathcal{H}$ obtained from the similarity transformation (6.1) applied to H of (4.6) is then

$$\mathcal{H} = -2 \wedge_{45} \sqrt{p_0^2 - p^2} + n = 0. \tag{6.14}$$

The operator $\mathcal{H}$ of (6.14) makes perfect sense in the momentum representation and it is clearly invariant under Lorentz transformations, as we should expect from the discussion in section 2.

Table 1.

n	(n_1, n_2)	s	λ	$M_\lambda = \frac{n}{2\lambda}$
1	(1/2, 1/2)	1/2	±1/2	±1
2	(1, 1)	1	±1	±1
		1	0	–
		0	0	–
	(1, 0)	0	±1	±1
		0	0	–
		1	0	–
	(0, 0)	0	0	–
3	(3/2, 3/2)	3/2	±3/2	±1
		3/2	±1/2	±3
		1/2	±1/2	±3
	(3/2, 1/2)	3/2	±1/2	±3
		1/2	±3/2	±1
		1/2	±1/2	±3
	(1/2, 1/2)	1/2	±1/2	±3
4	(2/2)	2	±2	±1
		2	±1	±2
		2	0	–
		1	±1	±2
		1	0	–
		0	0	–
	(2, 1)	2	±1	±2
		2	0	–
		1	±2	±1
		1	±1	±2
		1	0	–
		0	±1	±2
	(1, 1)	1	±1	±2
		1	0	–
		0	0	–
	(2, 0)	2	0	–
		1	±1	±2
		0	±2	±1
		0	0	–
	(1, 0)	1	0	–
		0	±1	±2
	(0, 0)	0	0	–

We note from (5.7) that $\wedge_{45}$ is a Hermitian matrix and thus it can be diagonalized with real eigenvalues. In fact as the permutation taking us from 12345 to 54321 is an element of the O(5) group, we see that eigenvalues of $\wedge_{45}$ are the same of $\wedge_{12}$ and the latter given by σ are integers or semi-integers positive or negative restricted by the relations (5.2)–(5.4). The eigenvalues of $2\wedge_{45}$ are all integers and we shall denote them by 2λ.

Turning now to the eigenvalues E^2, k^2 of p_0^2, p^2 discussed in the previous section and selecting a frame of reference in which $\wedge_{45}$ is diagonal, we can write

$$E^2 = k^2 + (n/2\lambda)^2 \tag{6.15}$$

which implies that the Bhabha equation does not represent a single mass but all those given by

$$M_\lambda \equiv n/2\lambda. \tag{6.16}$$

There is no problem with this result if the irrep (n_1n_2) of O(5) is made of semiintegers, because then λ is also a semiinteger and does not vanish. If (n_1n_2) are integers λ can take the value 0, but that state can be projected out as shown in [4, 15]. Thus in all representations (n_1n_2) of O(5) we can get the different masses that can take a particle satisfying the Bhabha equation (2.5).

It is of great interest to know the different values of spin and mass that a Bhabha particle can take. For the possible values of the spin we have to use the inequalities (5.2,3). For the masses associated with a given irrep (n_1n_2) of O(5) and definite spin s, we have to find the eigenvalues λ of the matrix $\wedge_{45}$ whose elements are given by (5.7) and then use (6.16). There are other more compact ways of getting the s and M_λ values associated with irrep (n_1n_2) of O(5) given in [4] and [15], but the ones mentioned in the previous phrase are the more direct ones and were used to obtain table 1 of the s, M_λ content in the lowest irreps (n_1n_2) of O(5) given below.

As a last point we note that in (2.5) the integer parameter n appears, which implies that all partitions $[h_1, h_2 \ldots h_n]$ with

$$h_1 \geqslant h_2 \geqslant \cdots \geqslant h_n \qquad h_1 + h_2 + \cdots + h_n = n \tag{6.17}$$

are possible irreps of the SU(4) group mentioned earlier, which is isomorphic to an O(6) group whose irreps are characterized $(q_1q_2q_3)$ related with the irreps of SU(4) [16] by

$$\begin{aligned} q_1 &= \tfrac{1}{2}(h_1 + h_2 - h_3 - h_4) \\ q_2 &= \tfrac{1}{2}(h_1 - h_2 + h_3 - h_4) \\ q_3 &= \tfrac{1}{2}(h_1 - h_2 - h_3 + h_4). \end{aligned} \tag{6.18}$$

The irreps of (n_1n_2) of O(5) must then satisfy the inequalities [16]

$$q_1 \geqslant n_1 \geqslant q_2 \geqslant n_2 \geqslant |q_3|. \tag{6.19}$$

Thus we see that once n is given the irreps (n_1n_2) of O(5) cannot be chosen at random.

We have shown through the supermultiplet representation of the equation (2.5) that it is actually characterized by the irreps (n_1n_2) of O(5). This result was known to Bhabha in his work of 1945, but our approach to it followed a very different path.

So far we have analysed the case of a free particle but in subsequent articles we plan to discuss cases in which it is in potentials of various forms.

Appendix

To determine the 4×4 $\mathcal{U}$ matrix satisfying (2.4) we first note that if the Lorentz transformation A is only a rotation, $\mathcal{U}$ will consist of two blocks in the diagonal associated with the spinor representation of the rotation, i.e. $D^{1/2}$.

We need then restrict ourselves only to boosts, and as they can be reduced by rotations to boosts in the $\nu = 3$ direction we only need to consider the $\mathcal{U}$ corresponding to the Lorentz transformation

$$A = \begin{pmatrix} c & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s & 0 & 0 & c \end{pmatrix} \tag{A.1}$$

where $c = \cosh\delta,\ s = \sinh\delta$ and δ is an arbitrary real parameter.

We now note that in terms of the γ's the spin operator [7] takes the form

$$S^{\mu\nu} = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{A.2}$$

and thus for the particular Lorentz transformations in (A.1) we have the result that the matrix corresponding to an infinitesimal transformation in the γ space is given by

$$S^{03} = \frac{i}{2}(\gamma^0\gamma^3 - \gamma^3\gamma^0) = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \tag{A.3}$$

and thus for the finite one corresponding to A in (A.1) we have

$$\mathcal{U} = \exp(i\delta S^{03}) = \begin{pmatrix} \overline{c}I & \overline{s}\sigma_3 \\ \overline{s}\sigma_3 & \overline{c}I \end{pmatrix} \tag{A.4}$$

where we used the fact that $\sigma_3^2 = I$ is a 2×2 unit matrix and denote by $\overline{c}, \overline{s}$ the functions

$$\overline{c} = \cosh(\delta/2) \qquad \overline{s} = \sinh(\delta/2). \tag{A.5}$$

Thus we have shown the existence of a $\mathcal{U}$ related to A and, in particular, if the boost is in an arbitrary directions given by the unit vector $\boldsymbol{b}$ instead of $\nu = 3$, then obviously $\mathcal{U}$ becomes

$$\mathcal{U} = \begin{pmatrix} \overline{c}I & \overline{s}(\boldsymbol{\sigma}\cdot\boldsymbol{b}) \\ \overline{s}(\boldsymbol{\sigma}\cdot\boldsymbol{b}) & \overline{c}I \end{pmatrix}. \tag{A.6}$$

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