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On exact Foldy–Wouthuysen transformation

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Abstract. New classes of external fields are found for which the exact Foldy–Wouthuysen transformation for the Dirac equation exists.

The Foldy–Wouthuysen transformation (FWT) [1] is one of the corner stones of relativistic quantum mechanics. It presents a straightforward and convenient way to obtain an adequate physical interpretation for relativistic wave equations. The very existence of an exact FWT for a considered relativistic problem justifies its quantum mechanical treatment, inasmuch as in this case transitions between positive and negative energy states are forbidden. Following [2] we say that the related Dirac sea is stable, and so it is possible to restrict ourselves to positive energy solutions.

The known problems, which admit the exact FWT, include minimal [3] and anomalous [4] interactions of spin-$\frac{1}{2}$ particles with a time-independent magnetic field, anomalous interaction of such particles with a time-independent electric field and with a pseudo-scalar field [2]. A specific generalization of the FWT for the Dirac equation with a half sum of scalar and zero component of four-vector potentials was proposed in [5]. Moreover, the analysis present in papers [2, 5] is based on custodial supersymmetry of the Dirac equation.

The aim of this paper is to present new classes of problems admitting the exact FWT.

Let us start with the Dirac Hamiltonian for a free particle

$$H = c\gamma_0\gamma_a p_a + \gamma_0 mc^2 \tag{1}$$

where γ_0, γ_a $(a = 1, 2, 3)$ are the Dirac matrices with diagonal γ_0.

The FWT reduces (1) to the diagonal form

$$H \to H' = UHU^{\dagger} = \gamma_0\sqrt{H^2} \tag{2}$$

here $H^2 = p^2c^2 + m^2c^4$, and the related operator U is found in paper [1]. In our notations it has the form

$$U = \exp\left(\frac{\gamma_a p_a}{p}\arctan\frac{p}{mc}\right).$$

It was proposed by Eriksen and Kolsrud [4] to use the two-step transformation (2) with

$$U = U_2U_1 \qquad U_1 = \frac{1}{\sqrt{2}}(1 + I\epsilon) \qquad U_2 = \frac{1}{\sqrt{2}}(1 + \gamma_0 I) \tag{3}$$

where $\epsilon = H/(H^2)^{1/2}$ is the sign energy operator and $I = \mathrm{i}\gamma_5\gamma_0$ is a unitary involution anticommuting with H and γ_0

$$I^\dagger I = II^\dagger = I^2 = 1 \qquad IH = -HI \qquad I\gamma_0 = -\gamma_0 I. \tag{4}$$

It is the operator U given in (3) which, in contrast with the original Foldy–Wouthuysen operator, admits the direct generalization for the case of anomalous interaction with a time-independent magnetic field [4].

We notice that relations (4) can be used to search for new types of external fields such that the related Dirac Hamiltonian for an interacting particle admits the exact FWT. The main idea which enables us to achieve this goal is that the corresponding involution I can be constructed using discrete symmetries of the Dirac equation studied in [6].

To demonstrate new possibilities which arise in this way let us consider the Dirac Hamiltonian for a particle interacting with an electrostatic field

$$H = c\gamma_0\gamma_a p_a + \gamma_0 mc^2 + \frac{e}{c}A_0. \tag{5}$$

It is generally accepted that this Hamiltonian admits only an approximate FWT [1]. Here we shall show that operator (5) also admits the exact FWT provided it has no zero eigenvalues and A_0 is an odd function of $\boldsymbol{x}$: $A_0(-\boldsymbol{x}) = -A_0(\boldsymbol{x})$. Familiar examples of odd A_0 are the potentials of the constant and dipole electric fields, when

$$A_0 = qe\boldsymbol{x}\cdot\boldsymbol{E} \qquad \text{and} \qquad A_0 = \frac{qe}{|\boldsymbol{x}-\boldsymbol{a}|} - \frac{qe}{|\boldsymbol{x}+\boldsymbol{a}|}$$

respectively ($\boldsymbol{E}$ and $\boldsymbol{a}$ are constant vectors).

The corresponding involution I is expressed via the space reflection operator P

$$I = \mathrm{i}\gamma_5\gamma_0 P \qquad P\Psi(x_0,\boldsymbol{x}) = \gamma_0\Psi(x_0,-\boldsymbol{x}). \tag{6}$$

Substituting (5) and (6) into (3) we obtain

$$UHU^\dagger = H' = \gamma_0\left(p^2c^2 + m^2c^4 + \frac{e^2}{c^2}A_0^2 + 2mce\gamma_0 A_0 + P\{eA_0, \boldsymbol{\sigma}\cdot\boldsymbol{p}\}\right)^{1/2} \tag{7}$$

where $\boldsymbol{\sigma} = \mathrm{i}\boldsymbol{\gamma}\times\boldsymbol{\gamma}/2$.

Formula (7) presents probably the first known example of the exact FWT for the Dirac Hamiltonian including a minimal interaction with an external electric field. Expanding H' in a power series of $1/c$ and neglecting the terms of orders $1/c^3$, we come to the approximate quasirelativistic Hamiltonian

$$H_{\mathrm{QR}} = \gamma_0\left(m^2c^4 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{eP\{\boldsymbol{\sigma}\cdot\boldsymbol{p}, A_0\}}{2mc^2}\right) - \frac{e\{p^2, A_0\}}{4m^2c} + \frac{e}{c}A_0$$

which is unitary equivalent to the corresponding Hamiltonian found by Foldy and Wouthuysen

$$WH_{\mathrm{QR}}W^\dagger = \gamma_0\left(mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}\right) + eA_0 - \frac{e}{8m^2c^2}[\boldsymbol{\sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{\pi}) + \mathrm{div}\boldsymbol{E}]$$

where

$$W = \exp\left(-\gamma_0 P\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2mc}\right) \qquad \text{and} \qquad E_a = -\frac{\partial A_0}{\partial x_a}.$$

Finally, let us consider the Hamiltonian of spin-$\frac{1}{2}$ particle of the following general form

$$H = c\gamma_0\gamma_a\pi_a + \gamma_0(mc^2 + V) + \gamma_5\varphi + \frac{k}{m}(\gamma_5\gamma_a H_a + \mathrm{i}\gamma_a E_a) + eA_0. \tag{8}$$

Here $\pi_a = p_a - eA_a(\boldsymbol{x})$, A_0 and A_a are components of a vector-potential, H_a and E_a are components of the vectors of magnetic and electric fields, V and φ are scalar and pseudoscalar potentials.

Hamiltonian (8) includes minimal and anomalous interactions with an electromagnetic field, and also scalar and pseudoscalar interactions. To construct the exact FWT for this Hamiltonian, let us search for involutions I (satisfying (4)) in the form

$$I = \Gamma_A R_A \tag{9}$$

(no sum over A) where Γ_A are numeric matrices, the multi-index A takes the values $A = 1, 2, 3, 12, 32, 23, 123$, R_A are operators of the discrete transformations of spatial variables $R_A\psi(x_0, \boldsymbol{x}) = \psi(x_0, r_A\boldsymbol{x})$, and

$$r_1\boldsymbol{x} = (-x_1, x_2, x_3) \qquad r_2\boldsymbol{x} = (x_1, -x_2, x_3) \qquad r_3\boldsymbol{x} = (x_1, x_2, -x_3)$$
$$r_{12}\boldsymbol{x} = -r_3\boldsymbol{x} \qquad r_{23}\boldsymbol{x} = -r_1\boldsymbol{x} \qquad r_{31}\boldsymbol{x} = -r_2\boldsymbol{x} \qquad r_{123}\boldsymbol{x} = -\boldsymbol{x}.$$

Requiring the anticommutativity of operators (7) with the free particle Hamiltonian (1), it is not difficult to specify explicit forms of matrices Γ_A and find all possible involutions (9) in the form

$$I = \gamma_0\gamma_a R_a \qquad a = 1, 2, 3 \tag{10a}$$
$$I = \mathrm{i}\gamma_a R_{bc} \qquad a \neq b,\ a \neq c,\ b \neq c \tag{10b}$$
$$I = \mathrm{i}\gamma_5 R_{123}. \tag{10c}$$

Substituting (8) and (10a)–(10c) into (2) we come to the following conditions (11a)–(11c) for A_0, $\boldsymbol{A} = (A_1, A_2, A_3)$ V and φ, respectively

$$A_0(r_a\boldsymbol{x}) = -A_0(\boldsymbol{x}) \qquad \boldsymbol{A}(r_a\boldsymbol{x}) = r_a\boldsymbol{A}(\boldsymbol{x}) \qquad V(r_a\boldsymbol{x}) = V(\boldsymbol{x})$$
$$\varphi(r_a\boldsymbol{x}) = \varphi(\boldsymbol{x}) \tag{11a}$$
$$A_0(r_{ab}\boldsymbol{x}) = -A_0(\boldsymbol{x}) \qquad \boldsymbol{A}(r_{ab}\boldsymbol{x}) = r_{ab}\boldsymbol{A}(\boldsymbol{x}) \qquad V(r_{ab}\boldsymbol{x}) = V(\boldsymbol{x})$$
$$\varphi(r_{ab}\boldsymbol{x}) = -\varphi(\boldsymbol{x}) \tag{11b}$$
$$A_0(-\boldsymbol{x}) = -A_0(\boldsymbol{x}) \qquad \boldsymbol{A}(-\boldsymbol{x}) = -\boldsymbol{A}(\boldsymbol{x}) \qquad V(-\boldsymbol{x}) = V(\boldsymbol{x})$$
$$\varphi(-\boldsymbol{x}) = \varphi(\boldsymbol{x}). \tag{11c}$$

Thus, we have found the exact FWT (generated by operators (3) and (10)) for the generalized Dirac Hamiltonian (8) which includes minimal, anomalous, scalar and pseudoscalar interactions with external fields. A sufficient condition for the existence of this transformation is that the related potentials satisfy at least one of the relations (11a), (11b), or (11c), and that the corresponding Hamiltonian has no zero eigenvalues.

Relations (11) define a wide class of potentials, including such interesting ones as the constant electric and magnetic fields, electric dipole field, electric field generated by two infinite charged strings with opposite charges, magnetic field generated by an infinite straight conductor and many, many others. Considering the superpositions of these fields, it is possible to generate examples of realistic physical systems with different types of symmetries enumerated in (11).

We see that using discrete involutive symmetries of the Dirac equation, it is possible to extend sufficiently the class of external fields admitting the exact FWT. Algebraic structures of these symmetries were studied in papers [6].

An exact FWT for a two-body equation is found in [7]. For a survey of FWT for higher spin and two particle equations refer to [8].

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