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Non-Lie and Discrete Symmetries of the Dirac Equation

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Abstract

New algebras of symmetries of the Dirac equation are presented, which are formed by linear and antilinear first–order differential operators. These symmetries are applied to decouple the Dirac equation for a charged particle interacting with an external field.

I. Introduction

Symmetries of differential equations have important applications in construction of conservation laws [1], separation of variables [2], reduction of nonlinear problems to more simple ones [3], etc. All that causes the continuous interest of physicists and mathematicians in the classical group-theoretical approach [4] and its modern generalizations.

Early in the seventies, W.I. Fushchych proposed the fruitful concept of non-Lie symmetries. It happens that even such well-studied subjects as the Maxwell and Dirac equations admit extended symmetry algebras which cannot be found using the classical Lie approach [5-7]. The distinguishing feature of these algebras is that they have usual Lie structures in spite of the fact that their basis elements are not Lie derivatives and belong to classes of higher-order differential operators or even integro-differential operators.

In recent paper [8] a new invariance algebra D of the Dirac equation was found. Being the algebra of the higher dimension than other known finite symmetry algebras of this equation, the algebra D is formed by discrete symmetries like parity or charge conjugation. This algebra has useful applications in searching for hidden supersymmetries and reduction of the Dirac equation for a particle interacting with various external fields [8].

In the present paper we continue the analysis of algebraic structures of discrete symmetries and study their connections with non-Lie symmetries of the Dirac equation. We find a finite dimensional symmetry algebra of the Dirac equation, which unites both the non-Lie [6, 7] and involutive discrete [8] symmetries. We also apply discrete symmetries to decouple the Dirac equation for a particle interacting with an external field.

II. Lie and non-Lie symmetries of the Dirac equation

Let us start with the free Dirac equation

$$L\psi = 0, \qquad L = \gamma^\mu p_\mu - m. \tag{2.1}$$

Here $p_\mu = i\dfrac{\partial}{\partial x^\mu}$, $\mu = 0, 1, 2, 3$, γ^μ are the Dirac matrices which we choose in the form

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix}, \quad \gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{2.2}$$

$\sigma_a (a = 1, 2, 3)$ are the Pauli matrices, I_2 is the 2×2 unit matrix.

We say a linear operator Q is a *symmetry* of equation (2.1) if there exists such an operator α_Q that

$$[Q, L] = \alpha_Q L. \tag{2.3}$$

In the classical Lie approach [4] symmetry operators are searched for in the form

$$Q = a^\mu p_\mu + b \tag{2.4}$$

where a^μ are functions of $x = (x_0, x_1, ...)$, b is a matrix dependent on x. The maximal invariance algebra of equation (2.1) in the class of operators (2.4) is the Poincaré algebra whose basis elements are

$$P_\mu = p_\mu, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{4}\left[\gamma_\mu, \gamma_\nu\right]. \tag{2.5}$$

In other words, any symmetry of the Dirac equation, which has form (2.4), is a linear combination of generators (2.5) (refer, e.g., to [9]). The related α_Q in (2.3) are equal to zero.

Supposing that coefficients a^μ in (2.4) are matrices, we find the simplest non-Lie symmetry algebra for equation (2.1) which is generated by the following operators [7, 9]

$$\begin{aligned} \Sigma_{\mu\nu} &= \frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right] + \frac{1}{m}\left(1 - i\gamma_4\right)\left(\gamma_\mu p_\nu - \gamma_\nu p_\mu\right), \\ \Sigma_1 &= \gamma_4 - \frac{i}{m}\left(1 - i\gamma_4\right)\gamma^\mu p_\mu. \end{aligned} \tag{2.6}$$

Operators (2.6) satisfy relations (2.3) for $\alpha_{\Sigma_{\mu\nu}} = \dfrac{1}{m}\left(\gamma_\mu p_\nu - \gamma_\nu p_\mu\right)$ and $\alpha_{\Sigma_1} = -\dfrac{1}{m}\gamma_4\gamma^\mu p_\mu$. Moreover, operators $\Sigma_{\mu\nu}$ commute with Σ_1 and form the Lie algebra isomorphic to so(1,3).

We notice that Lie symmetries (2.5) and non-Lie symmetries (2.6) can be united in frames of the 17-dimensional Lie algebra which includes (2.5) and (2.6) as subalgebras [9].

III. Algebras of discrete symmetries of the Dirac equation

It is well known that the Dirac equation is invariant w.r.t. specific discrete transformations like parity or charge conjugation. Let us analyze algebraic structures generated by these symmetries.

Consider reflections of independent variables $x = (x_0, x_1, x_2, x_3)$:

$$\begin{aligned} &\theta_0 x = (-x_0, x_1, x_2, x_3)\,, \quad \theta_1 x = (x_0, -x_1, x_2, x_3)\,, \quad \theta_2 x = (x_0, x_1, -x_2, x_3)\,, \\ &\theta_3 x = (x_0, x_1, x_2, -x_3)\,, \quad \theta x = (-x_0, -x_1, -x_2, -x_3)\,. \end{aligned} \tag{3.1}$$

The corresponding symmetry operators for equation (2.1) have the form

$$\Gamma_0 = \gamma_4\gamma_0\hat{\theta}_0, \quad \Gamma_1 = \gamma_4\gamma_1\hat{\theta}_1, \quad \Gamma_2 = \gamma_4\gamma_2\hat{\theta}_2, \quad \Gamma_3 = \gamma_4\gamma_3\hat{\theta}_3, \quad \Gamma_4 = i\gamma_4\hat{\theta} \tag{3.2}$$

where $\hat{\theta}_\mu$ and $\hat{\theta}$ are operators defined as follows:

$$\hat{\theta}_\mu\psi(x) = \psi\left(\theta_\mu x\right), \quad \hat{\theta}\psi(x) = \psi(-x). \tag{3.3}$$

Let us add the list of symmetries (3.2) by the following *antilinear* operator

$$\Gamma_5 = C = i\gamma_2 c \tag{3.4}$$

where c is the complex conjugation, $c\psi(x) = \psi^*(x)$.

Operators (3.2), (3.4) generate very interesting algebraic structures. First, they satisfy the Clifford algebra

$$\Gamma_k\Gamma_l + \Gamma_l\Gamma_k = 2g_{kl} \tag{3.5}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = g_{44} = g_{55} = 1$; $g_{kl} = 0$, $k \neq l$. Secondly, this Clifford algebra can be extended by adding the seventh basis element

$$\Gamma_6 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 = iC. \tag{3.6}$$

Finally, the enveloping algebra of this seven-dimensional Clifford algebra is isomorphic to the algebra $gl(8, R)$. In other words, there are 64 linearly independent products of the operators $\Gamma_f (f = 0, 1, ...6)$:

$$\left\{\Gamma_m, \quad \Gamma_m\Gamma_n, \quad \Gamma_k\Gamma_m\Gamma_n, \quad \hat{I}\right\}, \qquad k, l, m, = 0, 1, ...6 \tag{3.7}$$

($\hat{I}$ is the unit operator) which form a basis of the Lie algebra isomorphic to $gl(8, R)$. This isomorphism will be constructed explicitly in Section V.

Thus the discrete symmetries of the Dirac equation generate a very extended Lie algebra. Restricting ourselves to linear symmetries we come to the 16-dimensional Lie algebra including the identity operator $\hat{I}$ and the following 15 operators

$$\{\Gamma_a, \quad \Gamma_b\Gamma_c\}\,, \qquad a, b, c = 0, 1, ...4 \tag{3.8}$$

with Γ_a defined in (3.2). Operators (3.8) form a basis of the algebra $so(2, 4)$.

We notice that the Dirac equation for a charged particle interacting with an external field

$$\left(\gamma^\mu\pi_\mu - m\right)\psi = 0, \quad \pi_\mu = p_\mu - eA_\mu(x) \tag{3.9}$$

still admits some of symmetries (3.7) provided functions $A_\mu(x)$ have definite parities w.r.t. the related reflections (3.1) or their combinations. Moreover, if the corresponding symmetry (3.7) is diagonalizable, then equation (3.9) can be reduced to two uncoupled subsystems [8]. We will demonstrate in Section VI that for some classes of vector-potentials A_μ the Dirac equation can be reduced to *eight* uncoupled equations.

IV. The maximal present symmetry algebra for the Dirac equation

Thus there exist two symmetry algebras for the Dirac equation which are defined by relations (2.6), (3.7) and which are of different origin. Symmetries (2.6) are of the form of differential operators whereas (3.7) are functional operators of discrete transformations. Nevertheless, it is possible to find an algebraic structure which unify both of them.

First let us note that it is impossible to include all symmetries (2.6) and (3.8) into a finite-dimensional Lie algebra. Indeed, commutators of operators (2.6) and (3.2) generate second-order differential operators whose commutators give fourth-order differential operators, and so on. However the subset of symmetries (3.7) which commute with γ_4, i.e.,

$$\Gamma_k\Gamma_l, \quad \Gamma_4, \quad \Gamma_4\Gamma_k\Gamma_l, \quad k,l\neq 4 \tag{4.1}$$

can be united with operators (2.6) in framework of a 120-dimensional (!) Lie algebra with the following basis

$$\begin{aligned}
&Q_{4\mu,4\mu}=C\Gamma_\mu, \quad Q_{5\mu,5\mu}=C\Gamma_4\Gamma_\mu, \quad Q_{5\nu,5\mu}=g_{\mu\mu}C\Gamma_4\Gamma_\mu\Sigma_{\mu\nu},\\
&Q_{4\nu,4\mu}=g_{\mu\mu}C\Gamma_\mu\Sigma_{\mu\nu}, \quad Q_{4\nu,5\mu}=-\gamma_4 Q_{5\mu,5\nu}, \quad Q_{\mu\nu,\mu\nu}=\gamma_{\mu\nu}\hat\theta_\mu\hat\theta_\nu,\\
&Q_{5\nu,4\mu}=-\gamma_4 Q_{4\mu,4\nu}, \quad Q_{54,54}=i\Gamma_4, \quad Q_{54,\mu\nu}=\varepsilon_{\mu\nu\lambda\sigma}\gamma_\mu\gamma_\nu\hat\theta_\mu\hat\theta_\nu\Sigma_{\lambda\sigma},\\
&Q_{\mu\lambda,\mu\sigma}=g_{\mu\mu}\Sigma_{\mu\lambda}\gamma_\mu\gamma_\nu\hat\theta_\mu\hat\theta_\nu, \quad Q_{\sigma\lambda,\mu\nu}=\varepsilon_{\mu\sigma\nu\lambda}g_{\mu\mu}g_{\nu\nu}\Sigma_1\gamma_\mu\gamma_\nu\hat\theta_\mu\hat\theta_\nu,\\
&Q_{\mu\nu,mn}=\Sigma_{\mu\nu}, \quad Q_{54,mm}=i\Gamma_4.
\end{aligned} \tag{4.2}$$

Here $Q_{kl,mn}$ are tensors which are antisymmetric w.r.t. permutations of the first pair of indices and symmetric w.r.t. permutations of the second pair of indices and whose diagonal elements are equal, i.e., $Q_{kn,ll}=Q_{kn,mm}$ for any l and m. The Greek indices runs over the values $0,1,2,3$ and no summation over repeated indices is assumed.

Let us notice that algebra (4.2) can be extended by the following 136 elements

$$\begin{aligned}
&Q_{4\mu,\lambda\sigma}=C\Gamma_\mu\hat\theta_{\mu\lambda\sigma}, \quad Q_{4\mu,45}=C\Gamma_\mu\hat\theta_{\nu\lambda\sigma}, \quad \nu,\lambda,\sigma\neq\mu,\\
&Q_{5\mu,\lambda\sigma}=C\Gamma_4\Gamma_\mu\hat\theta_{\mu\lambda\sigma}, \quad Q_{5\mu,45}=C\Gamma_4\Gamma_\mu\hat\theta_{\nu\lambda\sigma}, \quad \nu,\lambda,\sigma\neq\mu,\\
&Q_{\mu\nu,4\lambda}=\gamma_\mu\gamma_\nu\hat\theta_\mu\hat\theta_\nu\hat\theta_{\mu\nu\lambda}, \quad Q_{\mu\nu,5\lambda}=\gamma_\mu\gamma_\nu\hat\theta_\mu\hat\theta_\nu\hat\theta_{\mu\nu\lambda}\Gamma_4\Sigma_1,\\
&Q_{\mu\nu,45}=\gamma_\mu\gamma_\nu\hat\theta_{\lambda\sigma}, \quad Q_{54,\mu\nu}=i\Gamma_4\hat\theta_{\mu\nu}
\end{aligned} \tag{4.3}$$

where

$$\hat\theta_{\mu\nu}=1+\frac{1}{m}(1-i\gamma_4)(\gamma^\mu p_\mu+\gamma^\nu p_\nu)\hat\theta_\mu\hat\theta_\nu, \quad \mu\neq\nu,$$

$$\hat\theta_{\mu\nu\lambda}=1+\frac{1}{m}(1-i\gamma_4)\left(\gamma^\mu p_\mu+\gamma^\nu p_\nu+\gamma^\lambda p_\lambda\right)\hat\theta_\mu\hat\theta_\nu\hat\theta_\lambda, \quad \mu\neq\nu,\mu\neq\lambda,\nu\neq\lambda$$

(no sum over repeated indices).

These elements are new symmetries of equation (2.1), which cannot be expressed via commutators of symmetries (2.6), (4.1).

Operators (4.2), (4.3) form a basis of the 256-dimensional real invariance algebra of the Dirac equation, defined over the field of real numbers. This algebra is characterized by the following commutation relations

$$\begin{aligned}
&[Q_{kl,mn}, Q_{k'l',m'n'}] = \\
&\quad -2[\delta_{mm'}\left(g_{kk'}Q_{ll',nn'} + g_{ll'}Q_{kk',nn'} - g_{kl'}Q_{lk',nn'} - g_{lk'}Q_{kl',nn'}\right) + \\
&\quad \delta_{nn'}\left(g_{kk'}Q_{ll',mm'} + g_{ll'}Q_{kk',mm'} - g_{kl'}Q_{lk',mm'} - g_{lk'}Q_{kl',mm'}\right) + \\
&\quad \delta_{mn'}\left(g_{kk'}Q_{ll',nm'} + g_{ll'}Q_{kk',nm'} - g_{kl'}Q_{lk',nm'} - g_{lk'}Q_{kl',nm'}\right) + \\
&\quad \delta_{nm'}\left(g_{kk'}Q_{ll',mn'} + g_{ll'}Q_{kk',mn'} - g_{kl'}Q_{lk',mn'} - g_{lk'}Q_{kl',mn'}\right)] + \\
&\quad g_{mnm'n'gf}\left(g_{kk'}Q_{ll',gf} + g_{ll'}Q_{kk',gf} - g_{kl'}Q_{lk',gf} - g_{lk'}Q_{kl',gf}\right) - \\
&\quad \tfrac{1}{2}\left(\delta_{mm'}\delta_{nn'} + \delta_{mn'}\delta_{nm'}\right)\left(g_{kk'}Q_{ll',ss} + g_{ll'}Q_{kk',ss} - g_{kl'}Q_{lk',ss} - g_{lk'}Q_{kl',ss}\right)
\end{aligned} \tag{4.4}$$

where $m \neq m'$, $n \neq n'$, $g_{mnm'n'gf}$ is the totally symmetric unit tensor whose nonzero components correspond to noncoinciding values of all indices, and no sum over s, g, f.

For $m = n$ ve have

$$[Q_{kl,nn}, Q_{k'l',m'n'}] = -2\left(g_{kk'}Q_{ll',m'n'} + g_{ll'}Q_{kk',m'n'} - g_{kl'}Q_{lk',m'n'} - g_{lk'}Q_{kl',m'n'}\right). \tag{4.5}$$

Taking into account both the invariance algebra of the two-component Klein-Gordon equation and equivalence of this equation to the Dirac one, it is possible to shaw that relations (4.2), (4.4) present the maximally extended finite symmetry algebra for equation (2.1).

Thus we found the most extended symmetry algebra for the Dirac equation, which includes non-Lie symmetries (2.6), discrete symmetries (4.1) and their combinations. Symmetries (4.2) form its 120-dimensional subalgebra and also satisfy relations (4.4), (4.5) for the restricted set of values of indices defined in (4.3). Another its interesting subalgebra is 56-dimensional one formed by the linear (i.e., without complex conjugation) operators which are presented as follows:

$$\{Q_{\mu\nu,\lambda\sigma}, \quad Q_{\mu\nu,54}, \quad Q_{54,\mu\nu}, \quad Q_{54,54}\}, \quad \mu,\nu,\lambda,\sigma = 0,1,2,3. \tag{4.6}$$

V. Commutative discrete symmetries

In order to describe all possible reductions of the Dirac for a charged particle interacting with an external field we shall prove that operators (3.7). It is easy to see that the four-component complex wave function of (3.9)

$$\Psi = column(\Psi_1, \Psi_2, \Psi_3, \Psi_4), \quad \Psi_k = \Psi_k^{(1)} + i\Psi_k^{(2)}, \quad k = 1,2,3,4 \tag{5.1}$$

is equivalent to the eight–component real function

$$\tilde{\Psi} = column\left(\Psi_1^{(1)}, \Psi_1^{(2)}, \Psi_2^{(1)}, \Psi_2^{(2)}, \Psi_3^{(1)}, \Psi_3^{(2)}, \Psi_4^{(1)}, \Psi_4^{(2)}\right). \tag{5.2}$$

In representation (5.2) the Dirac matrices (2.2) are extended to the 8×8 real matrices

$$\gamma_0 \to \tilde{\gamma}_0 = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}, \quad \gamma_a \to \tilde{\gamma}_a = \begin{pmatrix} 0 & -\tilde{\sigma}_a \\ \tilde{\sigma}_a & 0 \end{pmatrix}, \quad \gamma_4 \to \tilde{\gamma}_4 = \begin{pmatrix} I_4 & 0 \\ 0 & I_4 \end{pmatrix},$$

where I_4 is the 4×4 unit matrix,

$$\tilde{\sigma}_1=\begin{pmatrix}0 & I_2\\ I_2 & 0\end{pmatrix},\quad \tilde{\sigma}_2=\begin{pmatrix}0 & -i\sigma_2\\ i\sigma_2 & 0\end{pmatrix},\quad \tilde{\sigma}_3=\begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}. \tag{5.3}$$

All symmetries (3.2), (3.4), (3.6) are then expressed via the 8×8 real matrices as follows

$$\Gamma_\mu\to\tilde{\Gamma}_\mu=\tilde{\gamma}_4\tilde{\gamma}_\mu\theta_\mu,\quad \Gamma_4\to\tilde{\Gamma}_4=\tilde{\gamma}_4\theta,\quad \Gamma_5\to\tilde{\Gamma}_\mu=\tilde{\gamma}_5,\quad \Gamma_6\to\tilde{\Gamma}_6=\tilde{\gamma}_6 \tag{5.4}$$

where

$$\tilde{\gamma}_5=\begin{pmatrix}0 & -\gamma_3\\ \gamma_3 & 0\end{pmatrix},\quad \tilde{\gamma}_6=\begin{pmatrix}0 & -\gamma_1\\ \gamma_1 & 0\end{pmatrix}.$$

The set of 64 matrices

$$\left\{\tilde{\gamma}_m,\quad \tilde{\gamma}_m\tilde{\gamma}_n,\quad \tilde{\gamma}_k\tilde{\gamma}_m\tilde{\gamma}_n,\quad \hat{I}\right\},\quad k,l,m,=0,1,\ldots 6 \tag{5.5}$$

forms a basis in the space of real matrices of dimension 8×8, i.e., the basis of the algebra $gl(8,R)$. Relations (3.2), (3.4), (3.6), (5.4) establish the isomorphism of this algebra with the algebra of discrete symmetries of the Dirac equation (3.7). Using relations (5.4) we can represent these symmetries as operators with real matrix coefficients

$$\left\{\tilde{\Gamma}_m,\quad \tilde{\Gamma}_m\tilde{\Gamma}_n,\quad \tilde{\Gamma}_k\tilde{\Gamma}_m\tilde{\Gamma}_n,\quad \hat{I}\right\},\quad k,l,m,=0,1,\ldots 6. \tag{5.6}$$

Diagonalizing various of operators (5.6) we can reduce the corresponding Dirac equation (3.9) to uncoupled subsystems [8]. We note that only symmetric matrices are diagonalizable over the field of real numbers. Set (5.6) includes the 36 symmetric operators:

$$\left\{\tilde{\Gamma}_\alpha,\ \tilde{\Gamma}_\alpha\tilde{\Gamma}_a,\quad \tilde{\Gamma}_\alpha\tilde{\Gamma}_\beta\tilde{\Gamma}_a,\quad \tilde{\Gamma}_1\tilde{\Gamma}_2\tilde{\Gamma}_3,\quad \hat{I},\right\},\quad a=1,2,3,\ \alpha,\beta=0,4,5,6. \tag{5.7}$$

To describe all possible reductions of equation (3.9) is then equivalent to find all sets of commuting operators (5.7). Unfortunately, these operators do not form a Lie algebra and, thus, it is not possible to use standard algebraic methods for classifying these reductions. To overcome this difficulty we multiply any element of (5.7) by $\hat{\imath}=\Gamma_5\Gamma_6$ and obtain a basis of algebra the $sp(4,R)\subset gl(8,R)$:

$$\{\hat{\imath}\tilde{\Gamma}_\alpha,\quad \hat{\imath}\tilde{\Gamma}_\alpha\tilde{\Gamma}_a,\quad \hat{\imath}\tilde{\Gamma}_\alpha\tilde{\Gamma}_\beta\tilde{\Gamma}_a,\quad \hat{\imath}\tilde{\Gamma}_1\tilde{\Gamma}_2\tilde{\Gamma}_3,\quad \hat{\imath}\}. \tag{5.8}$$

Indeed, formula (5.8) defines 36 linearly independent real matrices whose products with the skew symmetric matrix $\hat{\imath}$, $\hat{\imath}^2=-1$, are hermitian.

Taking into account that algebra $sp(4,R)$ is of rank 3, we conclude that its commuting functionally independent elements form doublets or triplets only. The same is true for set (5.7), moreover, all its commuting subsets are given by the following formulae

$$\{\tilde{\Gamma}_\alpha,\tilde{\Gamma}_\beta\tilde{\Gamma}_a\},\quad \{\tilde{\Gamma}_\alpha\tilde{\Gamma}_a,\tilde{\Gamma}_\beta\tilde{\Gamma}_b\},\quad \{\tilde{\Gamma}_\sigma\tilde{\Gamma}_\alpha\tilde{\Gamma}_b,\tilde{\Gamma}_\sigma\tilde{\Gamma}_\beta\tilde{\Gamma}_c\},\quad \{\tilde{\Gamma}_\mu\tilde{\Gamma}_\nu\tilde{\Gamma}_a,\tilde{\Gamma}_1\tilde{\Gamma}_2\tilde{\Gamma}_3\}, \tag{5.9}$$

$$\{\tilde{\Gamma}_\mu,\tilde{\Gamma}_\nu\tilde{\Gamma}_a,\tilde{\Gamma}_\lambda\tilde{\Gamma}_b\},\quad \{\tilde{\Gamma}_6,\tilde{\Gamma}_0\tilde{\Gamma}_a,\tilde{\Gamma}_4\tilde{\Gamma}_b\},\quad \{\tilde{\Gamma}_\sigma\tilde{\Gamma}_\alpha\tilde{\Gamma}_a,\tilde{\Gamma}_\sigma\tilde{\Gamma}_\beta\tilde{\Gamma}_b,\tilde{\Gamma}_1\tilde{\Gamma}_2\tilde{\Gamma}_3\} \tag{5.10}$$

where $\alpha,\beta,\sigma=0,4,5,6,\ \mu,\nu,\lambda=0,4,5,\ a,b,c=1,2,3,\ \alpha\neq\beta,\ \beta\neq\sigma,\ \sigma\neq\alpha,\ \mu\neq\nu,$ $\nu\neq\lambda,\ \lambda\neq\mu,\ a\neq b,\ b\neq c,\ c\neq a.$

It is not difficult to calculate that there are 105 doublets of commuting operators (5.9) and 48 triplets (5.10). To reduce these numbers we use the equivalence transformation generated by $U = \dfrac{1}{\sqrt{2}}(1 + \Gamma_5\Gamma_6)$ (such an operator commutes with *arbitrary* vector–potential) and also delete the sets which include Γ_5 (they are valid only for trivial zero vector-potentials). As a result we obtain 51 doublets [8]

$$\begin{aligned}&\{\tilde{\Gamma}_4, \tilde{\Gamma}_\mu\}, \quad \{\tilde{\Gamma}_0, \tilde{\Gamma}_\mu\tilde{\Gamma}_a\} \quad \{\tilde{\Gamma}_\nu\tilde{\Gamma}_b, \tilde{\Gamma}_\mu\tilde{\Gamma}_a\}, \quad \{\tilde{\Gamma}_5\tilde{\Gamma}_b, \tilde{\Gamma}_6\tilde{\Gamma}_a\},\\ &\{\tilde{\Gamma}_\mu\tilde{\Gamma}_4\tilde{\Gamma}_a, \tilde{\Gamma}_1\tilde{\Gamma}_2\tilde{\Gamma}_3\}, \quad \{\Gamma_\mu\Gamma_\nu\Gamma_a, \Gamma_{\mu'}\Gamma_{\nu'}\Gamma_a\}, \quad a > b\end{aligned} \tag{5.11}$$

and 27 triplets:

$$\{\tilde{\Gamma}_\nu, \tilde{\Gamma}_\mu\tilde{\Gamma}_a, \tilde{\Gamma}_\lambda\tilde{\Gamma}_b\}, \quad \lambda \neq \mu, \quad \mu \neq \nu, \quad \nu \neq \lambda, \quad a \neq b. \tag{5.12}$$

VI. Reduction of the Dirac equation

Operators (5.7) and sets of operators (5.11), (5.12) can be used to reduce the Dirac equation (3.9), provided parities of A_μ are such that these operators are symmetries. Here we present an example of such a reduction.

Let parities of the vector-potential $A = (A_0, A_1, A_2, A_3)$ be described by the following relations

$$A(\theta\theta_0 x) = \theta\theta_0 A(x), \quad A(\theta_2 x) = \theta_2 A(x), \quad A(\theta\theta_3 x) = -\theta\theta_3 A(x),$$

where $\theta, \theta_0, \theta_2$, and θ_3 are reflections defined by relations (3.1). Then equation (3.9) admits three commuting symmetries

$$\Gamma_0 = C\gamma_0\hat{\theta}_0, \quad \Gamma_4\Gamma_2 = \gamma_4\gamma_2\hat{\theta}_2, \quad \Gamma_5\Gamma_3 = C\gamma_3\theta\hat{\theta}_3, \tag{6.1}$$

where $C, \hat{\theta}, \hat{\theta}_0, \hat{\theta}_2$ and $\hat{\theta}_3$ are operators defined in (3.3), (3.4).

Operators (6.1) form one of 27 triplets given by formula (5.12).

The commuting symmetries (6.1) can be simultaneously diagonalized. Indeed, using the operator

$$\begin{aligned}W &= \frac{1}{4}(1 + \gamma_0 C)\left(1 + \gamma_1\gamma_3\hat{\theta}\hat{\theta}_1\right)\left(1 + \gamma_0\gamma_4 C\hat{\theta}_1\hat{\theta}_3\right)\left(1 + \gamma_2\hat{\theta}_2\right),\\ W^{-1} &= \frac{1}{4}\left(1 - \gamma_2\hat{\theta}_2\right)\left(1 - \gamma_0\gamma_4 C\hat{\theta}_1\hat{\theta}_3\right)\left(1 - \gamma_1\gamma_3\hat{\theta}\hat{\theta}_1\right)(1 - \gamma_0 C)\end{aligned}$$

we obtain

$$W\Gamma_0 W^{-1} = \gamma_0\gamma_3 c, \quad W\Gamma_2\Gamma_4 W^{-1} = -i\gamma_4 c, \quad W\Gamma_5\Gamma_3 W^{-1} = c \tag{6.2}$$

where c is the complex conjugation operator.

The transformed Dirac operator $L' = WLW^{-1}$ has to commute with operators (6.2):

$$[L', c] = [L', \gamma_0\gamma_3] = [L', i\gamma_4] = 0. \tag{6.3}$$

It follows from (2.2), (6.3) that L' is a direct sum of four orthogonal operators with *real* coefficients. Thus the related equation

$$L'\psi' = 0, \quad \psi' = W\psi \tag{6.4}$$

is decoupled to eight subsystems for real and imaginary parts of ψ'. Indeed, by direct calculation we obtain

$$L' = i\gamma_4 \left(\hat{\theta}_0\hat{\theta}_3 \frac{\partial}{\partial x_0} - ce\hat{\theta}\hat{\theta}_0 A_0\right) - \gamma_0\gamma_3 \left(c\hat{\theta}_0\hat{\theta}_3 \frac{\partial}{\partial x_1} - e\hat{\theta}\hat{\theta}_0 A_1 + c\hat{\theta}_2 \frac{\partial}{\partial x_2} + e\hat{\theta}_0\hat{\theta}_1 A_3\right) + c\hat{\theta}_0\hat{\theta}_1 \frac{\partial}{\partial x_2} + e\hat{\theta}_2 A_2 \tag{6.5}$$

and equation (6.4) does decouple.

Explicit reductions which correspond to other sets of symmetries (5.12) can be found in analogy with (6.1)-(6.5).

VII. Discussion

We have shown that the discrete symmetries of the Dirac equation generate very rich algebraic structures, namely, the algebra $gl(8, R)$ (in the case of linear and antilinear symmetries (3.7)) and $so(3, 3)$ (in the case of linear symmetries (3.8)). The important for applications subalgebra $sp(4, R)$ is generated by operators (5.8). Finally, algebra (3.7) includes the subalgebra $so(1, 2)$ formed by operators $\{\Gamma_5, \Gamma_6, \Gamma_5\Gamma_6\}$. The invariance of the Dirac equation w.r.t. this subalgebra was discovered by Pauli, Gursey, Pursey and Plebanski [10].

We have also demonstrated that the discrete symmetries discussed in paper [8] can be unified with the non-Lie symmetries (2.6) [7] in frames of algebra (4.2), (4.3) which is the maximal finite Lie algebra of symmetries of the Dirac equation, known until now.

The distinguishing feature of symmetries (4.2), (4.3) is that all of them are involutive. This makes it possible to use them for reducing equation (3.9) (and of the Dirac-Pauli equation). A class of such reductions to uncoupled subsystems was found and discussed in paper [8]. Here this class is extended by including reductions to eight uncoupled subsystems.

One more perspective in the use of symmetries (4.2), (4.3) is connected with searching for hidden supersymmetries of equation (3.9) [8]. We plan to present an extension of the present results elsewhere.

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