On connection between the two-body Dirac oscillator and Kemmer oscillators

M. Bednar, J. Ndimubandi, and A.G. Nikitin

Abstract: We relate three different approaches to relativistic bosonic oscillators, and find non-Lie constants of motion for them. We show that all these approaches admit hidden parasupersymmetry and point out reducibility of the two-body Dirac oscillator.

Résumé : Nous comparons trois approches différentes au problème de l'oscillateur bosonique relativiste et trouvons des constantes du mouvement qui ne sont pas du type de Lie. Nous montrons que les trois méthodes admettent une parasupersymétrie cachée et soulignons que l'oscillateur de Dirac à deux corps est réductible.
[Traduit par la rédaction]

1. Introduction

The problem of the quantum relativistic harmonic oscillator has root in the attempts to find relativistic equations for interacting particles. The aim of such a question at the first quantized level is to look for relativistic wave equations which, in the nonrelativistic limit, give rise to the quantum harmonic oscillator.

A well-known example of such an equation is the Dirac oscillator [1]. Its modern treatment was proposed by Moshinsky and Sczcepaniak [2]; it has stimulated a lot of investigations connected with symmetries [3], supersymmetries (SUSY) [4, 5], possible generalizations to the cases of multiparticle systems [6–8], and particles of higher spins [9–12]. In particular, let us mention that the spin-one equations with oscillatorlike potentials [9] admit a relatively new kind of symmetry called parasupersymmetry (PSUSY) [13, 14].

The main goal of ref. 9 was to describe a consistent multidimensional physical model that generates PSUSY. As we prove in the following, it happens that such a model was proposed earlier by Moshinsky et al. [6] who were not considering symmetry aspects of their equations. In a relatively recent paper [11] the so called Duffin–Kemmer–Petiau (DKP) oscillator was proposed. We find that it also admits hidden PSUSY.

Of course it is interesting to compare the above mentioned models, which are selected by the same type of symmetry. In the present paper we analyze connections between them and demonstrate that

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(*i*) the two-body Dirac oscillator [6] is reducible and generates equations of motion for the pararelativistic oscillators [9];
(*ii*) the eigenvalue problems for the pararelativistic oscillator and DKP oscillator [11] are connected by the unitary transformation;
(*iii*) the pararelativistic oscillator is equivalent to the DKP oscillator supplemented by the additional constraint;
(*iv*) all the above-mentioned equations admit a common non-Lie constant of motion;
(*v*) these equations admit hidden PSUSY.

2. Bosonic oscillators

The two different possibilities to generalize the Dirac oscillator to the cases of spin-zero and spin-one particles were used in refs. 9 and 11. We start with the DKP oscillator proposed recently in ref. 11. Using the Heaviside units $h = c = 1$ and the notations of paper [9] we represent the corresponding equations in the form

$$L\psi \equiv \left(\beta_0 p_0 - \beta_a(p_a - i\omega\eta x_a) - m\right)\psi = 0 \tag{1}$$

where $p_0 = i\dfrac{\partial}{\partial x_0}$; $p_a = i\dfrac{\partial}{\partial x_a}$; $\eta = 2\beta_0^2 - 1$; $a = 1, 2, 3$; β_0 and β_{a} are 5×5 or 10×10 DKP matrices satisfying the algebra

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\nu\lambda}\beta_\mu, \qquad G = \{g_{\mu\nu}\} = diag\,(1, -1, -1, -1) \tag{2}$$

Equation (1) includes a specific interaction term linear in x. Setting $\omega = 0$, we come to the free KDP equation in the covariant formulation [15]. Moreover, the case of 5×5 matrices corresponds to scalar (spin-zero) particles while vector (spin-one) particles are described by (1) with 10×10 matrices.

The other (Hamiltonian) formulation of the DKP equation was used in the earlier paper [9] where the following pararelativistic oscillator was proposed

$$L_1\psi \equiv \left(i\frac{\partial}{\partial x_0} - H\right)\psi = 0, \qquad H = \left[\beta_{0,}\ \beta_a\right]\left(p_a - i\omega\eta x_a\right) + \beta_0 m \tag{3a}$$

$$L_2\psi \equiv (H\beta_0 - m)\psi = 0 \tag{3b}$$

Comparing these formulations we recognize that for the spin-zero case (1) is equivalent to system (3). Indeed, multiplying (3*a*) from the left by β_0 and adding the result to (3*b*) we come to (1). On the other hand, by multiplying (1) from the left by $(1 - \beta_0^2)$ and $\left\{\left[1 - \beta_a\dfrac{(p_a - i\eta\omega x_a)}{m}\right]\beta_0 - \dfrac{(1 - \beta_0^2)p_0}{m}\right\}$ we come to (3*b*) and (3*a*) correspondingly. In other words, (1) is a differential consequence of system (3), and system (3) is a differential consequence of (1).

For the case of vector particles (1) again is a differential consequence of the system (3) (this statement can be verified in analogy with the spin-zero case), and so any solution of system (3) satisfies (1). But now system (3) is not a consequence of (1), since it includes an implicit condition that is absent in (1) (this fact is proved in the following). In other words, there exist solutions of (1) that do not satisfy (3) and so these equations are not equivalent.

To understand the difference between (1) and (3) in the case of 10×10 β-matrices we consider the compatibility condition for system (3)

$$\left[L_1, L_2\right]\psi = 0 \tag{4}$$

Equation (4) is a differential consequence of (3). Evaluating the commutator we obtain

$$\left[L_1, L_2\right] = -\beta_0 \boldsymbol{\beta}\cdot\boldsymbol{\pi} L_2 - \boldsymbol{\beta}\cdot\boldsymbol{\pi}\beta_0\boldsymbol{\beta}\cdot\boldsymbol{\pi}, \qquad \boldsymbol{\pi} = \boldsymbol{p} - i\omega\eta\boldsymbol{x} \qquad \boldsymbol{\beta}\cdot\boldsymbol{\pi} = \beta_a\pi_a$$

thus, in accordance with ($3b$), the condition (4) can be reduced to the form

$$L_3\psi \equiv \boldsymbol{\beta}\cdot\boldsymbol{\pi}\beta_0\boldsymbol{\beta}\cdot\boldsymbol{\pi}\psi \equiv -2\left(1-\beta_0^2\right)\left[\beta_5, \boldsymbol{\beta}\right]\cdot\boldsymbol{L}\psi = 0 \tag{5}$$

where

$$\beta_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\beta^\mu\beta^\nu\beta^\rho\beta^\sigma, \qquad \boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{p}$$

Thus the compatibility condition for system (3) leads to the nontrivial constraint (5), which is absent in the approach based on the single equation (1). In other words, system (3) is not equivalent to (1) but can be reduced to the symmetrical form (1) supplemented by the additional condition (5).

It is not difficult to verify that condition (5) is compatible with (3), i.e., commutators of operators L_1, L_2, and L_3 reduce to linear combination of these operators. In other words, relation (4) is the necessary and sufficient condition for compatibility of the system (3).

Relation (5) admits a clear physical interpretation. To formulate it we notice that both (1) and (3), admit the hidden constant of motion

$$Q = \eta\left[2(\boldsymbol{S}\cdot\boldsymbol{J})^2 - 2\boldsymbol{S}\cdot\boldsymbol{J} - \boldsymbol{J}^2\right] + \boldsymbol{J}^2 \tag{6}$$

(where $\boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{p} + i\boldsymbol{\beta}\times\boldsymbol{\beta}$) whose eigenvalues are $q = 0$ or $2j(j+1)$, $j = 0,1,...$ [16]. We denote the corresponding eigenvectors by ψ_q. Indeed, Q commutes with $\boldsymbol{L}$, L_1, and L_2, so its eigenvalues are good quantum numbers that can be used to label solutions of (1) or (3). Moreover, if we define parity states in accordance with ref. 11, then the eigenvalue $q = 0$ corresponds to the parity $(-1)^j$, and $q = 2j(j+1)$ corresponds to the parity $-(-1)^j$.

Inasmuch as $L_3^2 \equiv -2\left(1-\beta_0^2\right)Q$, condition (5) implies that

$$\left(1-\beta_0^2\right)Q\psi = 0 \tag{7}$$

For the eigenvectors $\psi_{q=0}$ this condition is satisfied automatically, but for $\psi_{2j(j+1)}$ (3) and (7) have only trivial solutions. Thus, condition (5) suppresses eigenvectors of Q corresponding to nonzero eigenvalues. In other words, system (3) admits solutions with the definite ("natural" [11]) parities only, while (1) admits solutions with both types of parities.

3. The two-body Dirac oscillator

Let us investigate connections of (1) and (3) with the two-body Dirac oscillator [6–8] that we write in the form

$$H\psi = p_0\psi, \qquad H = \frac{1}{\sqrt{2}}(\boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^2)\,(\boldsymbol{p} - i\boldsymbol{x}\omega B) + m(\beta^1 + \beta^2) \tag{8}$$

where $\{\boldsymbol{\alpha}^1, \beta^1\}$ and $\{\boldsymbol{\alpha}^2, \beta^2\}$ are commuting sets of 16×16 Dirac matrices [6]

$$\boldsymbol{\alpha}^1=\begin{pmatrix}0 & \boldsymbol{\sigma}^1\\ \boldsymbol{\sigma}^1 & 0\end{pmatrix}\otimes\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},\qquad \boldsymbol{\alpha}^2=\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\otimes\begin{pmatrix}0 & \boldsymbol{\sigma}^2\\ \boldsymbol{\sigma}^2 & 0\end{pmatrix}$$

$$\beta^1=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\otimes\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},\qquad \beta^2=\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\otimes\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \tag{9}$$

and $B=\beta^1\beta^2$.

For $\omega=1$ (8) is reduced to equation (4.1) of ref. 6.

It happens that (1) and (3) are closely related to (8). Indeed, let us denote

$$\alpha_a^1=\gamma_0^{(1)}\gamma_a^{(1)},\qquad \alpha_a^2=\gamma_0^{(2)}\gamma_a^{(2)},\qquad \beta^1=\gamma_0^{(1)},\qquad \beta^2=\gamma_0^{(2)},\qquad \beta_\mu=\frac{1}{2}\left(\gamma_\mu^{(1)}+\gamma_\mu^{(2)}\right),\qquad \mu=0,1,2,3 \tag{10}$$

where $\{\gamma_\mu^{(1)}\}$and $\{\gamma_\mu^{(2)}\}$are commuting sets of Dirac matrices, satisfying the relations

$$\gamma_\mu^{(i)}\gamma_\nu^{(i)}+\gamma_\nu^{(i)}\gamma_\mu^{(i)}=2g_{\mu\nu},\qquad \left[\gamma_\mu^{(1)},\gamma_\nu^{(2)}\right]=0,\qquad i=1,2 \tag{11}$$

Then, making the similarity transformation $\psi\to\psi'=\gamma_0^{(2)}\psi$, $H\to H'=\gamma_0^{(2)}H\gamma_0^{(2)}$, we reduce (8) to the form

$$p_0\psi'=H'\psi',\qquad H'=\sqrt{2}\left[\hat{\beta}_0,\hat{\beta}_a\right]\left(p_a-i\hat{\eta}\omega x_a\right)+2m\hat{\beta}_0,\ \ \hat{\eta}=2\hat{\beta}_0^2-1 \tag{12}$$

In accordance with (11) the matrices β_μ (10) satisfy the DKP algebra (2). Moreover, these matrices can be reduced to a direct sum of 10×10, 5×5, and 1×1 matrices realizing irreducible representations of the Kemmer algebra (2) (moreover, the one-dimensional representation is trivial and is realized by zero elements) [15]. Indeed, starting with the realization (9) and making the transformation $\beta_\mu\to\beta'_\mu=U\beta_\mu U^\dagger$ where

$$\begin{aligned}U=(1-i)\Big[(&e_{1,1}+e_{1,13}+e_{2,2}+e_{2,14}+e_{3,3}+e_{3,15}-e_{10,8}+e_{10,12}-e_{11,4}-e_{11,16}+e_{13,15}\\ &-e_{13,9}+e_{14,6}-e_{14,10}+e_{15,7}-e_{15,11})/2\Big]\\ +(1+i)\Big[(&-e_{4,5}-e_{4,9}-e_{5,6}-e_{5,10}-e_{6,7}-e_{6,11}-e_{7,1}+e_{7,13}-e_{8,2}+e_{8,14}-e_{9,3}\\ &+e_{9,15}-e_{12,4}+e_{12,16}+e_{16,8}+e_{16,12})/2\Big]\end{aligned} \tag{13}$$

we obtain

$$\beta'=\begin{pmatrix}\beta_\mu^{(10)} & \cdot & \cdot\\ \cdot & \beta_\mu^{(5)} & \cdot\\ \cdot & \cdot & 0\end{pmatrix} \tag{14}$$

where $\beta_\mu^{(10)}$ and $\beta_\mu^{(5)}$ are the 10×10 and 5×5 KDP matrices

$$\beta_0^{(10)}=i(e_{1,7}+e_{2,8}+e_{3,9}-e_{7,1}-e_{8,2}-e_{9,3})$$

$$\beta_1^{(10)}=-i(e_{1,10}-e_{5,9}+e_{6,8}+e_{8,6}-e_{9,5}+e_{10,1})$$

$$\beta_2^{(10)}=-i(e_{2,10}+e_{4,9}-e_{6,7}-e_{7,6}+e_{9,4}+e_{10,2})$$

$$\beta_3^{(10)}=-i(e_{3,10}-e_{4,8}+e_{5,7}+e_{7,5}-e_{8,4}+e_{10,3})$$

$$\beta_0^{(5)}=-i(e_{1,2}-e_{2,1}),\qquad \beta_1^{(5)}=i(e_{1,3}+e_{3,1}),\qquad \beta_2^{(5)}=-i(e_{1,4}+e_{4,1}),\qquad \beta_3^{(5)}=i(e_{1,5}+e_{5,1})$$

e_{kl} denotes the unit matrix elements placed on the kth row and lth column and the dots denote zero matrices. In this way system (12) is reduced to uncoupled subsystems of ten equations for spin-one states, five equations for spin-zero states, and one equation with the trivial (zero) Hamiltonian.

We conclude from (3a), (12), and (14) that up to the constant factors $\sqrt{2}$ and 2 (which can be reduced to unities by changing of scaling), the two-body Dirac oscillator [6, 7] is reduced to a direct sum of the equations of motion (3a) for the DKP oscillators [9, 10]. Nevertheless, the DKP oscillator models include the additional condition (3b) and so are not equivalent to reduced versions of (12).

4. Connections between energy spectra

Now it is interesting to compare the mass spectra generated by (12) and the energy spectra of the DKP oscillator. The corresponding eigenvalue problems have the form

$$H'\psi' = \mu\psi' \tag{15}$$

where H' is the operator given in (12), and

$$\beta_0 E\psi = \left[\beta_a(p_a - i\omega\eta x_a) + m\right]\psi \tag{16}$$

It was demonstrated in ref. 6 that the eigenvalues μ satisfy the relations

$$\mu^2 = 0, \qquad s = 0, 1 \tag{17}$$

$$\mu^2 = 2M^2 + 4N\omega, \qquad s = 0 \tag{18a}$$

$$\mu^2 = 2M^2 + 4\,(N+1)\,\omega, \qquad s = 1 \tag{18b}$$

or

$$\mu^2\left[\mu^2 - 4(N+1)\omega - 2M^2\right]\left[\mu^2 - 4(N+2)\omega - 2M^2\right] = 32M^2\omega^2 j(j+1), \qquad s = 1 \tag{18c}$$

Here $M = \sqrt{2}m$, N is the main quantum number, $N = 2n + j$, $j = 0, 1, \ldots,$ $n = 0, 1, \ldots,$ and the relations (18b) and (18c) correspond to the parities $(-1)^j$ and $-(-1)^j$.

The eigenvalues E for (16) have the form (refer to equations (21), (31), (37) in ref. 11)

$$E^2 = m^2 + 2N\omega, \qquad s = 0 \tag{19a}$$

$$E^2 = m^2 + 2(N+1)\omega, \qquad s = 1 \tag{19b}$$

or are roots of the algebraic equation

$$m^2(E^2 - m^2 - 2(N+1)\omega)\,(E^2 - m^2 - 2(N+2)\omega) = 4E^2\omega^2 J(J+1) \tag{19c}$$

Moreover, (19b) and (19c) correspond to the parities $(-1)^j$ and$-(-1)^j$, respectively.

In accordance with the results given in Sect. 2 the eigenvalue problem for the pararelativistic oscillator (3) can be represented in the form (16) but the corresponding solutions have to be constrained by the additional condition (7). This condition nullifies solutions corresponding to the parities $-(-1)^j$ and so the admissible eigenvalues E has the form (19a) and (19b).[3]

Formulae (19) reduce to the form (18) if we denote

[3] Nonrelativistic energy eigenvalues for pararelativistic oscillator (3) found in ref. 10 (refer to formulae (3.31a) and (3.31b) therein) correspond to solutions that do not satisfy the compatibility condition (5) and so are forbidden.

$$E = iM, \qquad m = -i\mu/\sqrt{2} \tag{20}$$

This observation is in accordance with the fact that the change (20) together with the similarity transformation

$$\psi \to \psi' = \exp(i\beta_0\pi/2)\psi, \qquad L \to L' = \exp(i\beta_0\pi/2)L\exp(-i\beta_0\pi/2) \tag{21}$$

reduces (16) to the form (15). This transformation is nothing but a transition to the new realization of β-matrices

$$\beta'_0 = \beta_0, \qquad \beta'_a = i[\beta_0, \beta_a]$$

which also satisfy the DKP algebra (2).

5. Hidden PSUSY

All the above equations have a specific feature that is called hidden PSUSY [9]. Indeed, the Hamiltonians (3*a*), (12), and the operator L of (1) have the following structure

$$H = Q_1 + \beta_0 m, \qquad Q_1 = \left[\beta_0, \beta_a\right]\left(p_a - i\omega\eta x_a\right) \tag{22}$$

$$H = \sqrt{2}\,\hat{Q}_1 + 2\beta_0 m, \qquad \hat{Q}_1 = \left[\hat{\beta}_0, \hat{\beta}_a\right]\left(p_a - i\omega\eta x_a\right) \tag{23}$$

$$L = \beta_0 p_0 - i\tilde{Q}_1 - m, \qquad \tilde{Q}_1 = -i\beta_a\left(p_a - i\omega\eta x_a\right) \tag{24}$$

where Q_1, $\hat{Q}_1$, and $\tilde{Q}_1$ are parasupercharges, i.e., operators that satisfy the following double commutation relations [14]

$$\left[[Q_i, Q_j], Q_k\right] = 4\left(\delta_{jk}Q_i - \delta_{ik}Q_j\right)H_{\mathrm{PSS}}, \qquad [H_{\mathrm{PSS}}, Q_i] = 0, \qquad i, j, k = 1, 2 \tag{25}$$

For case (22) the second parasupercharge Q_2 and the corresponding (parasuper)Hamiltonian H_{PSS} have the form

$$Q_2 = \frac{i}{2}\left[\eta, Q_1\right] = i\left(\beta_0\beta_a + \beta_a\beta_0\right)\left(p_a - i\omega\eta x_a\right), \qquad H_{\mathrm{PSS}} = p^2 + \omega^2 x^2 - \left(\eta - 2 + 2\beta_5^2\right)\omega \tag{26}$$

The operators Q_2 and H_{PSS} related to (23) can be obtained from (26) by changing $\beta_a \to \hat{\beta}_a$ and $\beta_5 \to \hat{\beta}_5$.

The parasupercharge Q_1 (24) also satisfies relations (25) together with

$$\tilde{Q}_2 = i\eta\tilde{Q}_1, \qquad \tilde{H}_{\mathrm{PSS}} = p^2 + \omega^2 x^2 + \left(\eta - 2 + 2(\beta_0^2 - \beta_5^2)^2\right)\omega$$

The representation analogous to (22) has already been recognized for the Dirac Hamiltonian (either for a free [17] or for an interacting [18] particle). Such a representation is useful for searching for parasupersymmetries of the approximate nonrelativistic Hamiltonians and for constructing of the Foldy–Wouthuysen transformation [9, 10].

6. Discussion

Thus we demonstrated that the two-body Dirac oscillator can be reduced to the direct sum of (3*a*), and that the pararelativistic oscillator is equivalent to the DKP oscillator supplemented by the subsidiary condition (5). The related eigenvalue problems are connected by relations (20) and (21), ensuring the eigenvalues for the pararelativistic oscillator are given in (19*a*) and (19*b*).

The absence of the subsidiary condition (3*b*) deleting nonphysical solutions and appearence of the trivial Hamiltonian (refer to (12) and (14)) is the source of nonphysical eigenvalues (17). Let us recall that the problems connected with zero-energy solutions for two-particle equations were discussed elsewhere [19].

In accordance with the above, it is interesting to formulate reasonable constraints for the two-body oscillator that avoid the nonphysical solutions. Such constraints can be chosen in the form of relation (3*b*) (where H is the Hamiltonian (8), $\beta_0 = \beta_1 + \beta_2$, β_1 and β_2 are matrices defined in (9)), supplemented by the following condition:

$$\left[1 - \left(1 + \gamma^{(1)}_\mu \gamma^{(2)\mu}\right)\left(1 + 2\gamma^{(1)}_\mu \gamma^{(2)\mu}\right) + \left((3 + 2\gamma^{(1)}_\mu \gamma^{(2)\mu}\right)\gamma^{(1)}_\mu \gamma^{(2)\mu}\right]\psi = 0 \qquad (27)$$

where γ-matrices are defined in accordance with (10).

The operator in the square brackets is the projector to the subspace corresponding to the trivial (zero) β-matrices, refer to (14). The system of equations (8), (3*b*), and (27) is compatible and does not admit nonphysical solutions with $E = 0$. Refer to ref. 20 for more about constraints for two-particle equations.

We notice that a new version of the two-body Dirac oscillator proposed in ref. 8 (refer to equation (27) therein) also reduces to noncoupled subsystems of 10, 5, and 1 equations in the realization (10), (14).

All of (1), (3), and (12) have an evident symmetry under the rotation group O(3). Moreover, they are invariant with respect to the parity transformation

$$\psi\,(x_0, \boldsymbol{x}) \rightarrow \eta\;\psi\,(x_0, -\boldsymbol{x})$$

where η is the same matrix as used in (1), (3), and (12). An effective algorithm for construction of hidden constants of motion for such equations was proposed in ref. 16. Using this algorithm we found the symmetry (6).

The next note is connected with the hidden parasupersymmetries of the equations considered. Their presence for the pararelativistic oscillator was indicated in refs. 9 and 10. The above equivalence relations enable us to reformulate the results given in refs. 9 and 10 for the DKP and two-body oscillators.

Finally we point out that, in contrast with the Dirac oscillator [1], the equations considered above cannot be represented in a covariant form with an anomalous (Pauli) interaction.[4] Nevertheless, the covariant equations

$$\left(\beta^\mu p_\mu - m + \omega g S_{\mu\nu} F^{\mu\nu}\right)\psi = 0$$

where

$$S_{\mu\nu} = i\left[\beta_\mu, \beta_\nu\right], \qquad g = a + b\beta_5^2, \qquad F^{\mu\nu} = n^\mu x^\nu - n^\nu x^\mu$$

a, b, and n^μ are numerical parameters. The case $n^1 = n^2 = n^3 = 0$ also generates oscillatorlike spectra and so presents alternative possibilities in the generalization of the Dirac oscillator to the cases of scalar or vector particles. We will analyse these possibilities elsewhere.

[4] The representation proposed in ref. 10 (refer to equations (3*a*) and (3*b*) therein) for the pararelativistic oscillator is not covariant inasmuch as the corresponding Pauli term is not a Lorentz scalar but is a zero component of a four vector.

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