On extended supersymmetries and parasupersymmetries

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It is well known that supersymmetry has great perspectives in many branches of physics and mathematics. But till now the following fundamental question is not answered: if this fine kind of symmetry does be realized in Nature.

With some stipulations, a positive answer for this question was obtained on quantum mechanical level. Namely, it was recognized, that a number of realistic quantum mechanical problems generate exact SUSY, refer, e.g. to survey [1]. But at my knowledge, any problem with extended SUSY was not indicated yet.

In this topic we present a number of quantum mechanical problems generating extended SUSY and the relatively new kind of symmetry called parasupersymmetry (PSUSY) [2,3].

1. Let us start with the ordinary SUSY quantum mechanics whose equations of motion

$$H\Psi = E\Psi \tag{1}$$

admit two symmetry operators (supercharges) Q_1, Q_2, moreover,

$$\{Q_a, Q_b\}_+ = 2\delta_{ab}H, \quad [Q_a, H] = 0, \quad a, b = 1, 2. \tag{2}$$

The standard generalization of Witten superalgebra (2) is

$$Q_1 = \frac{1}{\sqrt{2}}\left(\sigma_1 p + \sigma_2 W\right), \qquad Q_2 = \frac{1}{\sqrt{2}}\left(\sigma_2 p - \sigma_1 W\right) \tag{3}$$

where $p = -i\frac{\partial}{\partial x}$, σ_1, σ_2 are the Pauli matrices, $W = W(x)$ is a superpotential.

Proposition 1. *Let W is an odd function, i.e., $W(-x) = -W(x)$. Then the superalgebra (2), (3) is reducible.*

Proof. For odd superpotentials there exist the invariant operator I commuting with Q_a and H

$$I = \sigma_3 R \tag{4}$$

where R is the space reflection operator: $R\Psi(x) = \Psi(-x)$. Transforming I to the diagonal numeric matrix

$$I \to I' = UIU^\dagger = \sigma_3, \quad U = \tfrac{1}{2}\left(1 - i\sigma_2\right)\left(1 + i\sigma_2 R\right).$$

we find immediately that $Q'_a = UQ_aU^\dagger$ and $H' = UHU^\dagger$ are reduced to direct sums of two orthogonal operators.

Proposition 2. *Let W is an even function, i.e., $W(-x) = W(x)$. Then the superalgebra (2), (3) can be extended by including the third supercharge*

$$Q_3 = i\sigma_1 R Q_1. \tag{5}$$

Proof. Using the relations $[\sigma_1 R, Q_2] = \{\sigma_1 R, Q_1\} = 0; \quad (\sigma_1 R)^2 = 1$ one convinced himself that operators (3), (5) satisfy algebra (2) for $a = 1, 2, 3$.

These propositions show a way for searching for extended SUSY in quantum mechanics. Namely, we can hope to find extended SUSY if the investigated problem is characterized by potentials having definite parities.

2. Let us consider the Dirac particle interacting with a time independent external magnetic field. The corresponding equation of motion has the form

$$L\psi \equiv (\gamma_\mu \pi^\mu - m)\,\psi = 0, \tag{6}$$

where $\pi_0 = p_0 = i\frac{\partial}{\partial x_0}, \quad \pi_a = p_a - eA_a(\mathbf{x}), \quad \gamma_\mu$ are Dirac matrices.

Equation (6) is equivalent to the second-order equation for two- component function

$$\begin{array}{c}(\pi_\mu \pi^\mu - m^2 + 2e\mathbf{S}\cdot\mathbf{H})\,\hat{\Phi} = 0,\\ (1-\gamma_5)\,\hat{\Phi} = 0\end{array} \tag{7}$$

where $\mathbf{S} = \frac{i}{2}\gamma\times\gamma, \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \quad \mathbf{H} = i\pi\times\pi.$

Relations between ψ and $\hat{\Phi}$ have the form

$$\hat{\Phi} = V\psi, \quad \psi = V^{-1}\hat{P}hi, \quad V = 1 + (1-\gamma_5)\,\gamma_\mu\pi^\mu/m, V^{-1} = V(-\pi_\mu). \tag{8}$$

For γ_5 diagonal function $\hat{\Phi}$ has two non-zero components only which we denote by Φ. Moreover, Φ satisfies the following equation

$$\begin{array}{c}(p_0^2 - m^2)\,\Phi = \hat{H}\Phi,\\ \hat{H} = \pi^{\mathbf{2}} - e\sigma\cdot\mathbf{H}, \quad \sigma = (\sigma_1, \sigma_2, \sigma_3)\,.\end{array} \tag{9}$$

Two-component equation (9) is much more convenient for searching of extended SUSY than Dirac equation (6). On the other hand using (8) we can establish one-to-one correspondence between symmetries of equations (9) and (6).

For any vector-potential $\mathbf{A}$ there exist the following supercharge for (9):

$$Q = \sigma\cdot\pi, \quad Q^2 = \hat{H}. \tag{10}$$

In other words, equation (9) (and the corresponding equation (6)) always generate $N = 1$ SUSY.

To find additional supercharges we have to impose some restrictions for the vector-potential. To formulate them let us consider the eight dimensional group generated by reflections of spatial variables:

$$\begin{gathered} r_1\mathbf{x} = (-x_1, x_2, x_3)\,, \quad r_2\mathbf{x} = (x_1, -x_2, x_3)\,, \quad r_3\mathbf{x} = (x_1, x_2, -x_3)\,, \\ r_{12}\mathbf{x} = (-x_1, -x_2, x_3)\,, \quad r_{31}\mathbf{x} = (-x_1, x_2, -x_3)\,, \quad r_{23}\mathbf{x} = (x_1, -x_2, -x_3)\,, \\ r\mathbf{x} = -\mathbf{x}, \quad I\mathbf{x} = \mathbf{x}. \end{gathered} \tag{11}$$

We say $\mathbf{A}(\mathbf{x})$ is an even function in respect with one of transformations (11) if one of the following relations is satisfied:

$$\mathbf{A}(r_a\mathbf{x}) = r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{ab}\mathbf{x}) = r_{ab}\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r\mathbf{x}) = r\mathbf{A}(\mathbf{x}) = -\mathbf{A}(\mathbf{x}). \tag{12}$$

$\mathbf{A}$ is add if the r.h.s. of (12) have the opposite signs.

If the vector-potential satisfies two or more relations (12) simultaneously, we come to the problem (9) with extended SUSY. Let

$$\mathbf{A}(r_1\mathbf{x}) = r_1\mathbf{A}(\mathbf{x}) \text{ and } \mathbf{A}(r_2\mathbf{x}) = r_2\mathbf{A}(\mathbf{x}), \tag{13}$$

then equation (9) admits three supercharges:

$$Q_1 = iR_1\sigma\cdot\pi, \quad Q_2 = iR_2\sigma\cdot\pi, \quad Q_3 = \sigma\cdot\pi. \tag{14}$$

Here and in the following R denote space reflection transformations for spinors Φ:

$$R_a\Phi(\mathbf{x}) = \sigma_a\Phi(r_a\mathbf{x}), \quad R_{ab}\Phi(\mathbf{x}) = \sigma_a\sigma_b\Phi(r_{ab}\mathbf{x}).$$

Let us present more examples of extended SUSY:

$$\left\{ \begin{array}{l} \mathbf{A}(r_{12}\mathbf{x}) = r_{12}\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{31}\mathbf{x}) = r_{31}\mathbf{A}(\mathbf{x}), \\ Q_1 = iR_{23}\sigma\cdot\pi, \quad Q_2 = iR_{31}\sigma\cdot\pi, \quad Q_3 = iR_{12}\sigma\cdot\pi; \end{array} \right. \tag{15}$$

$$\left\{ \begin{array}{l} \mathbf{A}(r_1\mathbf{x}) = r_1\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_2\mathbf{x}) = r_2\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_3\mathbf{x}) = r_3\mathbf{A}(\mathbf{x}), \\ Q_1 = iR_1\sigma\cdot\pi, \quad Q_2 = iR_2\sigma\cdot\pi, \quad Q_3 = iR_3\sigma\cdot\pi, \quad Q_4 = \sigma\cdot\pi; \end{array} \right. \tag{16}$$

$$\left\{ \begin{array}{l} \mathbf{A}(r_1\mathbf{x}) = -r_1\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_2\mathbf{x}) = -r_2\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_3\mathbf{x}) = -r_3\mathbf{A}(\mathbf{x}), \\ Q_1 = i\sigma_2 cR_1\sigma\cdot\pi, \quad Q_2 = i\sigma_2 cR_2\sigma\cdot\pi, \quad Q_3 = i\sigma_2 cR_3\sigma\cdot\pi, \quad Q_0 = \sigma\cdot\pi, \end{array} \right. \tag{17}$$

where c denotes the antilinear operator of complex conjugation: $c\Phi(\mathbf{x}) = \Phi^*(\mathbf{x})$.

We notice that in the cases (13), (15) and (16), (17) we have $N=3$ and $N=4$ extended SUSY correspondingly. Moreover, for the case of odd vector-potentials (refer to (17)) the corresponding supercharges generate the following superalgebra

$$\begin{gathered} \{Q_a, Q_b\}_+ = 2g_{ab}\hat{H}, \quad \left[Q_a, \hat{H}\right] = 0 \\ a, b = 0,1,2,3, \quad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1; \qquad g_{ab} = 0, \quad a \neq b. \end{gathered} \tag{18}$$

which is characterized by the metric tensor g_{ab}. We notice that it is impossible to reduce (18) to the form (2) by changing normalization constants for supercharges.

Imposing different combinations of conditions (12) (and the corresponding conditions for odd vector-potentials) we can extend the list of problems (13)-(17) generating extended SUSY. It is necessary to note that extended SUSY appears also in some problems which do not belong to the class (6). We can prove its existence for the Dirac oscillator [4], Dirac equation with a scalar potentials and many other problems.

3. The relatively new kind of symmetry called *parasupersymmetry* (PSUSY) is characterized by *threelinear* anticommutation [2] (or double commutation [3]) relations for parasupercharges

$$\begin{gathered}\left\{Q_a\left\{Q_b,Q_c\right\}-\delta_{bc}\hat{H}Q_a\right\}+(terms\ with\ permutations\ of\ a,b,c)=0;\\ \left[Q_a,\hat{H}\right]=0\end{gathered}\tag{19}$$

Of course it is interesting to discuss possible realizations of this fine symmetry in quantum mechanical problems. Here we present an example of problem with exact *extended* PSUSY.

Consider the Kemmer-Duffin equation with "correct" [5] anomalous interaction

$$\left[\beta^\mu\pi_\mu-m-(1-\beta_5^2)\,\frac{e^2}{2m}\left(S_{\mu\nu}F^{\mu\nu}+\frac{1}{4m^2}F_{\lambda\sigma}F^{\lambda\sigma}\right)\right]\Psi=0.\tag{20}$$

where β_μ are 10×10 matrices satisfying the Kemmer algebra, $F_{\mu\nu}=i[\pi_\mu,\pi_\nu]$.

In analogy with (6)-(14) it is possible to show that if the vector- potential A_μ corresponds to the "uniform" magnetic field, i.e., $A_0=A_3=0, A_1=A_1(x_1,x_2), A_2=A_2(x_1,x_2)$, and A_1, A_2 satisfy relations (13), equation (20) admits extended $N=4$ PSUSY.

Thus we demonstrated that extended SUSY and PSUSY are generated by many of quantum mechanical problems. These symmetries do be realized in Nature.

References

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