Higher symmetries and exact solutions of linear and nonlinear Schrödinger equation

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A new approach for the analysis of partial differential equations is developed which is characterized by a simultaneous use of higher and conditional symmetries. Higher symmetries of the Schrödinger equation with an arbitrary potential are investigated. Nonlinear determining equations for potentials are solved using reductions to Weierstrass, Painlevé, and Riccati forms. Algebraic properties of higher order symmetry operators are analyzed. Combinations of higher and conditional symmetries are used to generate families of exact solutions of linear and nonlinear Schrödinger equations.
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I. INTRODUCTION

Higher order symmetry operators (SOs) have many important applications in modern mathematical physics. These operators correspond to hidden symmetries of partial differential equations, including Lie–Bäcklund symmetries,[1,2] as well as super- and parasupersymmetries.[3–7]

Higher order SOs can be used to construct new conservation laws which cannot be found in the classical Lie approach.[3,8] These operators are applied to separate variables.[9] Moreover, one should use SOs whose order is higher than the order of the equation whose variables are separated.[10]

In the present paper we investigate higher order SOs of the Schrödinger equation, which are "non-Lie symmetries."[8,11] The simplest non-Lie symmetries are considered in detail and all related SOs are explicitly calculated. The potentials admitting these symmetries are found as solutions of the corresponding nonlinear compatibility conditions. It is shown that the higher order SOs extend the class of potentials which were previously obtained in the Lie symmetry analysis.

Algebraic properties of higher order SOs are investigated and used to construct exact solutions of the linear and related nonlinear Schrödinger equations. We propose a new method to generate extended families of exact solutions by using both the conditional symmetries[8,12–14] and higher order SOs.

The Schrödinger equation with a time-independent potential $V=V(x)$ is studied mainly. Time-dependent potentials $V=V(t,x)$ are discussed briefly in Sec. VI. By this, we recover the old result[15] connected with the Lax representation for the Boussinesq equation, and generate some other nonlinear equations admitting this representation.

The distinguishing feature of our approach is that coefficients of symmetry operators and the corresponding potentials are defined as solutions of differential equations which can easily be generalized to the case of multidimensional Schrödinger equation contrary to the method of inverse scattering problem.

This paper continues (and in some sense completes) our works[16–18] where non-Lie symmetries of the Schrödinger equation were considered. A detailed analysis of higher symmetries of multidimensional Schrödinger equations will be a subject of our subsequent paper.

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II. SYMMETRY OPERATORS OF THE SCHRÖDINGER EQUATION

Let us formulate the concept of higher order SO for the Schrödinger equation

$$L\Psi(t,x)=0,\quad L=i\partial_t-H,$$
$$H=\tfrac{1}{2}(-\partial_x^2+U(x)),\quad \partial_t\equiv\frac{\partial}{\partial t},\quad \partial_x\equiv\frac{\partial}{\partial x}. \tag{2.1}$$

In every sense of the word, a SO of equation (2.1) is any (linear, nonlinear, differential, integro-differential, etc.) operator Q transforming solutions into solutions. Restricting ourselves to linear differential operators of finite order n we represent Q in the form

$$Q=\sum_{i=0}^{n}(h_i\cdot p)_i,\quad (h_i\cdot p)_i=\{(h_i\cdot p)_{i-1},p\},\quad (h_i\cdot p)_0=h_i, \tag{2.2}$$

where h_i are unknown functions of (t,x), $\{A,B\}=AB+BA$, $p=-i\partial_x$.

Operator (2.2) includes no derivatives w.r.t. t which can be expressed as $\frac{1}{2}(p^2+U)$ on the set of solutions of Eq. (2.1).

Definition:[8] Operator (2.2) is a SO of order n of equation (2.1) if

$$[Q,L]=0. \tag{2.3}$$

Remark: The more general invariance condition[3] $[Q,L]=\alpha_Q L$, where α_Q is a linear operator, reduces to relation (2.3) if L and Q are operators defined in (2.1), (2.2). Terms proportional to $i(\partial/\partial t)$ cannot appear as a result of commutation of Q and L; hence, without loss of generality, $\alpha_Q=0$.

For $n=1,2$ SOs (2.2) reduce to differential operators of the first order and can be interpreted as generators of the invariance group of the equation in question. For $n>2$ these operators (which we call higher order SO) correspond to non-Lie[8,11] symmetries.

The Lie symmetries of equation (2.1) were described in Refs. 19-21 The general form of potentials admitting nontrivial (i.e., distinct from time displacements) symmetries is as follows:

$$U=a_0+a_1x+a_2x^2+\frac{a_3}{(x+a_4)^2}, \tag{2.4}$$

where $a_0,\dots,a_4$ are arbitrary constants. No other potentials admitting local invariance groups exist.

Group properties of equation (2.1) with potentials (2.4) were used to solve the equation exactly, to establish connections between equations with different potentials, to separate variables, etc.[9] Unfortunately, all these applications are valid for a very restricted class of potentials given by formula (2.4).

The class of admissible potentials can be essentially extended if we require that equation (2.1) admits higher order SOs.[17] The problem of describing such potentials (and the corresponding SOs) reduces to solving operator equations (2.2), (2.3). Evaluating the commutators and equating the coefficients for linearly independent differentials we arrive at the following system of determining equations (which is valid for *arbitrary* n):[5]

$$\partial_x h_n = 0, \quad \partial_x h_{n-1} + 2\partial_t h_n = 0,$$

$$\partial_x h_{n-m} + 2\partial_t h_{n-m+1} - \Sigma_{k=0}^{(n-2)/2}(-1)^k \frac{2(n-m+2+2k)!}{(2k+1)!(n-m+1)!} h_{n-m+2k+2}\partial_x^{2k+1}U = 0, \tag{2.5}$$

$$\partial_t h_0 + \Sigma_{p=0}^{(n-1)/2}(-1)^{p+1} h_{2p+1}\partial_x^{2p+1}U = 0,$$

where $m = 2,3,...,n$, and $[y]$ is the entire part of y.

Formulas (2.5) define a system of nonlinear equations in h_i and U. For $n=2$ the general solution for U is given by formula (2.4).

Let us consider the case $n=3$, which corresponds to the simplest non-Lie symmetry, in more detail. The corresponding system (2.5) reduces to

$$h_3' = 0, \quad h_2' + 2\dot{h}_3 = 0, \quad 2\dot{h}_2 + h_1' - 6h_3 U' = 0, \tag{2.6a}$$

$$2\dot{h}_1 + h_0' - 4h_2 U' = 0, \quad \dot{h}_0 - h_1 U' + h_3 U''' = 0, \tag{2.6b}$$

where the dots and primes denote derivatives w.r.t. t and x, respectively.

Excluding h_0 from (2.6b) and using (2.6a) we arrive at the following equation:

$$F(a,b,c;U,x) \equiv aU'''' - (2\ddot{a}x^2 + 6aU + c - 2\dot{b}x)U'' - 6(2\ddot{a}x + aU' - \dot{b})U'$$
$$- 12\ddot{a}U - 2(2\partial_t^4 ax^2 - 2\dddot{b}x + \ddot{c}) = 0, \tag{2.7}$$

where a,b,c are arbitrary functions of t.

Equation (2.7) is nothing but the compatibility condition for system (2.6). If the potential U satisfies (2.7) then the corresponding coefficients of the SO have the form

$$h_3 = a, \quad h_2 = -2\dot{a}x + b, \quad h_1 = g_1 + 6aU,$$

$$h_0 = -\frac{4}{3}\dddot{a}x^3 + 2\ddot{b}x^2 - 2\dot{c}x - 4\dot{a}\varphi + 4(b - 2\dot{a}x)U + d, \tag{2.8}$$

where

$$g_1 = 2\ddot{a}x^2 - 2\dot{b}x + c, \quad \varphi = \int U\, dx, \quad u = \varphi', \quad d = d(t). \tag{2.9}$$

III. EQUATIONS FOR POTENTIAL

Equation (2.7) was obtained earlier[17] (see Ref. 22) and, moreover, particular solutions for U were found.[17] Here, we analyze this equation in detail.

First of all, let us reduce the order of equation (2.7). Integrating it twice w.r.t. x and choosing the new dependent variable φ defined in (2.9) we obtain

$$a[\varphi''' - 3(\varphi')^2] - (g_1\varphi)' = \tfrac{1}{3}\partial_t^4 ax^4 - \tfrac{2}{3}\dddot{b}x^3 + \ddot{c}x^2 + dx + e. \tag{3.1}$$

Using the fact that φ depends on x only while $a,b,c,d,$ e are functions of t, it is possible to separate variables in (3.1). Indeed, dividing any term of (3.1) by $a \neq 0$, differentiating w.r.t. t and integrating over x we obtain the following consequence:

$$\frac{\dot{g}_1 a - g_1\dot{a}}{a^2}\varphi = \partial_t \frac{1}{a}\left(\frac{1}{15}\partial_t^4 ax^5 - \frac{1}{6}\dddot{b}x^4 + \frac{1}{3}\ddot{c}x^2 + \frac{1}{2}dx^2 + ex + f\right). \tag{3.2}$$

Consider equation (3.2) separately in two following cases:

$$\dot{g}_1 a - g_1 \dot{a} \neq 0, \tag{3.3a}$$

$$\dot{g}_1 a - g_1 \dot{a} = 0. \tag{3.3b}$$

Let condition (3.3a) be valid. Then dividing the l.h.s. and r.h.s. of (3.2) by $\partial_t (g_1/a)$ we come to the following general expression for φ:

$$\varphi = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = \frac{\alpha_4}{x+\alpha_5} + \frac{\beta_1 x + \beta_2}{x^2 + g b_3 x + \beta_4}, \tag{3.4}$$

where $\alpha_0, \ldots, \alpha_5$, $\beta_1, \ldots, \beta_4$ are constants.

It is possible to verify by a straightforward but cumbersome calculation that relation (3.4) is compatible with (3.1) only for $\beta_1 = \beta_2 = 0$. We will not analyze solutions (3.4) inasmuch as they correspond to potentials (2.4) and to SOs which are products of the usual Lie symmetries.[19–21]

If condition (3.3a) is valid, we obtain from equation (3.2)

$$\ddot{a} = a k_1, \quad \dot{b} = k_2 a, \quad c = k_3 a, \tag{3.4}$$

where k_1, k_2, k_3 are arbitrary constants. The corresponding equation (3.1) reduces to

$$\varphi''' - 3(\varphi')^2 - (G''\varphi)' = 2k_1 G + k_4 x + k_5, \tag{3.5}$$

where

$$G = \tfrac{1}{6} k_1 x^4 - \tfrac{1}{3} k_2 x^3 + \tfrac{1}{2} k_3 x^2, \quad G'' = g_1 = 2k_1 x^2 - 2k_2 x + k_3, \tag{3.6}$$

k_4 and k_5 are constants.

Let us prove that, up to equivalence, equation (3.5) can be reduced to one of the following forms:

$$U'' - 3U^2 + 3\omega_1 = 0, \tag{3.8a}$$

$$U'' - 3U^2 - 8\omega_2 x = 0, \tag{3.8b}$$

$$(U'' - 3U^2)' - 2\omega_3 (xU' + 2U) = 0, \tag{3.8c}$$

$$\varphi''' - 3(\varphi')^2 - 2\omega_4 (x^2 \varphi)' = \frac{1}{3}\, \omega_4^2 x^4 + \omega_5, \quad U = \varphi', \tag{3.8d}$$

where $\omega_1, \ldots, \omega_5$ are arbitrary constants. Indeed, by using invertible transformations

$$\varphi \to \varphi + C_1 x + C_2, \quad x \to x + C_3, \tag{3.9}$$

where $C_k (k=1,2,3)$ are constants, it is possible to simplify the r.h.s. of (3.5). These transformations cannot change the order of polynomial G, and so there exist four nonequivalent possibilities

$$k_1 = 0, \quad k_2 = 0, \quad k_4 = 0, \tag{3.10a}$$

$$k_1 = 0, \quad k_2 = 0, \quad k_4 \neq 0, \tag{3.10b}$$

$$k_1 = 0, \quad k_2 \neq 0, \tag{3.10c}$$

$$k_1 \neq 0. \tag{3.10d}$$

Setting in (3.9)

$$C_1 = -\tfrac{1}{6}k_3, \quad C_2 = C_3 = 0, \quad k_5 - \tfrac{1}{12}k_3^2 = \omega_1, \tag{3.11a}$$

$$C_1 = -\tfrac{1}{6}k_3, \quad C_2 = 0, \quad C_3 = -\frac{k_5}{k_4} + \frac{k_3^2}{12k_4}, \quad k_4 = 8\omega_2, \tag{3.11b}$$

$$C_1 = \frac{k_4}{4k_2}, \quad C_2 = \frac{k_5}{2k_2} + \frac{3k_4^2}{32k_2^3} + \frac{k_3k_4}{8k_2^2}, \quad C_3 = \frac{k_3}{2k_2} + \frac{3k_4}{4k_2^2}, \quad k_2 = -\omega_3, \tag{3.11c}$$

$$C_1 = -\frac{1}{6}\,k_3 + \frac{k_2^2}{12k_1}, \quad C_2 = -\frac{k_4}{4k_1} - \frac{k_2k_3}{6k_1} + \frac{k_2^3}{24k_1^2},$$

$$C_3 = \frac{k_2}{2k_1}, \quad k_1 = \omega_4, \quad k_5 - \frac{k_3^2}{12} + \frac{k_2k_4}{2k_1} + \frac{k_2^2k_3}{3k_1} - \frac{k_2^4}{16k_1^2} = \omega_5, \tag{3.11d}$$

for cases (3.10a)–(3.10d) correspondingly, we reduce (3.5) to one of the forms, (3.8a)–(3.8d) respectively.

From (2.2), (2.8), (3.4), (3.9)–(3.11) we find the corresponding symmetry operators

$$Q = p^3 + \tfrac{3}{4}\{U,p\} \equiv 2pH + \tfrac{1}{2}Up + \tfrac{i}{4}U', \tag{3.12a}$$

$$Q = p^3 + \tfrac{3}{4}\{U,p\} - \omega_2 t, \tag{3.12b}$$

$$Q = p^3 + \frac{3}{4}\,\{U,p\} + \omega_3(tH - \tfrac{1}{4}\{x,p\}), \tag{3.12c}$$

$$Q_\pm = \frac{1}{\sqrt{24}}\left[p^3 \pm \frac{i}{4}\,\omega\{\{x,p\},p\} + \frac{1}{4}\,\{3\varphi' - \omega^2x^2,p\} \pm \frac{i}{2}\,\omega\left(\varphi + 2x\varphi' - \frac{\omega^2}{3}\,x^3\right)\right]\exp(\pm i\omega t),$$

$$\omega = \sqrt{-\omega_4}, \tag{3.12d}$$

where U and φ are solutions of (3.2) and H is the related Hamiltonian (2.1).

Thus, *the Schrödinger equation (2.1) admits a third-order* SO *if potential U satisfies one of the equations (3.8)*. The explicit form of the corresponding SOs is present in (3.12).

IV. ALGEBRAIC PROPERTIES OF SOs

Let us investigate algebraic properties of SOs defined by relations (3.12). We shall see that these properties are predetermined by the type of equations (3.8) satisfied by U. By direct calculations, using (2.3), (2.1), and (3.12), we find the following relations:

$$[Q,\ H] = 0, \tag{4.1a}$$

$$Q^2 = 8H^3 - \frac{3}{2}\,\omega_1 H - \frac{C}{8} \tag{4.1b}$$

if the potential satisfies equation (3.8a). [C is the first integral of equation (3.8a), refer to (5.1)];

$$[Q,\ H] = i\omega_2 I, \quad [Q,\ I] = [H,\ I] = 0 \tag{4.2}$$

if the potential satisfies equation (3.8b);

$$[Q,\ H] = -i\omega_3 H \tag{4.3}$$

if the potential satisfies equation (3.8c), and

$$[H,\ Q_\pm] = \pm \omega Q_\pm, \tag{4.4a}$$

$$[Q_+,\ Q_-] = \omega\left(H^2 + \frac{1}{48}(2\omega^2 + \omega_5)\right) \tag{4.4b}$$

if the potential satisfies (3.8d).

It follows from (4.1)–(4.3) that non-Lie SOs Q and Hamiltonians H form consistent Lie algebras which can have rather nontrivial applications.

Formula (4.1b) presents an example of the general theorem[23,24] stating that commuting ordinary differential operators are connected by a polynomial algebraic relation with constant coefficients. In Sec. VII we use relations (4.1) to integrate the related equations (2.1).

Relations (4.2) define the Heisenberg algebra. The linear combinations $a_\pm = (1/\sqrt{2})(H \pm iQ)$ realize the unusual representation of creation and annihilation operators in terms of third-order differential operators.

In accordance with (4.3), Q plays a role of dilatation operator which continuously changes eigenvalues of H. Indeed, let

$$H\Psi_E = E\Psi_E, \tag{4.5}$$

then the function $\Psi' = \exp(i\lambda Q)\Psi_E$ (where λ is a real parameter) is also an eigenvector of the Hamiltonian H with the eigenvalue λE.

It follows from (4.4) that for $\omega_4 < 0$ the operators Q_+ and Q_- are raising and lowering operators for the corresponding Hamiltonian. In other words, if Ψ_E satisfies (4.5) then $Q_\pm \Psi_E$ are also eigenfunctions of the Hamiltonian which, however, correspond to the eigenvalues $E \pm \omega$

$$H(Q_\pm \Psi_E) = (E \pm \omega)(Q_\pm \Psi_E). \tag{4.6}$$

Relations (4.6) are typical for creation and annihilation operators of the quantum oscillator. This observation shows a way for constructing exact solutions of the Schrödinger equation whose potential satisfies relation (3.8d). Moreover, relations (4.4a) allow Q to be interpreted as a conditional symmetry;[8,12] such symmetries are of particular interest in the analysis of partial differential equations.[14,25,26] Thus third-order SOs of equation (2.1) generate algebras of certain interest. Moreover, algebraic properties of these SOs are the same for wide classes of potentials described by one of equations (3.8).

V. REDUCTION OF EQUATIONS FOR POTENTIALS

Let us consider equations (3.8) in detail and describe the corresponding classes of potentials. A solution of some of these nonlinear equations is a complicated problem which, however, can be simplified by using reductions to other well-studied equations.

A. The Weierstrass equation

Formula (3.8a) defines the Weierstrass equation whose solutions are expressed via either elementary functions or via the Weierstrass function, depending on values of the parameter ω_1 and the integration constant. Here, we represent these well-known solutions (refer, e.g., to the classic monograph of Whittaker and Watson[27]) in the form convenient for our purposes.

Multiplying the l.h.s. of (3.8a) by U' and integrating we obtain

$$\tfrac{1}{2}(U')^2 - U^3 + 3\omega_1 U = C, \tag{5.1}$$

where C is an integration constant which appeared above in (4.1b). Then by changing roles of dependent and independent variables it becomes possible to integrate (5.1) and to find U as an implicit function of x. We will distinguish five qualitatively different cases

$$C^2 - 4\omega_1^3 = 0, \quad C > 0, \tag{5.2a}$$

$$C^2 - 4\omega_1^3 = 0, \quad C < 0, \tag{5.2b}$$

$$C = \omega_1 = 0, \tag{5.2c}$$

$$C^2 - 4\omega_1^3 < 0. \tag{5.3a}$$

$$C^2 - 4\omega_1^3 > 0. \tag{5.3b}$$

For (5.2a)–(5.2c), solutions of (5.1) can be expressed via elementary functions, while (5.3a,b) generate solutions in elliptic functions.

For our purposes, it is convenient to transform (5.1) to another equivalent form. Using the substitution

$$U = V - \frac{\mu}{2}, \tag{5.4}$$

where μ is a real root of the cubic equation

$$\mu^3 - 3\omega_1 \mu + C = 0, \tag{5.5}$$

we obtain

$$\tfrac{1}{2}(V')^2 - V^3 - \overline{\omega_0} V^2 + 4\overline{\omega_1} V + 8\overline{\omega_0}\,\overline{\omega_1} = 0, \tag{5.6}$$

where $\overline{\omega_0} = \frac{3}{2}\mu$ and $\overline{\omega_1} = \frac{3}{4}(\omega_1 - \mu^2)$ are arbitrary real numbers.

The substitution (5.4), (5.5) transforms conditions (5.2), (5.3) to the following form:

$$\overline{\omega_1}(\overline{\omega_1} - \overline{\omega}_0^2)^2 = 0, \quad \overline{\omega_0} < 0, \tag{5.7a}$$

$$\overline{\omega_1}(\overline{\omega_1} - \overline{\omega}_0^2)^2 = 0, \quad \overline{\omega_0} > 0, \tag{5.7b}$$

$$\overline{\omega_1}(\overline{\omega_1} - \overline{\omega}_0^2)^2 = 0, \quad \overline{\omega_0} = 0, \tag{5.7c}$$

$$\overline{\omega_1}(\overline{\omega_1} - \overline{\omega}_0^2) \neq 0, \quad \overline{\omega_1} > 0, \tag{5.8a}$$

$$\overline{\omega_1}(\overline{\omega_1} - \overline{\omega}_0^2) \neq 0, \quad \overline{\omega_1} < 0. \tag{5.8b}$$

If relations (5.7a) are satisfied, then $\overline{\omega_1} = \overline{\omega}_0^2$ or $\overline{\omega_1} = 0$. Moreover, the corresponding solutions for V differ by a constant shift: $V \to V + 2\overline{\omega_0}$, $\overline{\omega_0} \to \overline{\omega_0}/2$. Without loss of generality we restrict ourselves to the former case, then solutions of equation (5.6) corresponding to conditions (5.7a–c) have the following forms:

$$V = \nu^2[2\tanh^2(\nu(x-k)) - 1], \quad \overline{\omega_0} = -\tfrac{1}{2}\nu^2, \quad \overline{\omega_1} = \tfrac{1}{4}\nu^4, \tag{5.9a}$$

$$V = \nu^2[2\coth^2(\nu(x-k)) - 1], \quad \overline{\omega_0} = -\tfrac{1}{2}\nu^2, \quad \overline{\omega_1} = \tfrac{1}{4}\nu^4, \tag{5.9a$'$}$$

$$V=\nu^2[2\tan^2(\nu(x-k)-1)],\quad \overline{\omega_0}=\tfrac{1}{2}\nu^2,\quad \overline{\omega_1}=\tfrac{1}{4}\nu^4, \tag{5.9b}$$

$$V=\frac{2}{(x-k)^2}. \tag{5.9c}$$

Here, k and ν are arbitrary real numbers.

For the cases (5.8) the general solution of (5.1) has the form

$$V=2\wp(x-k)+\tfrac{1}{2}\mu \tag{5.10}$$

where $\wp$ is a two-periodic Weierstrass function, which is meromorphic on all the complex plane. The invariants of this function are $g_2=-\frac{4}{3}(\overline{\omega_0^2}+3\overline{\omega_1})$ and $g_3=-\frac{4}{27}\overline{\omega_0}(\overline{\omega_0^2}-9\overline{\omega_1})$. Moreover, if condition (5.8a) holds, the corresponding solutions are bounded and can be expressed via the elliptic Jacobi functions

$$V=B\ \mathrm{cn}^2(Dx+k)+F, \tag{5.11a}$$

where

$$B=(e_3-e_2),\quad D=\sqrt{(e_1-e_3)/2},\ \ F=e_2 \tag{5.11b}$$

$e_1>e_2>e_3$ are real solutions of the cubic equation from the r.h.s. of (5.6).

We note that formulas (5.9) present the set of well-known potentials which correspond to the exactly solvable Schrödinger equations.[28] In accordance with the above, these equations admit extended Lie symmetries.

B. Painlevé and Riccati equations

Relation (3.8b) defines the first Painlevé transcendent. Its solutions are meromorphic on all the complex plane but cannot be expressed via elementary or special functions.

Equation (3.8c) is more complicated. However, by using the special change of variables and applying the Miura[29] ansatz, we shall reduce it to the Painlevé form also. Indeed, making the following change of variables:

$$U=-\sqrt[3]{\frac{\omega_3^2}{6}}\,V,\quad x=-\sqrt[3]{\frac{1}{6\omega_3}}\,y, \tag{5.12}$$

we obtain

$$V'''+VV'-\tfrac{1}{3}xV'-\tfrac{2}{3}V=0,\quad V'=\partial V/\partial y. \tag{5.13}$$

The ansatz

$$V=W'-\tfrac{1}{6}W^2 \tag{5.14}$$

reduces (5.13) to

$$(\partial_y-\tfrac{1}{3}W)(W'''-\tfrac{1}{6}W^2W'-\tfrac{1}{3}yW'-\tfrac{1}{3}W)=0.$$

Equating the expression in the second brackets to zero and integrating it we come to the second Painlevé transcendent

$$W''=\tfrac{1}{18}W^3+\tfrac{1}{3}yW+K, \tag{5.15}$$

where K is an arbitrary constant.

To make one more reduction of equation (3.8c) we take $U=\varphi'$. Then, integrating the resultant equation, we obtain

$$\varphi''' - 3(\varphi')^2 - 2\omega_3(x\varphi)' = C. \tag{5.16}$$

Then, defining

$$\varphi = 2\sqrt[3]{2\omega_3}\xi + \frac{1}{4}y^2 + \frac{C}{2\omega_3}, \quad y = \sqrt[3]{2\omega_3}x,$$

$$\hat{W} = \xi' - \xi^2 - \frac{1}{2}y, \quad \xi' = \frac{\partial \xi}{\partial y} \tag{5.17}$$

we represent (5.16) as

$$\hat{W}'' - 4\xi'\hat{W} + 2\xi\hat{W}' - y\hat{W} = 0. \tag{5.18}$$

The trivial solutions of (5.18) correspond to the following Riccati equation for ξ:

$$\xi' - \xi^2 - \tfrac{1}{2}y = 0. \tag{5.19}$$

It follows from the above that any solution of equations (5.15) or (5.19) generates a potential U defined by relations (5.12), (5.14), or (5.17). The corresponding Schrödinger equation admits a third-order SO.

The last of the equations considered, i.e., equation (3.8d), is the most complicated. The change

$$\varphi = 2f - \tfrac{1}{3}\omega_4 x^3 \tag{5.20}$$

reduces it to the following form:

$$f''' - 6(f')^2 + 4\omega_4(f'x^2 - xf) = \omega_4 + \tfrac{1}{2}\omega_5. \tag{5.21}$$

Multiplying (5.21) by f'' and integrating we obtain the first integral

$$\tfrac{1}{2}(f'')^2 - 2(f')^3 + 2\omega_4(f - xf')^2 - (\omega_4 + \tfrac{1}{2}\omega_5)f' = C \tag{5.22}$$

which is still a very complicated nonlinear equation.

Let us demonstrate that (5.21) can be reduced to the Riccati equation. To realize this we rewrite (5.21) as follows:

$$F'' + 2fF' - 4f'F = \tfrac{1}{2}\omega_5 - \omega_4, \tag{5.23}$$

where

$$F = f' - f^2 - \omega_4 x^2.$$

Choosing $\omega_5 = 2\omega_4$ we conclude that any solution of the Riccati equation

$$f' = f^2 + \omega_4 x^2 \tag{5.24}$$

generates a solution of equation (3.8d), given by relation (5.20).

One more possibility in solving of equation (3.8d) consists in its reduction to the Painlevé form. Making the change of variables $\varphi = \sqrt{-w_4}\chi$, $x = (1/\sqrt{-\omega_4})y$ and differentiating equation (3.8d) w.r.t. y, we obtain

$$(\widetilde{U}''-3\widetilde{U}^2)''+(6\widetilde{U}+6x\widetilde{U}'+2\widetilde{U}'')=4x^2, \tag{5.26}$$

where $\widetilde{U}=(\partial\chi/\partial_y)=-(1/\omega_4)U$.

Using the following generalized Miura ansatz.

$$\widetilde{U}=-V'+V^2+2Vy+y^2-1, \tag{5.27}$$

we reduce equation (5.26) to the form

$$\partial_y(\partial_y-2V-2y-2)(V'''-6V^2V'-4V_2-12yVV'-4yV-4V'y^2-2V')=0.$$

Equating the expression in the right brackets to zero, integrating and dividing it by $2V$, we come to the fourth Painlevé transcendent

$$V''=\frac{V'^2}{2V}+\frac{3}{2}V^3+8yV^2+(2y^2-1)V+\frac{b}{V}. \tag{5.28}$$

We note that the double differentiation and consequent change of variables

$$\varphi'=-\sqrt{\frac{\omega_4}{3}}\left(\Phi+\frac{1}{6}y^2\right),\quad x=\frac{1}{\sqrt[4]{4\omega_4}}y$$

transform equation (3.8d) to the form

$$\partial^4\Phi+\Phi''\Phi+\Phi'\Phi'-\tfrac{1}{3}(8\Phi+x^2\Phi''+7x\Phi')=0$$

which coincides with the reduced Boussinesq equation.[3,14] The procedures outlined above reduces the equation either to the fourth Painlevé transcendent (5.28) or to the Riccati equation (5.24).

Thus, the third-order SOs are admitted by a very extended class of potentials described above. We should like to emphasize that in general the corresponding Schrödinger equation does not possesses any nontrivial (distinct from time displacements) Lie symmetry.

VI. EQUATIONS FOR TIME-DEPENDENT POTENTIALS

Consider briefly the case of time-dependent potentials $U=U(x,t)$. The determining equations (2.6) are valid in this case also. Moreover, the compatibility condition for system (2.6) takes the form

$$F(a,b,c;x,U)+12a\ddot{U}-4(b-2\dot{a}x)\dot{U}'=0 \tag{6.1}$$

where $F(a,b,c;x,U)$ is defined in (2.7).

Equation (6.1) is much more complicated than (2.7) due to the time dependence of U, which makes it impossible to separate variables. For any fixed set of functions $a(t)$, $b(t)$, and $c(t)$, formula (6.1) defines a nonlinear equation for potential. Moreover, any of these equations admits the Lax representation

$$[H,\ Q]=i\,\frac{\partial Q}{\partial t}, \tag{6.2}$$

cf. (2.3). Refer to Refs. 30, 31 for the general results connected with arbitrary ordinary differential operators satisfying (6.2).

We will not analyze equations (6.1) here, but present a few simple examples concerning particular choices of arbitrary functions a, b, and c.

$a = \text{const}$, $b = c = 0$:

$$-12\ddot{U} + U'''' - 6(UU')' = 0; \tag{6.3}$$

a, b are constants, $c = 0$:

$$12\ddot{U} - (4b\dot{U} - U''' + 6UU')' = 0; \tag{6.4}$$

$\dot{a} = c = 0$, $\dot{b} = \omega_3 a$:

$$12\ddot{U} - 4(\omega_3 t - 2x)\dot{U}' + (U'' - 3U^2)'' + 2\omega_3(xU' + 2U)' = 0; \tag{6.5}$$

$a = \exp(t)$, $b = c = 0$:

$$12\ddot{U} + 8x\dot{U}' + (U'' - U^2)'' - 12(Ux)' - 2x^2U'' - 4x^2 = 0. \tag{6.6}$$

Formula (6.3) defines the Boussinesq equation. The Lax representation (6.2) for this equation is well known.[15] Formulas (6.4)–(6.6) present other examples of nonlinear equations admitting this representation and arise naturally under the analysis of third-order SOs of the Schrödinger equation.

VII. EXACT SOLUTIONS

Let us regard the case of potentials satisfying (3.8a) or (5.4), (5.6). Taking into account commutativity of the corresponding SO (3.12a) with Hamiltonian (2.1) it is convenient to search for solutions of the Schrödinger equation in the form

$$\Psi(t,x) = \exp(-iEt)\psi(x), \tag{7.1}$$

where $\psi(x)$ are eigenfunctions of the commuting operators H and Q

$$H\psi(x) = E\psi(x), \tag{7.2a}$$

$$Q\psi(x) = \lambda\psi(x). \tag{7.2b}$$

Using (7.2a), (3.12a), and (5.4) we reduce (7.2b) to the first-order equation

$$\left(2E + \frac{V}{2} + \overline{\omega_0}\right)\psi' = \left(\frac{1}{4}V' + i\lambda\right)\psi \tag{7.3}$$

whose general solution has the form

$$\psi = A\sqrt{V + 4E + 2\overline{\omega_0}}\,\exp\left(2i\lambda\int\frac{dx}{V + 4E + 2\overline{\omega_0}}\right), \tag{7.4}$$

where A is an arbitrary constant. Then, expressing ψ' via ψ in accordance with (7.3) and using (5.6), we reduce (7.2a) to the following *algebraic* relation for E and λ [compare with (4.1b)]:

$$\lambda^2 = 8E^2(E + \overline{\omega_0}). \tag{7.5}$$

Thus there exists a remarkably simple way to integrate the Schrödinger equation which admits a third-order SO. The integration reduces to the problem of solving the first-order ordinary differential equation (7.3) and algebraic equation (7.5).

Let us show that the existence of a third-order SO for the linear Schrödinger equation enables one to find exact solutions for the following *nonlinear* equation:

$$i\partial_t\tilde{\Psi}=\frac{1}{2}\,p^2\tilde{\Psi}+\frac{1}{2A^2}\,(\tilde{\Psi}^*\tilde{\Psi})\tilde{\Psi}. \tag{7.6}$$

Indeed, if $\lambda^2>0$, solutions (7.1), (7.4) satisfy the following relations:

$$\Psi^*\Psi=A^2(V+4E+2\overline{\omega_0}). \tag{7.7}$$

Using (7.2a) and (7.7) we make sure that the functions

$$\tilde{\Psi}=\exp(i\epsilon t)\,\psi(x),\quad \epsilon=-3E-\overline{\omega_0} \tag{7.8}$$

[where $\psi(x)$ are functions defined in (7.4)] are exact solutions of (7.6).

Thus we obtain a wide class of exact solutions of the nonlinear Schrödinger equation, which depend on arbitrary parameters ϵ, $\overline{\omega_0}$, $\overline{\omega_1}$, k [see (7.8), (7.4), (5.6), (5.8)]. Properties of these (and some more general) solutions are discussed in the following section.

VIII. LIE SYMMETRIES AND GENERATION OF SOLUTIONS

It is well known that equation (7.6) is invariant under the Galilei transformations (refer, e.g., to Refs. 2, 3)

$$x\to x'=x-vt,$$

$$\Psi(t,x)\to\Psi'(t,x')=\exp\left[i\left(vx-\frac{v^2}{2}+\varphi_0\right)\right]\Psi(t,x), \tag{8.1}$$

where v and φ_0 are real parameters. Using (8.1) it is possible to generate a more extended family of solutions starting with (7.8)

$$\overline{\Psi}=A\sqrt{V(x-k-vt)+4E+2\overline{\omega_0}}$$
$$\times\ \exp\left\{i\left[(2\epsilon-v^2)\,\frac{t}{2}+vx+\varphi_0+2\lambda\int_0^{x-k-vt}\frac{dy}{V(y)+4E+2\overline{\omega_0}}\right]\right\}. \tag{8.2}$$

Here, V is an arbitrary solution of equation (5.6), v, $\overline{\omega_0}$, $\overline{\omega_1}$, k, φ_0 and E are real parameters, λ and ϵ are defined in (7.5), (7.8).

In order for λ to be real we require $\epsilon\geqslant 0$, other parameters are arbitrary.

Solutions (8.2) are qualitatively different for different values of free parameters enumerated in (5.7). If $\overline{\omega_0}$ and $\overline{\omega_1}$ satisfy (5.7a) or (5.7c), possible V are given by formulas (5.9a), (5.9a′) or (5.9c). Solutions (8.2), (5.9a) are bounded for any x and t, whereas solutions (8.2), (5.9a′) and (8.2), (5.9c) are singular at $x-k-vt=0$. For $\overline{\omega_0}$ and $\overline{\omega_1}$ satisfying (5.7b) the modulus of the complex function (8.2), (5.9b) is periodic and singular at $x-k-vt=(2n+1)\pi/2\nu$. All the above mentioned singularities are simple poles. If $\overline{\omega_0}$ and $\overline{\omega_1}$ satisfy relations (5.8a), the solutions (8.2) are expressed via the two-periodic Weierstrass function $\wp$ [refer to (5.10)] and are, generally speaking, unbounded. But if we restrict ourselves to solutions (5.11) for potential, the corresponding solutions (8.2) are periodic and bounded.

To inquire into a physical content of the obtained solutions let us consider in more detail the cases (8.2), (5.9a) and (8.2), (5.11).

For potentials (5.9a) the corresponding relation (7.5) reduces to

$$\lambda^2 = 4E^2\epsilon, \quad \epsilon = 2E - \nu^2, \tag{8.3}$$

and the integral in (8.2) can be easily calculated. This enables us to represent solutions (8.2), (5.9a) as follows:

$$\tilde{\Psi} = \frac{A\nu}{\cosh[\nu(x-k-vt)]} \exp\left\{ i\left[\left(\frac{\nu^2 - v^2}{2} \right) t + vx + \varphi_0 \right] \right\}, \quad E=0; \tag{8.4}$$

$$\tilde{\Psi} = A\{\nu \tanh[\nu(x-k-vt)] \pm i\sqrt{\epsilon}\} \exp\left\{ i\left[\left(\frac{\nu^2 - v^2}{2} - 3E \right) t + (v \mp \sqrt{\epsilon})x + \varphi_0 \right] \right\}, \quad E \neq 0, \quad \epsilon \geqslant 0. \tag{8.5}$$

For potentials (5.11) we obtain from (8.2)

$$\tilde{\Psi} = \tilde{\Psi}_1 = A\sqrt{B}\ \mathrm{cn}[D(x-vt)+k] \exp[if_1(t,x)], \quad E=0; \tag{8.6}$$

$$\tilde{\Psi} = \tilde{\Psi}_2 = A\sqrt{B\ \mathrm{cn}^2[D(x-vt)+k]+F}\ \exp(if_2(t,x)], \quad E + \overline{\omega_0} = 0, \tag{8.7}$$

where

$$f_1(t,x) = f_2(t,x) + \frac{3}{2} Ft = \left(F - \frac{v^2}{2} \right) t + vx + \varphi_0,$$

B, D, and F are parameters defined in (5.11b).

For other values of E solutions (8.2), (5.11) are also reduced to the form (8.7) where the phase $f_2(t,x)$ is expressed via elliptic integrals.

Formula (8.4) presents a fast decreasing one-soliton solution.[31] Relation (8.5) defines a soliton solution whose behavior at $x \to \infty$ is typical of solitons with a finite density. Formulas (8.6), (8.7) describe ''cnoidal'' solutions for the nonlinear Schrödinger equation.

IX. CONDITIONAL SYMMETRY AND GENERATION OF SOLUTIONS

Let us return to the linear Schrödinger equation (2.1) with the potential U satisfying (3.8a). Generally speaking it possesses no nontrivial (distinct from time displacements) Lie symmetry. Nevertheless, its solutions can be generated within the framework of the concept of conditional symmetry.[2,3,12,14,32] Indeed, these solutions satisfy (7.7), and equation (2.1) with the additional condition (7.7) is invariant under the Galilei transformations (8.1) [i.e., condition (7.7) extends the symmetry of equation (2.1)].

This conditional symmetry enables us to generate new solutions. Starting with (7.1), (7.4) and using (8.1) we obtain

$$\Psi = A\sqrt{V(x-k-vt)+4E+2\overline{\omega_0}}$$
$$\times \exp\left\{ i\left[-(2E+v^2)\ \frac{t}{2} + vx + \varphi_0 + 2\lambda \int_0^{x-k-vt} \frac{dy}{V(y)+4E+2\overline{\omega_0}} \right] \right\}. \tag{9.1}$$

Functions (9.1) satisfy the Schrödinger equation with a potential $V(x-k-vt)$, where $V(x)$ is a solution of equation (5.6). In the particular case $E = -\overline{\omega_0}/2$ these functions are reduced to solutions (8.2) of the nonlinear equation (7.6).

One more generation of solutions can be made using a third-order SO. Inasmuch as $V(x)$ satisfies (5.6), then $V(x-vt)$ satisfies the Boussinesq equation (6.3). It means that the corresponding linear Schrödinger equation admits a third-order SO. In accordance with (2.2), (2.6) this SO can be represented in the form

$$Q = p^3 + \tfrac{1}{4}\{3V + 2\overline{\omega_0} + 6v^2, p\} + \tfrac{3}{2}vV$$
$$\equiv 2pH + \tfrac{1}{2}(V + 2\overline{\omega_0} + 6v^2)p + \tfrac{3}{2}vV + \tfrac{i}{4}V'. \tag{9.2}$$

Formula (9.2) generalizes (3.12a) to the case of time-dependent potential.

Acting by operator (9.2) on Ψ in (9.1) we obtain a new family of solutions

$$\Psi' = Q\Psi = a\psi + iv^2\Psi_1, \tag{9.3}$$

where $a = \lambda + 4Ev + \overline{\omega_0}v - 4v^3$, Ψ is the initial solution (9.1),

$$\Psi_1 = \frac{V' + 4i\lambda}{2(4E + V + 2\overline{\omega_0})}\,\Psi. \tag{9.4}$$

We note that if Ψ is a soliton solution

$$\Psi = \frac{\nu A}{\cosh[\nu(x - vt)]}\exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right] \tag{9.5}$$

[the corresponding potential is present in (5.9a)], then (9.4) is a soliton solution too:

$$\Psi_1 = \frac{\nu^2 A\ \sinh[\nu(x - vt)]}{\cosh^2[\nu(x - vt)]}\exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right]. \tag{9.6}$$

Starting with the potential (5.11) we obtain from (9.1) a particular solution

$$\Psi = A\sqrt{B\ \mathrm{cn}^2 z + F}\ \exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right], \quad z = D(x - vt). \tag{9.7}$$

The corresponding generated solution (9.4) reads

$$\Psi_1 = -\frac{ABD\ \mathrm{cn}\, z\ \mathrm{sn}\, z\ \mathrm{dn}\, z}{B\ \mathrm{cn}^2 z + 2F}\exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right]. \tag{9.8}$$

and is also bounded.

Acting by SO (9.2) on solutions (9.3), (9.8) we again obtain new solutions. Moreover, this procedure can be repeated. In particular, in this way it is possible to construct multisoliton solutions of the linear Schrödinger equation.

We see that higher order SOs present efficient possibilities for solving equations of motion and generating new solutions starting with known ones.

X. CONCLUSION

Higher order SOs present a powerful tool for analyzing and solving the Schrödinger equation. The concept of higher symmetries enables us to extend the class of privileged potentials (2.4) and to investigate invariance algebras of the equations whose potentials satisfy one of relations (3.8).

We note that potentials (5.9) can be represented in the form $V = W^2 + W'$, where $W = \nu\ \tanh[\nu(x - k)]$ for solution (5.9a) (superpotentials W for solutions (5.9a)–(5.9c) can also be easily calculated]. Moreover, the corresponding superpartners $\widetilde{V} = W^2 - W'$ reduce to constants, therefore it is possible to integrate easily the Schrödinger equation with potentials (5.9) using the Darboux transformation.[33]

It is worth noting that invariance condition (2.3) for operators (2.1), (3.12) can be treated as a zero curvature condition for equations associated with the eigenvalue problem for operator Q, or as the Lax condition where a role of the Lax operator L is played by a SO, refer to (6.2). The reasons stimulating our research of such a well-studied subject and distinguishing features of our approach are the following:

(1) The main goal of our paper is to present a constructive description of potentials for the Schrödinger equation which admit higher symmetries. In this way we extend the fundamental results[19–21] connected with the search for potentials admitting usual Lie symmetries.

To solve the deduced determining equations for potentials we use direct reductions to the Painlevé or Riccati forms. The obtained results can be used for analysis and solution of the Schrödinger equation as well as for construction of exact solutions of the Boussinesq equation, see item 5 in the following.

In the method of inverse problem, description of pairs of operators (2.1), (2.8) satisfying the Lax condition (6.2) is reduced to the Gelfand-Marchenko-Levitan equations[34] or to the Riemann problem[15,31] which can be solved explicitly for a restricted class of potentials.

(2) We use non-Lie symmetries of the Schrödinger equation for construction and generation of exact solutions. Moreover, we are interested not so much in finding *new solutions* as in developing a *new method* of their derivation, which consists in simultaneous using of higher order and conditional symmetries. Nevertheless, the cnoidal solutions (9.7), (9.8) and (8.6), (8.7) for the linear and nonlinear Schrödinger equations can be of interest for physicists as well as infinite series of soliton and cnoidal solutions generated by a repeated application of the procedure described in Sec. IX.

We believe that the combination ''higher order symmetries+conditional symmetries'' may be used effectively in the investigations and analysis of other equations of mathematical physics.

(3) Our approach admits a direct generalization to multidimensional Schrödinger equations. Note that higher symmetries of the three-dimension Schrödinger equation were investigated in Refs. 18, 35 for particular potentials.

(4) Algebraic relations (4.1)–(4.4) are valid for extended classes of potentials. They open additional possibilities in the application of algebraic methods to investigate the Schrödinger equation, in particular, the use of raising and lowering operators for this equation with potentials satisfying (3.8d). We note that relations (3.8d) are valid also for time-independent operators $\widetilde{Q}_{\pm} = \exp(\mp i\omega t)Q_{\pm}$, where $Q_{\pm}$ are given by relations (3.12d).

(5) Equations (3.8) which describe potentials that admit third-order symmetries are equivalent to the reduced versions of the Boussinesq equation, which appear under the similarity reduction[36] [this is the case for (3.8a,d)] and the reduction with using symmetries[14,25,26] [the last is valid for (3.8b,c)]. Thus the results obtained in Sec. V can be used to construct exact solutions of the Boussinesq equation.

A systematic study of higher symmetries of multidimensional Schrödinger equations is planned to be carried out elsewhere.

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