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Non-Lie Symmetries and Supersymmetries

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1 Introduction

Appeared more than one century ago, the classical Lie approach serves as a powerful tool in investigations of symmetries of partial differential equations. In the last three decades there appear essential generalizations of this approach. They are the modern version of the Lie-Bäcklund symmetries [1], the non-Lie approach [2–4], the conditional symmetry approach [5–7], etc.

I will speak about non-Lie symmetries of equations of mathematical physics. The term "non-Lie symmetries" has appeared in papers of Fushchych [2], the main ideas and basic results of the non-Lie approach are outlined in our books [3, 4].

To formulate the general idea of non-Lie symmetries, we consider a linear differential equation

$$L\left(x, \frac{\partial}{\partial_x}\right)\psi(x) = 0 \tag{1.1}$$

where L is a linear differential operator, $x = (x_1, x_2, ..., x_m)$, $\quad \psi = (\psi_1, \psi_2, ..., \psi_n)$ are sets of independent and dependent variables. In the Lie approach we search for generators of the invariance group of (1) in the form

$$Q = \zeta(x,\psi)\frac{\partial}{\partial x} + \eta(x,\psi)\frac{\partial}{\partial \psi} \tag{2.1}$$

where ψ and η are functions have to be determined. The procedure of finding these functions is well known and is outlined, e.g., in Olver's book [8].

The main idea of the non-Lie approach is to extend the class of symmetry operators (1.2). It can be done, e.g., by including terms with higher order differentials or even by considering integro- differential symmetry operators. In this way we find such symmetries of the equation considered which a priori cannot be found in the classical Lie approach.

In order this idea to be constructive, it is necessary to show a way for exact calculations of extended Lie symmetries. The algorithms for calculations and exact forms of these symmetries for the fundamental equations of quantum mechanics are outlined in our books [2, 3]. I remind that using the non-Lie approach a new eight-parameter symmetry group for the Maxwell equations was found [2]. The generators of this group belong to the class of integro-differential operators.

I will discuss non-Lie symmetries for the Schrődinger and Dirac equations. Being relatively simple models, these equations present a straightforward possibility to demonstrate the main ideas of the non-Lie approach. Moreover, in this way I will present some of our last results as application of this approach.

2 Symmetries of the one-dimensional Schrödinger equation

Let us start with the Schrődinger equation

$$\begin{aligned} &L\psi(t,x)=0, \qquad L=i\partial_t-H,\\ &H=\frac{1}{2}\left(-\partial_x^2+U(x)\right), \qquad \partial_t\equiv\frac{\partial}{\partial t}, \quad \partial_x\equiv\frac{\partial}{\partial x} \end{aligned} \tag{2.1}$$

where the potential $U=U(x)$ is an arbitrary function of the only spatial variable.

The problem of a complete description of Lie symmetries of equation (2.1) was solved in papers [9,10], where all the potentials U generating nontrivial symmetries have been found. The general form of these potentials is

$$U=a_o+a_1x+a_2x^2+\frac{a_3}{(x+a_4)^2} \tag{2.2}$$

where $a_0, ..., a_4$ are arbitrary parameters.

Group properties of equation (2.1), (2,2) were used to solve it exactly, to establish connections between equations with different types of potentials, to separate variables, etc. [11]. Unfortunately, all these applications are valid for a very restricted class of potentials given in (2.2).

But the class of privileged potentials can be extended if we suppose that (2.1) admits non-Lie symmetries. Consider higher order symmetry operators which we represent in the form

$$Q=\sum_{i=0}^{n}(h_i\cdot p)_i, \qquad (h_i\cdot p)_i=[(h_i\cdot p)_{i-1},p]_+, \qquad (h_i\cdot p)_0=h_i \tag{2.3}$$

where h_i are unknown functions of (t,x), $[A,B]_+=AB+BA$, $p=-i\dfrac{\partial}{\partial x}$.

Operators (2.3) do not include differentials in respect with t which are expressed via p^2+U on a set of solutions of (2.1).

Definition. Q is a *symmetry* of (2.1) if it commutes with L:

$$[Q,L]=0 \tag{2.4}$$

Substituting (2.3) into (2.4) we come to the following system of the determining equations

$$\begin{aligned} &\partial_x h_n=0, \qquad \partial_x h_{n-1}+2\partial_t h_n=0,\\ &\partial_x h_{n-m}+2\partial_t h_{n-m+1}-\\ &-\sum_{k=0}^{\left[\frac{m-2}{2}\right]}(-1)^k\frac{2(n-m+2+2k)!}{(2k+1)!(n-m+1)!}h_{n-m+2k+2}\partial_x^{2k+1}U=0,\\ &\partial_t h_0+\sum_{p=0}^{\left[\frac{n-1}{2}\right]}(-1)^{p+1}h_{2p+1}\partial_x^{2p+1}U=0 \end{aligned} \tag{2.5}$$

where $m=2,3,...,n$, $[y]$ is the entire part of y.

The system (2.5) describes coefficients of an n-order symmetry operator for arbitrary n. Moreover, the compatibility condition for this system is the equation for potentials admitting these symmetries. For $n = 2$ the general solution of (2.5) for U reduces to the form (2.2) and the corresponding symmetries reduce to the usual Lie symmetries if we take into account (2.1).

Consider the case $n = 3$ (the simplest non-Lie symmetry). The system (2.5) reduces to the form

$$\begin{aligned}
&h_3' = 0, \qquad h_2' + 2\dot{h}_3 = 0,\\
&2\dot{h}_2 + h_1' - 6h_3U' = 0,\\
&2\dot{h}_1 + h_0' - 4h_2U' = 0,\\
&\dot{h}_0 - h_1U' + h_3U''' = 0
\end{aligned} \tag{2.6}$$

where the dot and prime denote derivations in respect with t and x.

The compatibility condition for (2.6) can be present in one of the following forms

$$U'' - 3U^2 - 4\omega_0 U = 4\omega_1, \tag{2.7a}$$

$$U'' - 3U^2 = 8\omega_2 x, \tag{2.7b}$$

$$(U'' - 3U^2)' - 2\omega_3(xU' + 2U) = 0, \tag{2.7c}$$

$$\varphi''' - 3(\varphi')^2 - 4\omega_4(x^2\varphi)' = \tfrac{1}{3}\omega_4^2 x^4 + \omega_5, \tag{2.7d}$$

$$U''' = 0 \quad or \quad 2U + (x + \omega_6)U' = 0 \tag{2.7e}$$

where φ and U are connected by the relation $U = \varphi'$ and $\omega_1, ..., \omega_6$ are arbitrary constants.

Formula (2.7a) presents the Weierstrass equation whose solutions are expressed via elliptic integrals. The first integral of (2.7a) has the form

$$\tfrac{1}{2}(U')^2 - U^3 - 2\omega_0 U^2 - 4\omega_1 U = C \tag{2.8}$$

and a particular solution of (2.8) is

$$U = \nu^2(2\tanh^2(\nu x) - 1), \quad \omega_0 = -\tfrac{1}{2}\nu^2, \quad \omega_1 = \tfrac{1}{4}\nu^4, \quad c = \nu^6. \tag{2.9}$$

Formula (2.7b) defines the first Painlevé transcendent. Equation (2.8c) reduces to the second Painlevé transcendent

$$W'' = \tfrac{1}{18}W^3 + \tfrac{1}{3}yW + K \tag{2.9}$$

where

$$U = -\sqrt[3]{\frac{\omega_3^2}{6}}V \quad x = -\sqrt[3]{\frac{1}{6\omega_3}}y, \quad V = W' - \tfrac{1}{6}W^2, \quad W' = \frac{\partial W}{\partial y}.$$

Equation (2.7d) reduces to the Riccati form

$$f' = f^2 + x^2/4 \tag{2.10}$$

where $\varphi = 2f - \dfrac{\omega_4 x^3}{3}$.

Equations (2.7e) are easily integrated, their solutions have the form (2.2).

Thus the set of potentials admitting third-order symmetries is rather extended and described by nonlinear equations (2.7).

3 Algebraic properties of symmetry operators

Using definitions (2.3), (2.6) we find the explicit form of symmetries corresponding to solutions of (2.7):

$$Q = p^3 + \frac{3}{4}[U,p]_+ + \omega_0 p \tag{3.1a}$$

$$Q = p^3 + \frac{3}{4}[U,p]_+ - \omega_2 t, \tag{3.2b}$$

$$Q = p^3 + \frac{3}{4}[U,p]_+ + \omega_3(tH - \frac{1}{4}[x,p]_+), \tag{3.1c}$$

$$\begin{gathered} Q_\pm = \frac{1}{\sqrt{24}}\left[p^3 \mp \frac{i}{4}\omega[[x,p]_+,p]_+ - \left[\omega^2 x^2 - 3\varphi', p\right]_+ \mp \right. \\ \left. \mp \frac{i}{2}\omega\left(\varphi + 2x\varphi' - \frac{\omega^2}{3}x^3\right)\right] \exp(\pm i\omega t), \quad \omega = \sqrt{-\omega_1}. \end{gathered} \tag{3.1d}$$

Solutions of equations (2.7e) correspond to such third order symmetry operators which are products of the usual Lie symmetries.

Symmetries (3.1) are not generators of Lie groups. Nevertheless, these symmetries generate very interesting algebraic structures, satisfying the following relations

$$\begin{aligned} &[Q,H] = 0, \\ &Q^2 = 2H(2H+\omega_0)^2 - \omega_1(2H+\omega_0) - \frac{C}{8}; \end{aligned} \tag{3.2a}$$

$$[Q,H] = i\omega_2 I, \qquad [Q,I] = [H,I] = 0; \tag{3.2b}$$

$$[Q,H] = -i\omega_3 H; \tag{3.2c}$$

$$[H,Q_\pm] = \pm\omega Q_\pm, \qquad [Q_+,Q_-] = \omega H^2. \tag{3.2d}$$

Formula (3.2a) corresponds to a particular case of the general theorem [13] maintaining that commuting ordinary differential operators are connected by a polynomial algebraic relation. In the following we use (3.2a) to integrate the corresponding equations (2.1).

Relations (3.2b) define the Heisenberg algebra; relations (3.2c) mean that Q plays a role of dilatation operator which continuously changes eigenvalues of H. In accordance with (3.2d), $Q_\pm$ play a role of increasing and decreasing operators like it takes place for the harmonic oscillator problem.

4 Exact solutions, conditional symmetry and non-Lie generation of solutions

For the potentials satisfying (2.7a), it is convenient to search for solutions of (2.1) in the form

$$\psi(t,x) = \exp(-iEt)\psi(x), \tag{4.1}$$

where $\psi(x)$ are eigenfunctions of the commuting operators H and Q:

$$H\psi(x) = E\psi(x) \tag{4.2a}$$

$$Q\psi(x) = \lambda\psi(x). \tag{4.2b}$$

Using (4.2a) we reduce (4.2b) to the form

$$\left(2E + \frac{U}{2} + \omega_0\right)\psi' = \left(\tfrac{1}{4}U' + i\lambda\right)\psi. \tag{4.3}$$

Integrating (4.3) we obtain

$$\psi = A\sqrt{U + 4E + 2\omega_0}\,\exp\left(2i\lambda \int \frac{dx}{U + 4E + 2\omega_0}\right) \tag{4.4}$$

where A is an arbitrary constant and E, λ are connected by the relation

$$\lambda^2 = 2E(2E + \omega_0)^2 - \omega_1(2E + \omega_0) - \frac{C}{8}, \tag{4.5}$$

C is an arbitrary constant of (2.8).

Thus, the third-order symmetry presents a powerful tool to obtain exact solutions of the corresponding Schrődinger equation. Moreover, solutions (4.4) are easily generalized to satisfy the nonlinear equation

$$i\partial_t\Psi = \frac{1}{2}p^2\Psi + \tfrac{1}{2A^2}(\Psi^*\Psi)\Psi \tag{4.6}$$

if we set

$$\Psi = \exp(i\epsilon t)\,\psi(x), \quad \epsilon = -3E - \omega_0 \tag{4.7}$$

Let us present a new procedure of generation of solutions of (2.1) using the conditional symmetry approach [6]. We start with a nonlinear equation which is Galilei invariant. So we can make the usual Lie generation of solutions starting with (4.7):

$$\Psi = A\sqrt{U(x - vt) + 4E + 2\omega_0}\ \exp\left[i\left[(2\epsilon - v^2)\frac{t}{2} + vx + 2\lambda \int_0^{x-vt} \frac{dy}{U(y) + 4E + 2\omega_0}\right]\right], \tag{4.8}$$

v is a transformation parameter.

Functions (4.8) for the case of potentials satisfying (2.9) reduce to soliton solutions

$$\Psi = A\left[\nu\tanh[\nu(x - vt)] \pm i\sqrt{\epsilon}\right]\exp\left[i\left(\nu^2 - \frac{v^2}{2}\right)t + (v \mp \sqrt{\epsilon})x + \varphi_0\right]. \tag{4.9}$$

In spite of the fact that the corresponding linear Schrődinger equation does not possesses any nontrivial (different from time displacements) Lie symmetry, we can generate new solutions using the conditional symmetry [5–7]. Indeed, solutions (4.4) satisfy the relation

$$\psi^*\psi = A^2(U + 4E + 2\omega_0). \tag{4.10}$$

But equation (2.1), (2.7a) with additional condition (4.10) is invariant under the Galilei transformation. This circumstance enables us to generate a new solution

$$\psi = A\sqrt{U(x - vt) + 4E + 2\omega_0}\,\exp\left[i\left[(-E - v^2)\frac{t}{2} + vx + 2\lambda \int_0^{x-vt} \frac{dy}{U(y) + 4E + 2\omega_0}\right]\right]. \tag{4.11}$$

The corresponding Schrődinger equation includes a potential $U(x - vt)$ where $U(x)$ is a solution of (2.7a).

One more generation can be made using the third-order symmetry

$$Q = p^3 + \frac{1}{4}[3U + 2\omega_0 + 6v^2, p]_+ + \frac{3}{2}vU \tag{4.12}$$

which is admitted by the Schrődinger equation with $U = U(x - vt)$. Acting on (4.11) by (4.12) we obtain a new solution

$$\psi' = Q\psi = a\psi + iv^2\psi_1, \quad a = \lambda + 4Ev + \omega_0 v - 4v^3,$$

where

$$\psi_1 = \frac{U' + 4i\lambda}{2(4E + U + 2\omega_0)}\psi \tag{4.13}$$

We notice that if ψ is the soliton solution

$$\psi = \frac{\nu A}{\cosh[\nu[x - vt)]} \exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right],$$

then the generated solution ψ_1 is a soliton solution also

$$\psi_1 = \frac{\nu^2 A \sinh[\nu(x - vt)]}{\cosh^2[\nu(x - vt)]} \exp\left[i\left(-\frac{iiv^2}{2}t + vx + \varphi_0\right)\right].$$

Thus non-Lie symmetries present new possibilities in solving the Schrődinger equation and generating new solutions [4].

5 Symmetries in supersymmetric quantum mechanics

The motion equation of supersymmetric quantum mechanics [14] has the form

$$L\psi \equiv \left(i\frac{\partial}{\partial t} - H\right)\psi = 0, \quad H = \frac{1}{2}\left(p^2 + W^2\right) + \sigma_3 W' \tag{5.1}$$

where $\psi = \psi(t, x)$ is a two-component wave function, σ_3 is the Pauli matrix, $W(x)$ is a superpotential.

Equation (5.1) has two specific symmetries (supercharges)

$$Q_1 = \frac{1}{\sqrt{2}}\left(\sigma_1 p + \sigma_2 W\right), \qquad Q_2 = \frac{1}{\sqrt{2}}\left(\sigma_2 p - \sigma_1 W\right) \tag{5.2}$$

which are valid for any superpotential. Moreover, these operators satisfy the following superalgebra

$$\begin{aligned} &[Q_a, Q_b]_+ = 2\delta_{ab}H, \quad a, b = 1, 2, \\ &[Q_a, H] = 0. \end{aligned} \tag{5.3}$$

Investigations of Lie and non-Lie symmetries of equation (5.1) are complicated by the matrix form of the Hamiltonian H. The corresponding symmetries can be expanded using the complete set of the Pauli matrices

$$Q = \sum_{\mu=0}^{3}\sum_{i=0}^{n}\left(h_i^{(\mu)} \cdot p_i\right)_i \tag{5.4}$$

where the notations (2.3) are used. Then starting with definition (2.4) we come to the following determining equations for unknown functions $h_i^{(\alpha)}, \alpha = 1, 2, \quad h_i^{\pm} = \frac{1}{2}\left(h_i^0 \pm h_i^3\right)$ and superpotentials W [15]:

$$
\begin{aligned}
&\partial_x h_n^{(\pm)} = 0,\\
&\partial h_{n-1}^{(\pm)} + 2\partial_t h_n^{(\pm)} = 0,\\
&\partial h_{n-m}^{(\pm)} + 2\partial_t h_{n-m+1}^{(\pm)} -\\
&\sum_{s=0}^{[(m-2)/2]} (-1)^s \frac{2(n-m+2+2s)!}{(2s+1)!(n-m+1)!} h_{n-m+2s+2}^{(\pm)} \partial_x^{2s+1}\left(W^2 \pm W'\right) = 0,\\
&\partial_t h_0^{(\pm)} + \sum_{p=0}^{[(n-1)/2]} (-1)^{p+1} h_{2p+1}^{(\pm)} \partial_x^{2p+1}\left(W^2 \pm W'\right) = 0;\\
&\partial_x h_n^{(\alpha)} = 0,\\
&\partial_x h_{n-1}^{(\alpha)} + 2\partial_t h_n^{(\alpha)} + 2i(-1)^\alpha h_n^{(\alpha)} \partial_x W,\\
&\partial_x h_{n-2}^{(\alpha)} + 2\partial_t h_{n-2}^{(\alpha)} + 2i(-1)^\alpha h_{n-1}^{(\alpha)} \partial W - 2n h_n^{(\alpha)} \partial_x W^2 = 0,\\
&\partial_x h_{n-2p-1}^{(\alpha)} + 2\partial_t h_{n-2p}^{(\alpha)} -\\
&\sum_{s=0}^{p-1} (-1)^s \frac{2(n+1+2s-2p)!}{(2s+1)!(n-2p)!} h_{n+1+2s-2p}^{(\alpha)} \partial_x^{2s+1} W^2 + 2i(-1)^\alpha h_{n-2p}^{(\alpha)} \partial_x W +\\
&i\sum_{s=0}^{p-1} (-1)^{\alpha+s+p} \frac{(n-2s)!}{(n-2p)!(2p-2s-1)!(p-s)} h_{n-2s}^{(\alpha)} \partial_x^{2p-2s+1} W = 0,\\
&p = 1, 3, \ldots, \left[\frac{n-1}{2}\right],\\
&\partial_x h_{n-2p}^{(\alpha)} + 2\partial_t h_{n-2p+1}^{(\alpha)} -\\
&\sum_{s=0}^{p-1} (-1)^s \frac{2(n+2+2s-2p)!}{(2s+1)!(n-2p+1)!} h_{n+2+2s-2p}^{(\alpha)} \partial_x^{2s+1} W^2 + 2i(-1)^\alpha h_{n-2p+1}^{(\alpha)} \partial_x W -\\
&i\sum_{s=0}^{p-2} (-1)^{\alpha+s+p} \frac{(n-2s-1)!}{(n-2s+1)!(2p-2s-3)!(p-s-1)} h_{n-2s-1}^{(\alpha)} \partial_x^{2p-2s-1} W = 0,\\
&p = 2, 4, \ldots, \left[\frac{n}{2}\right],\\
&\partial_t h_0^{(\alpha)} + \sum_{q=0}^{[(n-1)/2]} (-1)^{q+1} h_{2q+1}^{(\alpha)} \partial_x^{2q+1} W^2 + i\sum_{q=0}^{[n/2]} (-1)^{q+\alpha} h_{2q}^{(\alpha)} \partial_x^{2q+1} W = 0.
\end{aligned}
\tag{5.5}
$$

Equations (5.5) present a complete description of Lie and non-Lie symmetries of arbitrary order n. For exact solutions of (5.5) and the discussion of the corresponding symmetries refer to [15]. Here we present only the conclusion [15,16] that all physically interest ("privileged") potentials generate nontrivial Lie or non-Lie symmetries.

Such an analysis of Lie and non-Lie symmetries was extended to the case of parasupersymmetric quantum mechanics [17].

We also analyzed non-Lie symmetries of Wess-Zumino supersymmetric quantum mechanics [18, 19]. All these symmetries belong to the envelopping algebra generated by the first order summetries obtained by Arai [19].

6 Symmetries of three-dimension Schrődinger equation

Let us start with the free equation

$$\left(i\frac{\partial}{\partial t}-\frac{\mathbf{p}^2}{2m}\right)\psi=0,\qquad \psi=\psi(t,x) \tag{6.1}$$

where

$$\mathbf{x}=(x_1,x_2,x_3)\,,\quad \mathbf{p}^2=p_1^2+p_2^2+p_3^2,\quad p_a=-i\frac{\partial}{\partial x_a}. \tag{6.2}$$

It is well known that the maximal Lie symmetry of (6.1) is described by the 12-parameter Schrődinger group. The analysis of the non-Lie symmetries is complicated by multidimensionality of independent variables. This complication is overcome using the approach of generalized Killing tensors [20].

Let us represent a n-order symmetry in the form

$$Q=\sum_{i=0}^{n}\left[\left[\ldots\left[K^{a_1a_1\ldots a_j},\partial_{a_1}\right]_+,\partial_{a_2}\right]_+,\ldots\partial_{a_j}\right]_+, \tag{6.3}$$

Then the invariance condition (2.4) leads to the following determining equations

$$\begin{aligned}
&\partial^{(a_{j+1}}K^{a_1a_2\ldots a_j)}=-2m(j+1)\dot{K}^{a_1a_2\ldots a_{j+1}},\quad j=0,1,\ldots n-1,\\
&\partial^{(a_{n+1}}K^{a_1a_2\ldots a_n)}=0,\\
&\dot{K}=0,\quad j=0
\end{aligned} \tag{6.4}$$

where the complete symmetrization is imposed over the indices in brackets.

A differential consequence of (6.4) is

$$\partial^{(a_{j+1}}\partial^{a_{j+2}}\ldots\partial^{a_{j+s}}K^{a_1a_2\ldots a_j)}=0,\quad s=n-j+1. \tag{6.5}$$

We call solutions of (6.5) generalized Killing tensors of order s and rank j.

For the general solution of (6.4), (6.5) see [20,21]. The corresponding number of linearly independent n-order symmetries is

$$N_n=\frac{1}{4!}(n+1)(n+2)^3(n+3) \tag{6.6}$$

and all these symmetries belong to the envelopping algebra of the Lie algebra of the Schrődinger group.

For symmetries of the Schrődinger equation with a potential $V=V(\mathbf{x})$

$$\left[i\frac{\partial}{\partial t}-\frac{1}{2}\left(\mathbf{p}^2+V\right)\right]\psi=0$$

we obtain the following determining equations [22]

$$\begin{aligned}
&\partial^{(a_{n+1}}K^{a_1a_2\ldots a_n)}=0,\\
&2\dot{K}^{a_1a_2\ldots a_{2m}}+\frac{1}{2m}\partial^{(a_{2m}}K^{a_1a_2\ldots a_{2m-1})}+\\
&\qquad\sum_{k=m}^{[(n-1)/2]}(-1)^{m+k+1}\frac{2(2k+1)!}{(2k-2m+1)!(2m)!}U_k^{a_1a_2\ldots a_{2m}},\\
&2\dot{K}^{a_1a_2\ldots a_{2l+1}}+\frac{1}{2l+1}\partial^{(a_{2l+1}}K^{a_1a_2\ldots a_{2l})}\\
&\qquad+\sum_{k=l+1}^{[n/2]}(-1)^{k+l}\frac{2(2k)!}{(2k-2l-1)!(2l+1)!}W_k^{a_1a_2\ldots a_{2l+1}}
\end{aligned} \tag{6.7}$$

where

$$
\begin{aligned}
&m = 0, 1, ..., [n/2]\,, \quad l = 0, 1, ..., [(n-1)/2]\,,\\
&U_k^{a_1 a_2 \dots a_{2m}} = K^{a_1 a_2 \dots a_{2m} b_1 b_2 \dots b_{2k-2m+1}} \partial_{b_1} \partial_{b_2} \dots \partial_{b_{2k-2m+1}} V,\\
&W_k^{a_1 a_2 \dots a_{2l+1}} = K^{a_1 a_2 \dots a_{2l+{}^{\backprime}1} b_1 b_2 \dots b_{2k-2l-1}} \partial_{b_1} \partial_{b_2} \dots \partial_{b_{2k-2l-1}} V.
\end{aligned}
$$

These equations define as coefficients of the n-order symmetry (6.3) as potentials V admitting it. In paper [22] we solved (6.7) for the case of three-dimension potentials of the type (2.2) and found complete sets of the corresponding n-order symmetries.

7 Symmetries of the Dirac equation

Searching for non-Lie symmetries of the Dirac equation

$$
L\psi = (\gamma_\mu p^\mu - m)\,\psi = 0 \tag{7.1}
$$

needs a combination of the approaches mentioned in Sections 4 and 5, inasmuch as L is a matrix operator with differentials in respect to four variables. For the corresponding determining equations refer to [4]. Here we present the exact number of n-order symmetries

$$
\begin{aligned}
N_n = \frac{5}{4!}(n+1)(n+2)(2n+3)(n^2+3n+4) -\\
\frac{1}{6(2n+1)(13n^2+19n+18)} - \frac{1}{2}[1-(-1)^n].
\end{aligned} \tag{7.2}
$$

In particular,

$$
N_0 = 1, \quad N_1 = 25, \quad N_2 = 154. \tag{7.3}
$$

The first-order symmetries are exhausted by the following 25 representativities

$$
\begin{aligned}
&P_\mu = p_\mu, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu},\\
&W_{4\mu} = \frac{i}{2}\gamma_4(p_\mu - m\gamma_\mu),\\
&W_{\mu\nu} = \frac{i}{2}\gamma_4(\gamma_\mu p_\nu - \gamma_\nu p_\mu),\\
&B = i\gamma_4(D - m\gamma_\mu x^\mu),\\
&A_\mu = \frac{i}{2}\gamma_4 \varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho}\gamma^\sigma + \frac{1}{2}\gamma_\mu.
\end{aligned} \tag{7.4}
$$

Non-Lie symmetries (7.4) do not form a Lie algebra but include subsets generating superalgebras [23]. We present one of them

$$
Q_a = 2W_{4a} + \epsilon_{abc} W_{bc}, \quad a = 1, 2, 3 \tag{7.5}
$$

Symmetries (7.5) satisfy relations (5.3) together with $H = \mathbf{p}^2 + m^2$. In other words, *non-Lie symmetries of the Dirac equation generate the symmetry superalgebra of supersymmetric quantum mechanics.*

In conclusion we notice that the above results admit extensions to much more complicated systems having arbitrary numbers of independent and dependent variables. Non-Lie symmetries of two-particle and arbitrary spin particle equations were analyzed in [24, 25].

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