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On Parasupersymmetries and Relativistic Descriptions for Spin one Particles: II. The Interacting Context with (Electro)Magnetic Fields

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Abstract

This second part belongs to a series of two papers devoted to a constructive review of the relativistic wave equations for vector mesons due to the recent impact of spin one developments in connection with parasupersymmetric quantum mechanics. Here, the mesons are interacting with external (electro)magnetic fields but the simplest context of homogeneous constant magnetic fields directed along the z-axis is particularly studied. Discussions on reality of energy eigenvalues, on causal propagation and on gyromagnetic ratios are especially presented. Supersymmetries and parasupersymmetries are analysed with respect to new pseudosupersymmetries suggested by these developments in one particular context.

I. Introduction

In the first paper [1] of this series, we have systematically revisited the symmetric and Hamiltonian forms of *relativistic* wave equations describing *free* spin one particles. The motivation of such a study was effectively based on the recent interest enhanced by developments [2, 3] in parasupersymmetrical quantum mechanics (PSSQM) as already recalled in [1]. One of the main characteristics presented in that first part was the analysis of the 16-dimensional reducible representation of the Lie algebra $sl(2, \mathrm{C})$ in connection with the different (symmetric) formulations for nonzero rest mass vector mesons proposed in the literature: they effectively cover the wellknown wave equations due to BARGMANN-WIGNER [4] or DE BROGLIE [5], STUECKELBERG [6], KEMMER-DUFFIN-PETIAU [7] (hereafter called the Kemmer equation) and to HAGEN-HURLEY [8]. In that *free* context, we have proposed a *covariant* wave equation

$$(\beta^{\mu} p_{\mu} - m)\, \Psi(x) = 0 \tag{1.1}$$

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where the β^μ ($\mu = 0, 1, 2, 3$)-matrices satisfy the following relations that we call the TZOU relations [9]:

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu + \beta^\nu \beta^\mu \beta^\lambda + \beta^\lambda \beta^\mu \beta^\nu + \beta^\mu \beta^\lambda \beta^\nu + \beta^\nu \beta^\lambda \beta^\mu = 2g^{\mu\nu}\beta^\lambda + 2g^{\nu\lambda}\beta^\mu + 2g^{\mu\lambda}\beta^\nu \quad (1.2)$$

where the $g^{\mu\nu}$'s refer to the Minkowskian metric $G \equiv \{g^{\mu\nu} | g^{00} = -g^{ii} = 1\}$, all the conventions adopted here being identical to those given in [1]. Depending on specific choices of β^μ-matrices belonging to superpositions of irreducible representations contained in the above 16-reducible one, we have thus characterized all these symmetric formulations describing vector mesons through wave functions with decreasing numbers of components such as 16 [4, 5], 11 [6], 10 [7] or 7 [8].

One of the main purposes of this second paper is to propose and to discuss a generalization of eq. (1.1) to the *interacting* context dealing with vector mesons of charge e in external electromagnetic fields, an already fundamental application at the first quantized level with a view to extend it, later, to quantum field theory through interacting Lagrangians constrained by gauge invariance principles. In fact, we have to recall, at the start, that there are many problems related to such a programme. In particular, three of them can immediately be pointed out here

i) the appearance of possible complex energy eigenvalues [10–12] for (electro)magnetic fields of certain magnitudes,

ii) the causality of propagation [13–15],

iii) the acceptance of possible nonminimal interaction terms and the corresponding value of the gyromagnetic ratio $g = 2$ or $g = \frac{1}{s}$ [16, 17] where s is the spin of the particle.

Through our proposal, such problems will be considered in the following with a particular emphasis in the context of the Kemmer equation generalizing eq. (1.1), the latter taking the new form [18]

$$\left\{\beta^\mu \pi_\mu - m + P_K \left[\frac{e}{2m} S_{\mu\nu} F^{\mu\nu} - \frac{\lambda e^{2\alpha}}{2^\alpha m^{4\alpha-1}} (F_{\mu\nu} F^{\mu\nu})^\alpha\right]\right\} \Psi(x) = 0 \quad (1.3a)$$

where P_K is a Kemmer projector ($P_K^2 = P_K$) defined by

$$P_K \equiv \beta_\mu \beta^\mu - 2 \quad (1.3b)$$

which also appears as given by

$$P_K = 1 - \beta_5^2. \quad (1.3c)$$

Here, $\Psi(x)$ is a ten-component wave function describing charged spin one particles of nonzero rest mass m in external electromagnetic fields $F \equiv (\vec{E}, \vec{B}) \equiv \{F_{\mu\nu}\}$ where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \quad x \equiv (x^\mu) \equiv (x^0, x^1, x^2, x^3) \equiv (t, \vec{x}) \quad (1.4)$$

while the matrices β^μ, β_5 are such that the Tzou relations (1.2) reduce to the Kemmer ones [7, 19]

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\lambda\nu}\beta^\mu \quad (1.5)$$

and that [20]

$$\beta_5 = \frac{i}{4}\,\varepsilon_{\mu\nu\varrho\sigma}\,\beta^\mu\beta^\nu\beta^\varrho\beta^\sigma, \quad \varepsilon_{0123}=1. \tag{1.6}$$

In eq. (1.3), we also notice the usual minimal electromagnetic coupling through the substitution

$$p_\mu \rightarrow \pi_\mu = p_\mu - e\,A_\mu(x), \tag{1.7}$$

$A(x)\equiv\{A^\mu(x)\}$ being the fourpotential fixed up to a gauge transformation on the scalar part $A^0\equiv V$ and the vector part $\vec{A}(x)$. This minimal coupling is supplemented by extra terms characterizing anomalous (nonminimal) couplings where we recognize the covariant form of the famous Pauli term when the relativistic spin tensor $S_{\mu\nu}$ is given by

$$S_{\mu\nu} = i\,[\beta_\mu, \beta_\nu]. \tag{1.8}$$

Finally, there is also a nonlinear scalar term depending on a new parameter λ and on an exponent α which will be discussed hereafter.

The study of eq. (1.3) through many elements will be developed in *Section 2* and will represent the main part of this paper when we limit ourselves to the simplest case of homogeneous constant magnetic fields and to the special choice of the exponent $\alpha=\frac{1}{2}$. In strong relation with the free context and its associated wave equations discussed in the first part of this series, we will then come back on the Hagen-Hurley (*Section 3*), Bargmann-Wigner or de Broglie (*Section 4*) and Stueckelberg (*Section 5*) formulations. Finally, in *Section 6*, we will summarize our results, give the main conclusions in connection with the three above-mentioned problems (i, ii, iii) and point out some further remarks.

As already mentioned, our conventions will be identical to those given in the first part of this series: moreover, for brevity, we will refer to formulas of this first part by adding a capital I inside their reference numbers, so that no confusion could appear with the present formulas.

2. The Kemmer Formulation with (Electro)Magnetic Interactions

Taking advantage of our general developments, it appeared that the Kemmer formulation (described by a ten-component wave function) revealed itself as the richest one in order to show the way of reasoning for all the descriptions. Thus, essentially for pedagogical reasons, we propose to start our study by this Kemmer generalization of eq. (1.1) to the interacting case, i.e. when charged, massive, vector mesons are subject to electromagnetic external fields. In order to fix our specific considerations (when necessary), we will use the explicit 10-dimensional representation (I.2.20) corresponding to a direct sum of the $sl(2, \mathbb{C})$-representations $D(1,0)$, $D(0,1)$ and $D(\frac{1}{2},\frac{1}{2})$ [see (I.2.22)].

In that context, the matrix $\beta_5\equiv(1.6)$ takes the form

$$\beta_5 = i(-e_{1,4}+e_{4,1}-e_{2,5}+e_{5,2}-e_{3,6}+e_{6,3}) \tag{2.1}$$

leading to

$$\beta_5^2 = 3-\beta_\mu\beta^\mu = e_{1,1}+e_{2,2}+e_{3,3}+e_{4,4}+e_{5,5}+e_{6,6} \tag{2.2a}$$

and, evidently, to

$$P_K = e_{7,7} + e_{8,8} + e_{9,9} + e_{10,10}. \tag{2.2b}$$

Let us recall here that the notation $e_{j,k}$ refers to a d-dimensional matrix ($j, k = 1, 2, \ldots, d$) containing all zero elements except the one located at the intersection of the j^{th} line and k^{th} column which is equal to unity.

We propose to come, first, on a short historical survey (§ 2.A) completing a very recent one [12, 15] in order to connect, as a second part, these developments to parasupersymmetric properties [2, 3] by visiting some energy spectra (§ 2.**B**). Finally (§ 2.C), our third step is devoted to the introduction of new nonlinear interaction terms in covariant form and to their implications in a resulting new context dealing with pseudosupersymmetries in particular.

2.A. Comments on a historical survey

The best review of the description of vector mesons in external electromagnetic fields can undoubtedly be obtained by superposing some sections of two contributions due to VIJAYALAKSHMI et al. [15] and to DAICIC-FRANKEL [12]. The discussions on the appearance of complex energies (for intense magnetic fields) and on the causality of propagation are quoted there in connection with the superposition of minimal coupling terms with anomalous (magnetic moment) coupling ones. For brevity, we refer to these references [12, 15, and references therein] while here we want to present a new characterization of the above difficulty on complex energies by giving an appropriate generalization of the equation (1.1) containing the minimal coupling as well as an anomalous magnetic moment coupling in the Kemmer case.

So, let us analyse the relativistic Kemmer equation

$$\left[\beta^\mu \pi_\mu - m + P_K \frac{e}{2m} S_{\mu\nu} F^{\mu\nu}\right] \Psi(x) = 0, \tag{2.3}$$

which corresponds to eq. (1.3) when $\lambda = 0$. It is the more general CPT-invariant and covariant equation corresponding to the gyromagnetic ratio $g = 2$ which has already been proposed: it includes the minimal electromagnetic coupling (1.7) and an anomalous coupling expressed in terms of the tensors (1.4) and (1.8). It is easy to convince ourselves that such a system of ten equations reduces to the CORBEN-SCHWINGER equations [21] on the four components $\varphi_0 = \Psi_{10}$, $\varphi_1 = \Psi_7$, $\varphi_2 \equiv \Psi_8$, $\varphi_3 \equiv \Psi_9$ or to the SHAY-GOOD formulation [22] on the six components $\Psi_1, \ldots, \Psi_6$. Moreover, expressed in Tamm-Sakata-Taketani Hamiltonian form [25], it leads to a Hamiltonian which is unitary equivalent to the DAICIC-FRANKEL one [12] as it is easily verified. Due to the wellknown properties of the preceding developments, we immediately deduce that the formulation based on eq. (2.3) satisfies the causality principle but leads to an energy spectrum containing possible complex energy eigenvalues (for some values of the magnetic field, in particular). Indeed, if we limit ourselves to the (very often studied) simplest context of homogeneous constant *magnetic* fields, i.e. $F \equiv [\vec{E} = \vec{0}, \vec{B} \equiv (0, 0, B)]$ derived from the gauge symmetrical potential [26]

$$A_0 = A_3 = 0, \quad A_1 = -Bx_2, \quad A_2 = -Bx_1, \quad F_{12} = B, \tag{2.4a}$$

and implying that

$$\pi_0 = p_0, \quad \pi_3 = p_3, \quad \pi_1 = p_1 + eBx_2, \quad \pi_2 = p_2 - eBx_1, \tag{2.4b}$$

our equation (2.3) becomes

$$\left[\beta^{\mu}\pi_{\mu}-m+P_K\frac{eB}{m}\Sigma_3\right]\Psi(x)=0 \tag{2.5}$$

where, inside the so-chosen Kemmer representation, the matrix Σ_3 is given through eq. (1.8) by

$$\Sigma_3\equiv S_{12}=-i(e_{1,2}-e_{2,1}+e_{4,5}-e_{5,4}+e_{7,8}-e_{8,7}). \tag{2.6}$$

Let us also notice that the 6-dimensional Tamm-Sakata-Taketani formulation associated with eq. (2.5) is

$$i\frac{\partial\chi(x)}{\partial t}=H_{\mathrm{TST}}\,\chi(x), \tag{2.7a}$$

$$H_{\mathrm{TST}}=m(I_3\otimes\sigma_2)+\frac{\vec{\pi}^2}{2m}I_3\otimes(\sigma_2+i\sigma_1)+\frac{1}{m}eB(S_3\otimes\sigma_2)-$$
$$-\frac{i}{m}\sum_{j,k=1}^{3}\pi_j\pi_k(S_jS_k\otimes\sigma_1) \tag{2.7b}$$

where we refer to direct products between 3 by 3 (unit I_3 and) S_1, S_2, S_3 matrices belonging to the $D^{(1)}$-representation of $su(2, \mathbb{C})$ with the usual Pauli matrices. Let us notice that it coincides with the hamiltonian (I.3.12) in the *free* case, the identification of the corresponding 6 by 6 matrices being evident. Such a formulation takes also the Zaitsev-Feynman-Gell Mann form [23, 24]

$$(\pi^{\mu}\pi_{\mu}-m^2-2eBS_3\otimes I_2)\,\chi(x)=0 \tag{2.8}$$

showing that the gyromagnetic ratio implied in these developments is $g=2$, a characteristic value which will be discussed in the following (cf. Section 6).

From eqs. (2.7) *or* (2.8), it is easy to confirm by using JOHNSON-LIPPMANN arguments [27] that we get relativistic energy eigenvalues such that

$$E_n^2=m^2+2eB\left(n+\frac{1}{2}+s\right) \tag{2.9}$$

where n is the (principal) quantum number ($n=0, 1, 2, \ldots$) associated with the resulting one-dimensional harmonic oscillator issued from the so-called perpendicular contribution [27] and where s denotes the eigenvalues $(0, \pm 1)$ of the (diagonalized) matrix $S_3\otimes I_2$. For specific values such as $n=0$, $s=-1$, we thus point out possible negative eigenvalues E_0^2 (depending on the relativistic strength of the magnetic field): the energy eigenvalues of our problem become complex for

$$B\geq\frac{m^2}{e} \tag{2.10}$$

as already noticed since a long time by different authors [10, 11, 21].

Let us now enlighten these difficulties through parasupersymmetric arguments.

2.B. Some light from parasupersymmetry to spin one descriptions

PSSQM has effectively led to two approaches which, in particular, differ from permitting [2] or not [3] *negative* eigenvalues of the nonrelativistic corresponding parasuperhamiltonians [28]. Let us try to connect the RUBAKOV-SPIRIDONOV characteristics [2] with the difficulty pointed out in Section 2.A when the nonrelativistic limit of these developments is considered. In fact, in correspondence with the (relativistic) equation (2.8) which explicitly writes

$$p_0^2\, \chi(x) = (m^2 + \pi_1^2 + \pi_2^2 + p_3^2 - 2eBS_3 \otimes I_2)\, \chi(x) \tag{2.11}$$

and with the (relativistic) energies (2.9), we can point out the *nonrelativistic* ($m \gg p$) spectrum

$$E_n^{\mathrm{NR}} = \frac{eB}{m}\left(n + \frac{1}{2} - s\right) = \omega\left(n + \frac{1}{2} - s\right), \tag{2.12}$$

a spectrum typically associated with the RUBAKOV-SPIRIDONOV approach to PSSQM [2] when a parasupersymmetric harmonic oscillatorlike system is studied. It is characterized by an angular frequency $\omega = \frac{eB}{m}$ and assignes, at the start, a negative energy eigenvalue to the fundamental groundstate. The Rubakov-Spiridonov nonrelativistic parasuperhamiltonian associated with the eigenvalues (2.12) is

$$H_{\mathrm{PSS}} = \frac{1}{2m}(\pi_1^2 + \pi_2^2) - \omega S_3 \tag{2.13}$$

where S_3 is taken in its diagonal form $S_3 = \mathrm{diag}(1, 0, -1)$. Then, we can define two parasupercharges Q_1 and Q_2 given by

$$Q_1 = \frac{1}{\sqrt{2m}}(S_1 \pi_1 + S_2 \pi_2), \quad Q_2 = \frac{1}{\sqrt{2m}}(S_1 \pi_2 + S_2 \pi_1) \tag{2.14}$$

and show that we easily recover the typical structure relations of the RUBAKOV-SPIRIDONOV algebra [2], i.e.

$$\begin{aligned}
&[H_{\mathrm{PSS}}, Q_a] = 0, \quad Q_a^3 = Q_a H_{\mathrm{PSS}}, \qquad a = 1, 2,\\
&\{Q_1^2, Q_2\} + Q_1 Q_2 Q_1 = Q_2 H_{\mathrm{PSS}}, \quad \{Q_2^2, Q_1\} + Q_2 Q_1 Q_2 = Q_1 H_{\mathrm{PSS}}.
\end{aligned} \tag{2.15}$$

All these properties confirm the association of Rubakov-Spiridonov nonrelativistic developments with the relativistic equation (2.11) and its possible complex energy eigenvalues for magnetic fields characterized by a fixed strength according to eq. (2.10). The origin of such difficulties appears as enlightened by such an association.

On this basis, we can try to exploit the other approach [3] of PSSQM, which is precisely characterized by nonnegative energy eigenvalues of the parasuperspectrum corresponding to harmonic oscillatorlike system [3, 28]. Let us, indeed, recall that, with respect to the Rubakov-Spiridonov spectrum (2.12), the Beckers-Debergh one takes the form

$$E_n^{\mathrm{NR}} = \frac{eB}{m}\left(n + \frac{1}{2} + \frac{c}{2}\right), \quad c = 1, -1, 1, \tag{2.16}$$

and leads to threefold degeneracies already discussed. The main result is that it corresponds to a fundamental groundstate with zero energy ensuring exact parasupersymmetry [3] and, consequently, excluding negative eigenvalues. Then, the critical point connected with the use of the Beckers-Debergh approach in PSSQM in order to solve the above difficulty is that we have to go the way back from nonrelativistic developments. A recent approach solving such a problem has already been proposed [29]: it leads to remarkable results obtained by JOHNSON and LIPPMANN [27] in the sense that the particular solution [29] requires not only the usual operators π_a $(a=1, 2)$ given in eqs. (2.4b) but also the Johnson-Lippmann Π_a $(a=1, 2)$ defined by

$$\Pi_a = p_a + e A_a \Rightarrow \Pi_1 = p_1 - e B x_2, \quad \Pi_2 = p_2 + e B x_1 . \tag{2.17}$$

For brevity we refer to the original reference [19] for further details.

2.C. From spin one relativistic descriptions to pseudosupersymmetry

Let us now propose another method based on a generalization of the equation (2.3) or, at the moment, of the equation (2.5) when we restrict the interaction context as already mentioned to constant magnetic fields directed along the z-axis [18]. Let us modify eq. (2.11) by adding a new term characterized by a real parameter λ as follows

$$p_0^2\, \chi(x) = [m^2 + \pi_1^2 + \pi_2^2 + p_3^2 - 2 e B S_3 \otimes I_2 + \lambda e B]\, \chi(x), \tag{2.18}$$

so that we now obtain, in correspondence with eqs. (2.9) and (2.12),

$$E_n^2 = m^2 + 2 e B \left(n + \frac{1}{2} - s + \frac{\lambda}{2} \right) \tag{2.19a}$$

and

$$E_n^{\mathrm{NR}} = \frac{e B}{m} \left(n + \frac{1}{2} - s + \frac{\lambda}{2} \right), \quad \frac{e B}{m} = \omega . \tag{2.19b}$$

The introduction of this new parameter λ permits a simple discussion in order to avoid negative values of E_0^{NR} in correspondence with $n=0$ and $s=1$: this will always be the case for $\lambda \geq 1$ in the above considerations.

In fact, the new term introduced in eq. (2.18) corresponds to a further nonlinear interaction term in eq. (2.5). We effectively propose to write the new general Kemmer equation on the form

$$\left\{ \beta^\mu \pi_\mu - m + P_K \left[\frac{e B}{m}\, \Sigma_3 - \lambda \frac{e^{2\alpha}}{m^{4\alpha - 1}}\, B^{2\alpha} \right] \right\} \Psi(x) = 0 \tag{2.20}$$

where, besides λ, we have introduced a supplementary real parameter α which takes the value $\alpha = \frac{1}{2}$ for discussing our equation (2.18). By noticing that, in the special context of the magnetic field B, we have

$$(F_{\mu\nu} F^{\mu\nu})^\alpha \equiv 2^\alpha B^{2\alpha}, \tag{2.21}$$

we point out that the most general *covariant* form corresponding to our equation (2.20) is effectively

$$\left\{\beta^{\mu}\pi_{\mu}-m+P_{K}\left[\frac{e}{2m}S_{\mu\nu}F^{\mu\nu}-\frac{\lambda e^{2\alpha}}{2^{\alpha}m^{4\alpha-1}}(F_{\mu\nu}F^{\mu\nu})^{\alpha}\right]\right\}\Psi(x)=0 \tag{2.22}$$

which ensures not only the reality of the energy eigenvalues but also the causal propagation as it can be tested through the method of characteristics [13] for any value of the parameter α. This equation coincides with our original proposal mentioned in the introduction (see eq. (1.3)).

In order to keep the contact with previous results, let us once again consider the particular equation (2.20). Let us then notice that it leads to a new kind of Zaitsev-Feynman-Gell Mann equation:

$$(\pi^{\mu}\pi_{\mu}-m^{2}+2eB\Sigma_{3})\chi(x)-\lambda m^{2}\left(\frac{eB}{m^{2}}\right)^{2\alpha}\chi(x)=0 \tag{2.23}$$

which evidently reduces to eq. (2.18) for the particular value $\alpha=\frac{1}{2}$ and confirms – according to eqs. (2.19) – that, for $\alpha=\frac{1}{2}$, the value $\lambda\geqq 1$ guarantees the reality of the energy eigenvalues. By the same way of reasoning, we also deduce that the constraint

$$\lambda\geqq\frac{1}{4}\left(\frac{m^{2}}{eB}\right)^{2(\alpha-1)} \tag{2.24}$$

works for $\alpha\geqq 1$ ensuring also real eigenvalues. Moreover, it is interesting to notice that, for $\alpha=1$, and, consequently, $\lambda\geqq\frac{1}{4}$, or equation (2.22) or (2.23) cannot be included into the family discussed by VIJAYALAKSHMI et al. [15], so that we have to look at its causal character by ourselves. Through the method of characteristics [13], we are led to lightlike normals and to the expected property which can also be extended for general electromagnetic fields through steps associated with the study of the so-called parallel ($F_{\parallel}$) and perpendicular ($F_{\perp}$) electromagnetic fields [30]. Let us finally stress out that, in that special context $\alpha=1$, the λ-constraint does not depend on the external field strengths while the general Kemmer equation (2.20) is quadratic in these quantities.

Such encouraging results have then to be analysed in connection with PSSQM-arguments and, more precisely, with the BECKERS-DEBERGH [3] approach due to the evident nonnegative character of the energy eigenvalues in the nonrelativistic limit of eq. (2.23) or (2.18). If we priviledge the specific allowed values $\alpha=\frac{1}{2}$ and $\lambda=1$ [18], it is interesting to point out that the resulting nonrelativistic (Johnson-Lippmann type) Hamiltonian takes now the form

$$H^{\mathrm{NR}}=\frac{1}{2m}(\pi_{1}^{2}+\pi_{2}^{2})+\frac{\omega}{2}(I_{3}-2S_{3}),\qquad \omega=\frac{eB}{m}, \tag{2.25}$$

(compare with eq. (2.13)) and can once again results from two charges called here Q'_{1} and Q'_{2}. These charges generate with H^{NR} a new structure of the type

$$[H^{\mathrm{NR}},Q'_{a}]=0,\quad Q'^{3}_{a}=Q'_{a}H^{\mathrm{NR}},\qquad a=1,2, \tag{2.26a}$$

$$Q'^{2}_{a}Q'_{b}=Q'_{b}Q'^{2}_{a}=-Q'_{a}Q'_{b}Q'_{a}=Q'_{b}H^{\mathrm{NR}},\qquad a\neq b. \tag{2.26b}$$

This is neither a Lie parasuperalgebra [31], nor a Lie superalgebra [32] but is called a Lie pseudosuperalgebra [18] subtending pseudosupersymmetries and, physically, corresponding to a model where usual bosons and (new) pseudofermions are superposed. Some connections and differences with other recent proposals due to GREENBERG [33] or to KHARE et al. [34] have already been pointed out [35]. A positive consequence of such particular developments issued – let us remember it – from the interaction of vector mesons in magnetic fields, is the discovery of a (maybe) interesting new field that we call "*pseudosupersymmetric quantum mechanics*". Let us only mention here, in comparison with PSSQM which is evidently subtended by parastatistics [36], that we have found characteristics leading to a new kind of statistics, consistently called "*pseudostatistics*", and describing the above mentioned pseudofermions. In particular, the creation ($b^\dagger$) and annihilation (b) operators of these pseudofermions are such that

$$b^2 = (b^\dagger)^2 = 0, \quad b\,b^\dagger b = b, \quad b^\dagger b\,b^\dagger = b^\dagger. \tag{2.27}$$

These relations show, in particular, that pseudofermions are neither fermions, nor parafermions, nor quons [33], nor $p=2$-orthofermions [34], but that they can be interesting through welcome properties such as the one showing that they lead to small violations of the Pauli principle. Pseudosupersymmetry will appear as a new concept located, let us say, "*between supersymmetry and parasupersymmetry*": one of its first already positive effect is thus simply connected with reality of (relativistic) energies and causality requirements in the study of vector mesons in external electromagnetic fields when we study the particular context $\alpha = \frac{1}{2}$ and $\lambda = 1$.

Let us end this section by mentioning that the pseudostatistical contents are nothing else but a superposition of $p=0$-parafermions *and* $p=1$-(para)fermions corresponding to a direct sum of $s=0$ *and* $s=\frac{1}{2}$ (para)particles. Such a property can be understood by noticing that the pseudofermionic operators b and $b^\dagger$ satisfying eqs. (2.27) can be cast in the following forms through a unitary transformation:

$$U b U^\dagger = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), \quad U b^\dagger U^\dagger = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right), \tag{2.28}$$

where we recognize a direct sum of diagonal blocks referring to these two kinds of parafermions. The corresponding properties (2.26) are evidently verified and the associated nonrelativistic Hamiltonian takes the form

$$(H^{\mathrm{NR}})' = U H^{\mathrm{NR}} U^\dagger = \omega \left(\begin{array}{c|cc} a^\dagger a + 2 & & \\ \hline & a^\dagger a + 1 & \\ & & a^\dagger a \end{array}\right) = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}, \tag{2.29}$$

where a and $a^\dagger$ are the corresponding bosonic operators appearing in the charges. We evidently recover the supersymmetric Hamiltonian (and spectrum) through the above 2 by 2-submatrix h_1. From h_0, we deduce for oscillatorlike considerations that

$$h_0 = \omega(a^\dagger a + 2) = \frac{1}{2}(p^2 + \omega^2 x^2 + 3\omega) \tag{2.30}$$

showing that the total Hamiltonian (2.29) is *not* equivalent to a direct sum of a trivial $p=0$-parasupersymmetry *and* an expected ($p=1$)-supersymmetry, although the equivalence is effective at the level of the charges.

3. The Hagen-Hurley Equation with External Magnetic Fields

The 7-component description of spin one particles proposed by HAGEN and HURLEY [8] corresponds to the direct sum $D\left(\frac{1}{2}, \frac{1}{2}\right) \oplus D(1, 0)$ (see eq. (I.2.25)) and can be characterized by eq. (1.1) within a representation of the matrices β_μ given by eq. (I.2.24) and satisfying the TZOU relation (1.2). These results, typical of our part I, have now to be handled in order to test if the general covariant equation (1.3) is still valid for our interacting context but in a consistent way with the free one. In fact, this happens if we replace the Kemmer projector by the following Hagen-Hurley one P_{HH}:

$$P_{\mathrm{HH}} = \beta_\mu \beta^\mu - 3 = e_{1,1} + e_{2,2} + e_{3,3}. \tag{3.1}$$

The corresponding equation (3.1) then writes

$$\left\{\beta^\mu \pi_\mu - m + P_{\mathrm{HH}}\left[\frac{e}{2m} S_{\mu\nu} F^{\mu\nu} - \frac{\lambda e^{2\alpha}}{2^\alpha m^{4\alpha-1}} (F_{\mu\nu} F^{\mu\nu})^\alpha\right]\right\} \Psi_{\mathrm{HH}}(x) = 0 \tag{3.2}$$

and reduces the Hagen-Hurley developments to the same discussion as the one presented in Section 2.C. We thus recover the correspondence between the Kemmer and Hagen-Hurley formulations in this interaction case.

An interesting point is to discuss the meaning of the projector P_{HH} in terms of a β_5-matrix which could be introduced in the Tzou algebra associated with this Hagen-Hurley formulation. So, what matrix could play the role of β_5 in the present developments remembering that the definition (1.6) is only valid in the Kemmer algebra – called $\mathscr{K}(4)$ – characterized by the relations (1.5)? Let us recall here that relatively elaborate algebraic developments due to KEMMER [7] and others [37–40, and references therein] are already subtended by the *simple* relations (1.5): in particular, the matrix $\beta_5 \equiv (1.6)$ then appears as a unique fifth element leading to a Kemmer algebra $\mathscr{K}(5)$ (with $g_{55} = +1$, for example). Through the Tzou relations (1.2), only similar but orientied questions can actually be solved and we want to adopt such a point of view in all the present and following sections. So, let us ask for the form of a fifth element β_5 of the Tzou algebra by taking the explicit 7-dimensional representation (I.2.24) in our developments. If we parametrize such a β_5-matrix by 49 unknowns, we finally get *three* independent forms given by

$$\beta_5^{(1)} = i(e_{2,3} - e_{3,2} - i e_{4,7} - i e_{7,4} - e_{5,6} + e_{6,5}), \tag{3.3a}$$

$$\beta_5^{(2)} = i(e_{1,3} - e_{3,1} - e_{4,6} + e_{6,4} + i e_{5,7} + i e_{7,5}), \tag{3.3b}$$

$$\beta_5^{(3)} = i(e_{1,2} - e_{2,1} - e_{4,5} + e_{5,4} - i e_{6,7} - i e_{7,6}). \tag{3.3c}$$

Let us point out that, if, in the Kemmer context, the fifth element leads from $\mathscr{K}(4)$ to $\mathscr{K}(5)$, we can argue that, in the Tzou context, the matrices $\beta_5^{(1)}$, $\beta_5^{(2)}$ and $\beta_5^{(3)}$ lead from an algebra $\tau(4)$ to a new one called $\tau(7)$ characterized by seven fundamental elements, i.e. the four β_μ- and the three $\beta_5^{(i)}$-matrices, when the metric is chosen in the following form $(+, -, -, -, +, +, +)$. In terms of the matrices (3.3), we immediately get

$$P_{\mathrm{HH}} = 3 - \beta_5^{(1)^2} - \beta_5^{(2)^2} - \beta_5^{(3)^2} \tag{3.4}$$

and realize in that way the above connection.

4. The Bargmann-Wigner or de Broglie Context

Through the fusion method developed by DE BROGLIE [6] and discussed by TZOU [9], we learned [1] that the free BARGMANN-WIGNER [4] and de Broglie formulations are equivalent: they are characterized by a 16-component wave function $\Psi(x)$ and 16 by 16 Tzou matrices satisfying the relations (1.1) and (1.2) respectively. If each of the preceding (10-component) Kemmer and (7-component) Hagen-Hurley contexts refer to spin one particles only, let us point out that this (16-component) de Broglie description contains, from the start, parasitic (5-component) spin zero and (1-component) trivial formulations. This has easily been noticed through the relations (I.2.3)–(I.2.5) and the resulting direct sum of $sl(2, \mathbb{C})$-irreducible representations leading to the matrices

$$\beta_\mu = \begin{pmatrix} \beta_\mu^{(1)} & 0 & 0 \\ 0 & \beta_\mu^{(0)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.1}$$

where $\beta_\mu^{(1)}$, $\beta_\mu^{(0)}$ refer to representations of Kemmer matrices for spin one (10 by 10) and for spin zero (5 by 5) respectively. In order to fix our choices (when necessary), let us remember that we will use the explicit 10-dimensional representation (I.2.20) for the spin 1-Kemmer description and the following 5-dimensional one for the spin 0-Kemmer description:

$$\begin{aligned} \beta_0^{(0)} &= i(e_{2,1} - e_{1,2}), \quad \beta_1^{(0)} = i(e_{1,3} + e_{3,1}), \\ \beta_2^{(0)} &= i(e_{1,4} + e_{4,1}), \quad \beta_3^{(0)} = i(e_{1,5} + e_{5,1}), \end{aligned} \tag{4.2}$$

so that we have an explicit representation of the matrices $\beta_\mu \equiv (4.1)$ which evidently satisfy the structure relations (1.5).

Once again but now in this Bargmann-Wigner or de Broglie context, we claim that we can propose and ad-hoc covariant equation of the type (1.3) or (3.2) solving our problems. This equation reads:

$$\left\{ \beta^\mu \pi_\mu - m + P_{\rm BW} \left[\frac{e}{2m} S_{\mu\nu} F^{\mu\nu} - \frac{\lambda e^{2\alpha}}{2^\alpha m^{4\alpha-1}} (F_{\mu\nu} F^{\mu\nu})^\alpha \right] \right\} \Psi_{\rm BW}(x) = 0 \tag{4.3}$$

where we have introduced the new Bargmann-Wigner projector $P_{\rm BW}$ constructed as follows:

$$P_{\rm BW} = -\frac{1}{8} (\beta^\mu \beta_\mu)(\beta^\mu \beta_\mu - 1)(\beta^\mu \beta_\mu - 2) \left(\beta^\mu \beta_\mu - \frac{13}{3} \right) \tag{4.4}$$

becoming simply

$$P_{\rm BW} = e_{7,7} + e_{8,8} + e_{9,9} + e_{10,10} + e_{11,11}. \tag{4.5}$$

The study of eq. (4.3) follows in a complete parallel way to the ones developed for the Kemmer or Hagen-Hurley formulations in the preceding sections and we get equivalent results.

Let us now inquire about possible connections between our projector (4.4) and acceptable β_5-matrices in this 16-dimensional context. We first notice that the definition (1.6) cannot be trivially extended to the 16-dimensional case although we know [41, 42] that the matrix $\beta_5^{(0)}$ is identically equal to zero in the spin 0 formulation. Indeed, a general search for a fifth 16 by 16 matrix satisfying the structure relations (1.5) leads to the matrix

$$\beta_5 = i(e_{4,1} - e_{1,4} - e_{2,5} + e_{5,2} - e_{3,6} + e_{6,3} + e_{11,16} - e_{16,11}). \tag{4.6}$$

We thus conclude that it is not possible to express our projector P_{BW} in terms of the square of such a matrix.

5. The Stueckelberg Formulation

We also learned [1] that the STUECKELBERG formulation [6], referring to "interference zones" as quoted by TZOU [27], corresponds to the direct sum (I.2.16) and contains the simultaneous descriptions of spins 0 *and* 1 through an 11-component wave function and 11 by 11 matrices given in eqs. (I.2.15) and satisfying the Tzou relations (1.2). The corresponding equation (1.1) appears as a difficult one to be generalized to the interaction context for disconnecting the different spin descriptions. We can suggest a new projector called P_{ST} given by

$$P_{ST} = \frac{1}{12}(\beta_\mu \beta^\mu - 2) S_{\mu\nu} S^{\mu\nu} \tag{5.1}$$

where $S_{\mu\nu}$ has been defined in eq. (1.8). Let us mention that, within the representation (I.2.15), we have already given the Stueckelberg spin components S_{12}, S_{23} and S_{31} in eqs. (I.2.18). The other three components take then the forms

$$\begin{aligned} S_{01} &= e_{2,7} + e_{7,2} - e_{3,6} - e_{6,3} - e_{8,9} - e_{9,8}, \\ S_{02} &= -e_{1,7} - e_{7,1} + e_{3,5} + e_{5,3} - e_{8,10} - e_{10,8}, \\ S_{03} &= e_{1,6} + e_{6,1} - e_{2,5} - e_{5,2} - e_{8,11} - e_{11,8}, \end{aligned} \tag{5.2}$$

so that we obtain

$$\begin{aligned} S_{\mu\nu} S^{\mu\nu} = {} & 8(e_{1,1} + e_{2,2} + e_{3,3} + e_{5,5} + e_{6,6} + e_{7,7}) \\ & + 6(e_{8,8} + e_{9,9} + e_{10,10} + e_{11,11}). \end{aligned} \tag{5.3}$$

The projector P_{ST} finally is

$$P_{ST} = e_{8,8} + e_{9,9} + e_{10,10} + e_{11,11} \tag{5.4}$$

and the corresponding Stueckelberg covariant equation writes

$$\left\{ \beta^\mu \pi_\mu - m + P_{ST} \left[\frac{e}{2m} S_{\mu\nu} F^{\mu\nu} - \frac{\lambda e^{2\alpha}}{2^\alpha m^{4\alpha - 1}} (F_{\mu\nu} F^{\mu\nu})^\alpha \right] \right\} \Psi_{ST}(x) = 0. \tag{5.5}$$

Once again, we recover its correspondence with the other formulations in this interacting case.

Let us now come on the question concerning the existence of a matrix β_5 in the Tzou algebra characterized by the relations (1.2) but by using the explicit 11-dimensional representation (I.2.15). If we parametrize the matrix β_5 by 121 unknowns and require once again that $g_{55} = +1$, we finally obtain *four* independent forms given by

$$\beta_5^{(1)} = i(e_{1,10} - e_{2,9} + e_{4,11} + e_{7,8} - e_{8,7} + e_{9,2} - e_{10,1} + e_{11,4}), \tag{5.6a}$$

$$\beta_5^{(2)} = i(-e_{1,11} + e_{3,9} + e_{4,10} + e_{6,8} - e_{8,6} - e_{9,3} - e_{10,4} + e_{11,1}), \tag{5.6b}$$

$$\beta_5^{(3)} = i(-e_{2,11} + e_{3,10} - e_{4,9} - e_{5,8} + e_{8,5} + e_{9,4} - e_{10,3} + e_{11,2}), \tag{5.6c}$$

$$\beta_5^{(4)} = i(e_{4,8} + e_{5,9} + e_{6,10} + e_{7,11} + e_{8,4} + e_{9,5} + e_{10,6} + e_{11,7}). \tag{5.6d}$$

In conclusion, we learn that, if, at the start, the four matrices β_μ given in the realization (I.2.15) generate a Tzou algebra that we call $\tau(4)$, the eight matrices $\{\beta_\mu, \beta_5^{(1)}, \beta_5^{(2)}, \beta_5^{(3)}, \beta_5^{(4)}\}$ are the fundamental elements of an algebra $\tau(8)$ with the choice of a metric consistently determined as $(+, -, -, -, +, +, +, +)$. As an opposite result with respect to the one obtained in the Hagen-Hurley context (see eq. (3.4)), it is not possible here to express our projector P_{ST} in terms of the squares of these β_5-matrices.

6. Summary, Conclusions and Comments

Let us *summarize* the results obtained in Sections 2, 3, 4 and 5 considering, respectively, the four formulations due to KEMMER [7], HAGEN-HURLEY [8], DE BROGLIE [5] and STUECKELBERG [6] and describing, essentially, vector mesons when they interact with external magnetic fields which are constant and homogeneous in Minkowskian space. Each of these formulations has its own characteristics already pointed out in I when the *free* context is under study: two of them (Kemmer and Hagen-Hurley) refer to spin one particles only although the two others (de Broglie and Stueckelberg) contain simultaneous information on spin zero *and* one particles, these superpositions leading to a more difficult separation between the spins zero and one. When the *interacting* context is considered, we have also shown in *all* the formulations that the minimal electromagnetic coupling supplemented by only an anomalous magnetic moment coupling lead to difficulties in the (relativistic) energy spectrum: it contains unphysical eigenvalues when the strength of the magnetic field becomes such that the inequality (2.10) is satisfied. Such a defect being *common to the four* formulations, we have then proposed to eliminate it by adding a further nonlinear coupling and to relate these considerations when the magnetic field is constant and homogeneous to very recent developments referred to as parasupersymmetry [2, 3] and pseudosupersymmetry [18, 35], two concepts extending the very constructive and wellknown supersymmetry [43, 44] mainly applied in quantum mechanics. In fact, the Kemmer formulation (Section 2) was handled with more details in order to explain the main features – defects and qualities – contained in our approach. The other three formulations were developed with the specific aim to come back on equations already encountered in the Kemmer case (up to certain numbers of components in the wave functions).

Let us *conclude* that all these considerations are subtended by the covariant equation (1.3) with specific projectors in each case and where, in order to be as general as possible, we have introduced the parameters λ and α in the new *nonlinear* coupling term. Such a proposal eliminates the problem (i) quoted in the introduction, i.e. the one pointing out the appearance of possible complex energy eigenvalues in the corresponding spectrum when we limit ourselves to constant and homogeneous magnetic fields. In fact, we have shown that the values $\lambda = 1$ and $\alpha = \frac{1}{2}$ are already sufficient for our purpose. We have also shown that our equation satisfies the principle of causality eliminating in that way the problem (ii) also quoted in the introduction, this result being obtained for any nonzero value of λ or α. In particular, let us insist on the fact that, for $\alpha = 1$ and $\lambda \geqq \frac{1}{4}$, our results significantly improve the Vijayalakshmi et al. discussion [15]. Finally, by maintaining the anomalous magnetic moment coupling term in our equation (1.3) and its implications, we have ensured the value $g=2$ of the gyromagnetic ratio for vector mesons interacting with magnetic fields according to recent results [17] but obtained in field theory through Lagrangian considerations, a result which can be combined with the one obtained for electrons, in particular, and which illustrates the $(g=2)$-value for all "truly elementary (pointlike) particles of any spin" [17]. Once again, this deals with the problem (iii) quoted in the introduction and with the inclusion of nonminimal coupling terms: the guiding mark of this $(g=2)$-value is located in all the Zaitsev-Feynman-Gell Mann equations that we have obtained in our developments. These results are consistent with those obtained by DAICIC-FRANKEL [12]. DURAND [45] and WEAVER [46] besides the FERRARA et al. [17] ones.

Moreover, we notice that the equations without redundant components for *arbitrary* spin and nonzero mass particles [47, 48] also lead to the value $g=2$ when the electromagnetic minimal coupling principle is implied [49].

Let us end this Section by a few algebraic comments. If the study of the *free* case has been particularly interesting for enlightening the different formulations as issued from direct sums of $sl(2, \mathbb{C})$-irreducible representations (explaining the spin contents as well as the numbers of associated components in the wave functions), the *interacting* case has also had an important impact from an algebraic point of view. Indeed, the search for a fifth element in the corresponding Tzou algebras was a positive point leading to welcome information on the basis elements of such algebras: the Hagen-Hurley (Section 3) and Stueckelberg (Section 5) representations are particularly instructive at this point of view by using specific finite dimensional realizations of the matrices. By passing, let us mention the interesting algebraic problem of studying in an abstract way the Tzou algebras characterized by the structure relations (1.2), such a study having, in our opinion, to be developed through an ad hoc algorithm. Let us also recall the rich parallelism between Dirac (spin $\frac{1}{2}$) developments and Kemmer (spin 1) ones based on Clifford algebras, i.e. on $\mathscr{C}\ell_2 \otimes \mathscr{C}\ell_2$ with $N=2^4=4^2$ in the Dirac case and on $\mathscr{C}\ell_4 \otimes \mathscr{C}\ell_4$ with $N=2^8=16^2$ in the Kemmer one, such a parallelism being thus simply related to parastatistical considerations for the order $p=1$ in the Dirac case and the order $p=2$ in the Kemmer one, these last remarks having already been exploited in different contexts [50].

In order to be as complete as possible, we finally want to point out that if, in Part I of this series, we have studied some explicit parasupersymmetries associated with the symmetric forms of the spin one particle relativistic *free* descriptions (see Section I.4), it is evident that we could also obtain some corresponding results from our new equations (1.3) in each description when the (electro)magnetic interaction is included. Here, we have preferred to insist on the fact that, in Part II of this series, parasupersymmetry has helped us to solve the three main problems already pointed out in the Introduction when the *interacting* context is considered.

References

[1] J. BECKERS and N. DEBERGH and A. G. NIKITIN, Fortschr. Phys. **43** (1995) 67.
[2] V. A. RUBAKOV and V. P. SPIRIDONOV, Mod. Phys. Lett. **A3** (1988) 1337.
[3] J. BECKERS and N. DEBERGH, Nucl. Phys. **B340** (1990) 767; J. Math. Phys. **31** (1990) 1513.
[4] V. BARGMANN and E. P. WIGNER, Proc. Nat. Ac. Sci. **34** (1948) 211.
[5] L. DE BROGLIE, *Théorie générale des particules à spin – Méthode de fusion* (Paris, 1943).
[6] E. C. G. STUECKELBERG, Helv. Phys. Acta **11** (1938) 225.

[7] G. PETIAU, Acad. Roy. Belgique, Classe Sci., Mémoire Coll. in 8°, **16** (1936) Fasc. 2;
R. J. DUFFIN, Phys. Rev. **54** (1938) 1114;
N. KEMMER, Proc. Roy. Soc. London, Ser. **A 173** (1939) 91.
[8] C. R. HAGEN and W. J. HURLEY, Phys. Rev. Lett. **24** (1970) 1381.
[9] K. H. TZOU, Comptes Rendus Acad. Sci. (Paris) **244** (1957) 2137.
[10] G. VELO and D. ZWANZIGER, Phys. Rev. **186** (1969) 1337; **188** (1969) 2218.
[11] W. TSAI and A. YILDIZ, Phys. Rev. **D4** (1971) 3643;
T. GOLDMAN and W. TSAI, Phys. Rev. **D4** (1971) 3648;
W. TSAI, Phys. Rev. **D4** (1971) 3652;
T. GOLDMAN, W. TSAI and A. YILDIZ, Phys. Rev. **D5** (1972) 1926.
[12] J. DAICIC and N. E. FRANKEL, J. Phys. **A 26** (1993) 1397.
[13] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, vol. **2** (Wiley, New York, 1962).
[14] A. S. WIGHTMAN, *Troubles in the external field problem for invariant wave equations* (Gordon & Breach, New York, 1971).
[15] B. VIJAYALAKSHMI, M. SEETHARAM and P. M. MATTHEWS, J. Phys. **A 12** (1979) 665.
[16] L. P. SINGH and C. R. HAGEN, Phys. Rev. **D9** (1974) 898, 910.
[17] S. FERRARA, M. PORRATI and V. L. TELEGDI, Phys. Rev. **D 46** (1992) 3529.
[18] J. BECKERS and N. DEBERGH, *On new (super)symmetries from vector mesons interacting with external electromagnetic fields*, Liège preprint PTM 93/07 (1993).
[19] H. UMEZAWA, *Quantum Field Theory* (North-Holland, Amsterdam, 1956).
[20] S. A. BLUDMAN, Phys. Rev. **107** (1957) 1163;
J. BECKERS, Physica **28** (1962) 1019.
[21] H. CORBEN and J. SCHWINGER, Phys. Rev. **58** (1940) 953.
[22] D. SHAY and R. H. GOOD Jr., Phys. Rev. **179** (1969) 1410.
[23] G. A. ZAITSEV, Zurn. Exp. Teor. Fiz. (JETP) **28** (1955) 530.
[24] R. P. FEYNMAN and M. GELL-MANN, Phys. Rev. **109** (1958) 193.
[25] I. E. TAMM, DOKL. ACAD. SCI. USSR **29** (1940) 551;
S. SAKATA and M. TAKETANI, Proc. Phys. Math. Soc. Japan **92** (1940) 757.
[26] L. D. LANDAU and E. LIFSHITZ, *Théorie des champs* (MIR, Moscou, 1970).
[27] M. H. JOHNSON and B. A. LIPPMANN, Phys. Rev. **76** (1949) 828.
[28] J. BECKERS and N. DEBERGH, *Parasupersymmetries and Lie superalgebras*, Proceedings MOSCOW 1990; eds. V. V. Dodonov and V. I. Manko, Lecture Notes in Physics **382** (Springer Verlag, Berlin, 1991), pp. 414–417.
[29] N. DEBERGH, J. Phys. **A 27** (1994) L213.
[30] H. BACRY, PH. COMBE and J. L. RICHARD, Nuovo Cimento **A 67** (1970) 267;
J. BECKERS and V. HUSSIN, Phys. Rev. **D 29** (1984) 2814.
[31] J. BECKERS and N. DEBERGH, J. Phys. **A 23** (1990) L751S, L1073.
[32] V. G. KAC, Adv. Math. **26** (1977) 8; Comm. Math. Phys. **53** (1977) 1.
[33] O. W. GREENBERG, *Quons, an interpolation between Bose and Fermi oscillators, in* Proc. of Workshop on Harmonic Oscillators, NASA Conference Publications **3197** (1993).
[34] A. KHARE, A. K. MISHRA and G. RAJASEKARAN, Internat. J. Mod. Phys. **A 8** (1993) 1245.
[35] J. BECKERS and N. DEBERGH, *From relativistic vector mesons in constant magnetic fields to nonrelativistic (pseudo)supersymmetries*, Liège preprint PTM 93/08 (1993).
[36] H. S. GREEN, Phys. Rev. **90** (1953) 270;
O. W. GREENBERG and A. M. L. MESSIAH, Phys. Rev. **B 138** (1965) 1155;
Y. OHNUKI and S. KAMEFUCHI, *Quantum Field Theory and Parastatistics* (University of Tokyo Press, Tokyo, 1982).
[37] J. K. LUBANSKI, Physica **9** (1942) 310.
[38] H. GARNIR, Acad. Roy. Belgique, Classe Sci., Mémoire Coll. in 8°, **23** (1949) Fasc. 9.
[39] I. FUJIWARA, Prog. Theor. Phys. **10** (1953) 589.
[40] E. FISCHBACH, M. M. NIETO and C. K. SCOTT, J. Math. Phys. **14** (1973) 1760.
[41] J. BECKERS, N. DEBERGH and A. G. NIKITIN, J. Phys. **A 25** (1992) 6145.
[42] HARISH-CHANDRA, Proc. Roy. Soc. **A 186** (1946) 502.
[43] E. WITTEN, Nucl. Phys. **B 188** (1981) 513.
[44] A. LAHIRI, P. K. ROY and B. BAGCHI, Internat. J. Mod. Phys. **A 5** (1990) 1383.
[45] E. DURAND, Phys. Rev. **D 11** (1975) 3405.
[46] D. L. WEAVER, Phys. Rev. **D 14** (1976) 2824: J. Math. Phys. **19** (1978) 88.

[47] D. L. WEAVER, C. L. HAMMER and R. H. GOOD, Phys. Rev. **B135** (1964) 241;
P. M. MATHEWS, Phys. Rev. **B143** (1966) 978.
[48] W. I. FUSHCHICH, A. L. GRISHCHENKO and A. G. NIKITIN, Theor. Math. Phys. **8** (1971) 766;
A. G. NIKITIN and W. I. FUSHCHICH, Theor. Math. Phys. **44** (1981) 584.
[49] A. G. NIKITIN, Ukrain. Fiz. Zurn. **19** (1974) 1000.
[50] J. BECKERS, N. DEBERGH and A. G. NIKITIN, J. Math. Phys. **33** (1992) 3387.

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