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On Parasupersymmetries and Relativistic Descriptions for Spin one Particles: I. The Free Context

J. Beckers[a], N. Debergh[a)†] and A. G. Nikitin[b]

[a] Theoretical and Mathematical Physics,
Institute of Physics (B.5),
University of Liège,
B-4000 LIEGE 1 (Belgium)

[b] Institute of Mathematics,
Academy of Sciences
Repin Street, 3
KIEV 4 (Ukraine)

Abstract

This series of two papers is devoted to a constructive review of the relativistic wave equations for *vector* mesons due to the recent impact of spin one developments in connection with parasupersymmetric quantum mechanics. The free case as well as the interacting context with an electromagnetic field will be successively visited and discussed. Their associated parasupersymmetric properties will be pointed out.

In this first part, the *free* context is presented by studying systematically the (symmetric) forms of wave equations subtended by a 16-dimensional reducible representation of the Lie algebra $sl(2, \mathbf{C})$ or, evidently, so (3, 1), this representation playing a well known role in $p = 2$-parastatistical developments. Their hamiltonian forms are also discussed and some second order descriptions are finally reviewed.

I. Introduction

Since the very constructive idea and concept of supersymmetry [1] born in particle physics, the area covered by the so-called "*supersymmetric quantum mechanics*" [2] (SSQM) is becoming richer and larger [3]. Essentially developed as a *nonrelativistic* item, SSQM has given back, in a certain sense, interesting information at the *relativistic* level such as the one corresponding to the Dirac theory for spin one-half particles. For example, we know that the squares of Dirac Hamiltonians are intimately connected with nonrelativistic supersymmetric Hamiltonians [4]: let us recall that the specific context of harmonic oscillators has been particularly developed and exploited [5, 6], so that the difficult approach of relativistic oscillators gets, in that way, a specific answer. In all the above-mentioned topics, supersymmetry (and consequently SSQM) is (are) dealing with the superposition of usual bosons *and* spin one-half fermions.

From the *parastatistical* point of view [7, 8], let us remember that spin one-half fermions are the simplest parafermions which can be studied: they correspond to the so-called $p = 1$-order of paraquantization [8]. In order to take care of a statistics other

† Chercheur Institut Interuniversitaire des Sciences Nucléaires, Bruxelles.

than the Fermi-Dirac one, we can consider $p=2$-parafermions and study their superposition with usual bosons as recently proposed by RUBAKOV and SPIRIDONOV [9]: such a superposition leads to the so-called "*parasupersymmetric quantum mechanics*" (PSSQM) which has also received a large amount of contributions in different aspects [10–14].

Since the start, PSSQM has dealt with *spin one* developments – let us recall that the $su(2, \mathbb{C})$-representation associated with the study of $p=2$-parafermions [8] is the 3-dimensional one $D^{(1)}$ – so that its *nonrelativistic* characteristics have immediately been related [14–17] to *relativistic* formulations such as the DUFFIN-KEMMER-PETIAU equations [18–20] which will be simply called "*the Kemmer equation*" in the following. Let us insist more particularly on our study [15] of pararelativistic (harmonic) oscillators where 10 by 10 Kemmer matrices play the essential role. At our knowledge, the connection between third order commutation relations, typical for $p=2$-parafermions, *and* standard Kemmer relations has been originally given by BOGOLUBOV et al. [21]. All these considerations are, in a certain sense, the analogues of those developed in the supersymmetric context and related to the Dirac theory as mentioned above. But, if, on the one hand, we know that the Dirac equation is unique for describing relativistic $m \neq 0$ spin one-half particles, we have to notice, on the other hand, that there exists a very large number of different wave equations for describing relativistic spin-one particles. So, if there are simple connections and discussions within the supersymmetric context and SSQM, there are also more complicated ones within the parasupersymmetric context and PSSQM.

Such general remarks thus ask not only for a good understanding of the concept and properties subtended by "*parasupersymmetry*" in general but also for a new insight in the possible relativistic descriptions of (free or interacting) spin one particles at the level of the *first* quantization. Let us mention in addition that the Poincaré invariance of parasuperfields has already been developed elsewhere [22].

The main purpose of this series of two papers is thus connected with a general come back on the existing different spin-one relativistic descriptions in order to examine their own characteristics in terms of parasupersymmetric arguments. In this first part, we want to reconsider the *free* case corresponding to the description of relativistic nonzero mass Kemmer particles. Its contents are distributed as follows. In *Section 2*, we present an original review of the different *first order* wave equations for *free* relativistic spin-one particles, first, in their symmetric forms (§ 2.A) and, second, in their Hamiltonian forms (§ 2.B). In *Section 3*, we complete our programme by considering some *second order* wave descriptions developed, once again, for *free* spin-one particles. *Section 4* is then devoted to the study of the parasupersymmetries associated with this free context, once again intimately pointed out from symmetric and Hamiltonian forms of the wave equations.

In the following, we are working in Minkowski space characterized by the metric $G \equiv \{g_{\mu\nu} | g_{00} = -g_{ii} = 1; \mu, \nu = 0, 1, 2, 3; i = 1, 2, 3\}$ and in the system of natural units $(c=1, \hbar=1)$.

2. On Free Vector Mesons through First Order Equations

Depending on the subject which has to be discussed, we evidently know that the descriptions of relativistic particles can put in evidence a *symmetric* character of the wave equation when the time and space coordinates are on an equal footing (with a view to obtain a covariant formulation) or can point out a *Hamiltonian* form useful for the corresponding developments but in a close manner with its classical counterpart, etc.

For the description of *vector* mesons, we want to characterize these two (equivalent) points of view and plan to distinguish them into the two main following subsections 2.A and 2.B when only *first order* wave equations are under study. Let us mention that, besides general treatises [23], there are review papers [24–26 and references therein] with specific purposes dealing with the description of spin one theories.

2.A. Symmetric formulations of first order wave equations for vector mesons

General relativistic free wave equations have already been discussed in an extensive way [23] and their symmetric forms have been suggested mainly through the requirement of Lorentz covariance which is subtended by the Lie algebra $sl(2, \mathbb{C})$ or so $(3, 1)$ as it is well known. Here we propose to consider the covariant (symmetric) form

$$(\beta^\mu p_\mu - m)\,\Psi(x) = 0 \tag{2.1}$$

where as usual $p_\mu \equiv i\partial_\mu \equiv i\,\dfrac{\partial}{\partial x^\mu}$, $x \equiv (x^\mu) \equiv (x^0, x^i) \equiv (x^0, \vec{x})$, m being the nonzero rest mass of the corresponding particles while the β^μ are matrices satisfying the following relations:

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu + \beta_\nu\beta_\mu\beta_\lambda + \beta_\lambda\beta_\mu\beta_\nu + \beta_\mu\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda\beta_\mu = 2g_{\mu\nu}\beta_\lambda + 2g_{\nu\lambda}\beta_\mu + 2g_{\mu\lambda}\beta_\nu . \tag{2.2}$$

We have discovered these relations in an old paper [27] where Tzou studied the superposition of spins 0- *and* 1-descriptions through different matrix representations. Then we were interested in his 16-dimensional realization by noticing its reducible character, its specific interest with $p = 2$-parastatistics [21] and its spin content when we are considering the following product in the $so(3, 1)$- *or* $sl(2, \mathbb{C})$-context [28]:

$$\left[D\left(0, \frac{1}{2}\right) \oplus D\left(\frac{1}{2}, 0\right)\right] \otimes \left[D\left(0, \frac{1}{2}\right) \oplus D\left(\frac{1}{2}, 0\right)\right] =$$
$$= D(0, 0) \oplus D\left(\frac{1}{2}, \frac{1}{2}\right) \oplus D(1, 0) \oplus D(0, 1) \oplus D'\left(\frac{1}{2}, \frac{1}{2}\right) \oplus D'(0, 0). \tag{2.3}$$

Such a product corresponds to the construction of β-matrices from 4 by 4 Dirac matrices in the sense that we can consider

$$\beta_\mu = \frac{1}{2}\left(\gamma_\mu^{(1)} + \gamma_\mu^{(2)}\right), \quad \gamma_\mu^{(1)} = \gamma_\mu \otimes 1_4, \quad \gamma_\mu^{(2)} = 1_4 \otimes \gamma_\mu, \tag{2.4}$$

the direct products leading to the extended dimension. We also have to notice that

$$[\gamma_\mu^{(1)}, \gamma_\nu^{(2)}] = 0, \quad \{\gamma_\mu^{(1)}, \gamma_\nu^{(1)}\} = \{\gamma_\mu^{(2)}, \gamma_\nu^{(2)}\} = 2g_{\mu\nu}1_4, \quad \mu, \nu = 0, 1, 2, 3. \tag{2.5}$$

From the point of view of Clifford algebra [29], we are considering the direct product of two Cl_4 (leading to a $Cl_8 : N = 2^8 = (\underline{16})^2$) in the same way that Dirac matrices are resulting from the *direct* product of two Cl_2 (leading to a $Cl_4 : N = 2^4 = (\underline{4})^2$) with the analogue of eq. (2.3) on the form

$$D\left(\frac{1}{2}, 0\right) \otimes D\left(0, \frac{1}{2}\right) = D\left(\frac{1}{2}, \frac{1}{2}\right). \tag{2.6}$$

From now, it is already evident that the matrices (2.4) satisfy not only the relations (2.2) but a subclass given by the simplest relations

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = g_{\mu\nu} \beta_\lambda + g_{\lambda\nu} \beta_\mu, \tag{2.7}$$

which are known as the famous KEMMER relations [19, 23].

With such information, let us reconsider the different (symmetric) formulations on the form (2.1) which have been proposed. We present them by refering to wave functions with decreasing numbers of components starting with 16 and going to 7 as follows in the subsequent sections.

a. The Bargmann-Wigner or de Broglie equations

The so-called BARGMANN-WIGNER equation [30] is clearly equivalent to that obtained by the fusion method developed by DE BROGLIE [31] and discussed by TZOU [27]. It is given by the required form (2.1) with the matrices (2.4) of dimension 16, $\Psi(x)$ being a 16-component wave function. From the relations (2.4) and (2.5), we notice that the product (2.3) leads to the reducible direct sum of 10 by 10, 5 by 5 and 1 by 1 matrices respectively associated with spin 1-developments through $D(\frac{1}{2}, \frac{1}{2}) \oplus D(1, 0) \oplus D(0, 1)$, with spin 0-developments through $D'(0, 0) \oplus D'(\frac{1}{2}, \frac{1}{2})$ and with the trivial context $D(0, 0)$. These characteristics superpose the spin 1 and spin 0 descriptions without "*interference zones*" as pointed out by TZOU [27]. By using the notations $e_{j,k}$ for 16 by 16 matrices containing all zero elements except the ones at the intersection of the jth line and kth column which are equal to unity, we mention the following representation of the two sets of matrices $\gamma_\mu^{(1)}$ and $\gamma_\mu^{(2)}$:

$$\begin{aligned}
\gamma_0^{(1)} = {} & (-e_{1,13} - e_{2,14} - e_{3,15} + e_{4,16} + e_{5,9} + e_{6,10} + e_{7,11} + \\
& + e_{8,12} + e_{9,5} + e_{10,6} + e_{11,7} + e_{12,8} - e_{13,1} - e_{14,2} - e_{15,3} + e_{16,4}), \\
\gamma_1^{(1)} = {} & i(-e_{1,16} - e_{2,11} + e_{3,10} - e_{4,13} + e_{5,12} + e_{6,15} - e_{7,14} - \\
& - e_{8,9} - e_{9,8} + e_{10,3} - e_{11,2} + e_{12,5} - e_{13,4} - e_{14,7} + e_{15,6} - e_{16,1}), \\
\gamma_2^{(1)} = {} & i(e_{1,11} - e_{2,16} - e_{3,9} - e_{4,14} - e_{5,15} + e_{6,12} + e_{7,13} - \\
& - e_{8,10} - e_{9,3} - e_{10,8} + e_{11,1} + e_{12,6} + e_{13,7} - e_{14,4} - e_{15,5} - e_{16,2}), \\
\gamma_3^{(1)} = {} & i(-e_{1,10} + e_{2,9} - e_{3,16} - e_{4,15} + e_{5,14} - e_{6,13} + e_{7,12} - \\
& - e_{8,11} + e_{9,2} - e_{10,1} - e_{11,8} + e_{12,7} - e_{13,6} + e_{14,5} - e_{15,4} - e_{16,3})
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\gamma_0^{(2)} = {} & (e_{1,13} + e_{2,14} + e_{3,15} + e_{4,16} + e_{5,9} + e_{6,10} + e_{7,11} - \\
& - e_{8,12} + e_{9,5} + e_{10,6} + e_{11,7} - e_{12,8} + e_{13,1} + e_{14,2} + e_{15,3} + e_{16,4}), \\
\gamma_1^{(2)} = {} & i(-e_{1,16} + e_{2,11} - e_{3,10} + e_{4,13} - e_{5,12} + e_{6,15} - e_{7,14} - \\
& - e_{8,9} - e_{9,8} - e_{10,3} + e_{11,2} - e_{12,5} + e_{13,4} - e_{14,7} + e_{15,6} - e_{16,1}), \\
\gamma_2^{(2)} = {} & i(e_{1,11} + e_{2,16} - e_{3,9} - e_{4,14} + e_{5,15} + e_{6,12} - e_{7,13} + \\
& + e_{8,10} - e_{9,3} + e_{10,8} + e_{11,1} + e_{12,6} - e_{13,7} - e_{14,4} + e_{15,5} + e_{16,2}), \\
\gamma_3^{(2)} = {} & i(-e_{1,10} + e_{2,9} + e_{3,16} - e_{4,15} - e_{5,14} + e_{6,13} + e_{7,12} + \\
& + e_{8,11} + e_{9,2} - e_{10,1} + e_{11,8} + e_{12,7} + e_{13,6} - e_{14,5} - e_{15,4} + e_{16,3}).
\end{aligned} \tag{2.9}$$

It is straightforward to analyse the spin contents of this formulation by remembering that the tensor operators included in the (covariant) Lorentz generators of $so(3, 1)$ are given [16] by

$$M^K_{\mu\nu} = L_{\mu\nu} + S^K_{\mu\nu}, \quad L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \tag{2.10}$$

with the spin part $\vec{S} \equiv (S_1, S_2, S_3) \equiv (S^K_{23}, S^K_{31}, S^K_{12})$ obtained here from

$$S^K_{\mu\nu} = i[\beta_\mu, \beta_\nu] \tag{2.11}$$

$$= \frac{i}{4}[\gamma^{(1)}_\mu, \gamma^{(1)}_\nu] + \frac{i}{4}[\gamma^{(2)}_\mu, \gamma^{(2)}_\nu]. \tag{2.12}$$

b. The Stueckelberg equation

Originally proposed by STUECKELBERG [32], the following system of first order equations

$$\partial^\mu \Psi_\mu = m\Psi_4, \quad \partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu = m\Psi_{[\mu\nu]},$$
$$\partial_\mu \Psi_4 + \partial_\lambda \Psi_{[\lambda\mu]} = m\Psi_\mu, \tag{2.13}$$

which are dealing with eleven unknown functions, can be cast into the form (2.1) with 11 by 11 matrices and the 11-component wave function

$$\Psi(x)^T = (\Psi_{[23]}, \Psi_{[31]}, \Psi_{[12]}, \Psi_4, -\Psi_{[01]}, -\Psi_{[02]}, -\Psi_{[03]}, -\Psi_0, \Psi_1, \Psi_2, \Psi_3), \tag{2.14}$$

the $\Psi_{[\mu\nu]}$-components being skewsymmetric ones. The corresponding 11 by 11 matrices are given by

$$\beta_0 = i(-e_{4,8} + e_{5,9} + e_{6,10} + e_{7,11} + e_{8,4} - e_{9,5} - e_{10,6} - e_{11,7}),$$
$$\beta_1 = i(-e_{2,11} + e_{3,10} + e_{4,9} - e_{5,8} - e_{8,5} + e_{9,4} + e_{10,3} - e_{11,2}),$$
$$\beta_2 = i(e_{1,11} - e_{3,9} + e_{4,10} - e_{6,8} - e_{8,6} - e_{9,3} + e_{10,4} + e_{11,1}),$$
$$\beta_3 = i(-e_{1,10} + e_{2,9} + e_{4,11} - e_{7,8} - e_{8,7} + e_{9,2} - e_{10,1} + e_{11,4}). \tag{2.15}$$

We also observe that these matrices satisfy our general relations (2.2) and (evidently) correspond in (2.3) to the reduced direct sum

$$D(1,0) \oplus D(0,1) \oplus D'\left(\frac{1}{2}, \frac{1}{2}\right) \oplus D'(0,0). \tag{2.16}$$

Effectively, these developments coincide with those first pointed out by TZOU [27] where, in his representation, he noticed the presence of "*interference zones*".

Let us point out that the spin content is not here given by the $S^K_{\mu\nu}$'s defined in terms of the matrices β_μ by eq. (2.11). The Lorentz invariance of eq. (2.1) with the matrices β_μ satisfying the relations (2.2) implies that we have

$$[\beta_\mu, S_{\lambda\nu}] = i(g_{\mu\lambda}\beta_\nu - g_{\mu\nu}\beta_\lambda). \tag{2.17}$$

Such conditions with the representation (2.15) fix the set of components $S_{\mu\nu}$. We get, in particular, that

$$S_{12} = i(-e_{1,2} + e_{2,1} - e_{5,6} + e_{6,5} - e_{9,10} + e_{10,9}),$$
$$S_{23} = i(-e_{2,3} + e_{3,2} - e_{6,7} + e_{7,6} - e_{10,11} + e_{11,10}),$$
$$S_{31} = i(e_{1,3} - e_{3,1} + e_{5,7} - e_{7,5} + e_{9,11} - e_{11,9}), \tag{2.18}$$

so that we also confirm the simultaneous descriptions of spins 1 and 0:

$$\vec{S}^2 \equiv 2(1_{11} - e_{4,4} - e_{8,8}) \tag{2.19}$$

through the Stueckelberg equation rewritten on the form (2.1).

c. The Kemmer equation

This equation [18–20] is identically on the form (2.1) with matrices satisfying the relations (2.7) and, consequently, the relations (2.2). A well-adapted representation is given for spin 1-developments in terms of 10 by 10 matrices as follows

$$\begin{aligned}
\beta_0 &= i(-e_{1,7} - e_{2,8} - e_{3,9} + e_{7,1} + e_{8,2} + e_{9,3}),\\
\beta_1 &= i(-e_{1,10} - e_{5,9} + e_{6,8} + e_{8,6} - e_{9,5} - e_{10,1}),\\
\beta_2 &= i(-e_{2,10} + e_{4,9} - e_{6,7} - e_{7,6} + e_{9,4} - e_{10,2}),\\
\beta_3 &= i(-e_{3,10} - e_{4,8} + e_{5,7} + e_{7,5} - e_{8,4} - e_{10,3}).
\end{aligned} \tag{2.20}$$

The Lorentz invariance of eq. (2.1) is once again compatible with the spin tensor given by the $S^K_{\mu\nu}$'s defined in terms of the β_μ's by eq. (2.11), so that we get, in particular, that

$$\begin{aligned}
S_{12} &= -i(e_{1,2} - e_{5,4} - e_{8,7} - e_{2,1} + e_{4,5} + e_{7,8}),\\
S_{23} &= -i(e_{2,3} - e_{6,5} - e_{9,8} - e_{3,2} + e_{5,6} + e_{8,9}),\\
S_{31} &= -i(e_{3,1} - e_{4,6} - e_{7,9} - e_{1,3} + e_{6,4} + e_{9,7}),
\end{aligned} \tag{2.21}$$

showing that it is convenient for the description of spin 1 particles. Such considerations correspond in (2.3) to the reduced direct sum

$$D\left(\frac{1}{2}, \frac{1}{2}\right) \oplus D(1, 0) \oplus D(0, 1). \tag{2.22}$$

Let us point out that evidently parallel characteristics can be obtained from 5 by 5 matrices but describing spin 0 particles and corresponding in (2.3) to the direct sum

$$D(0, 0) \oplus D\left(\frac{1}{2}, \frac{1}{2}\right). \tag{2.23}$$

d. The Hagen-Hurley equation

A 7-component description on spin 1 particles has also been proposed [33]. It can be written on the form (2.1) through 7 by 7 matrices β_μ given as

$$\begin{aligned}
\beta_0 &= i(e_{1,4} + e_{2,5} + e_{3,6} - e_{4,1} - e_{5,2} - e_{6,3}),\\
\beta_1 &= i(e_{1,7} + ie_{2,6} - ie_{3,5} + ie_{5,3} - ie_{6,2} + e_{7,1}),\\
\beta_2 &= i(-e_{1,6} + e_{2,7} + ie_{3,4} - ie_{4,3} + ie_{6,1} + e_{7,2}),\\
\beta_3 &= i(ie_{1,5} - ie_{2,4} + e_{3,7} + ie_{4,2} - ie_{5,1} + e_{7,3}),
\end{aligned} \tag{2.24}$$

satisfying once more the structure relations (2.2) and corresponding in (2.3) to the reduced direct sum

$$D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(1,0). \tag{2.25}$$

The spin content is once again issued from eq. (2.17) and leads also to an evident spin 1-interpretation, the components S_{12}, S_{23}, S_{31} appearing as direct sums of two identical spin matrices belonging to the $D^{(1)}$-representation of $su(2, \mathsf{C})$, up to the seventh diagonal matrix element (which is zero). This gives

$$\vec{S}^2 \equiv 2(\mathbf{1}_7 - e_{7,7}) \tag{2.26}$$

with the correct meaning of the eigenvalues.

e. Some preliminary remarks

We have thus collected four different first order formulations refering with specific arguments to spin 1-descriptions of nonzero rest mass particles: they are respectively characterized by wave functions with 16, 11, 10 or 7 components and by matrix realizations respectively given in eqs. (2.9), (2.15), (2.20) and (2.24). A remarkable fact is that they all satisfy the general structure relations (2.3). Another point is to notice that some of the above formulations automatically contain *or* are well-adapted for the description of the spin 0 context. In each case, we have to recall that all these formulations contain *redundant* components which finally have to be eliminated according to the well known property that only $2(2s+1)$ components are needed for describing free spin s-particles with nonzero rest masses.

In order to complete the Lorentz invariance arguments, we immediately point out that, through the contents (2.3), (2.8), (2.16) and (2.22), the invariance under parity is guaranteed as it is the case in the Dirac context characterized by (2.6). Consequently, the content (2.25) also shows that the Hagen-Hurley equation is not parity invariant, the latter property asking for a doubling of such a formulation. Besides the answer given by (2.22) and the Kemmer formulation, this asks for a 14-component wave function and an equation (2.1) refering to the direct sum

$$D\left(\frac{1}{2},\frac{1}{2}\right)\oplus D(1,0)\oplus D(0,1)\oplus D'\left(\frac{1}{2},\frac{1}{2}\right). \tag{2.27}$$

It is then intimately connected with a description proposed by Lomont and Moses [34] leading to spin constraints such as (2.17) rather than to components $S_{\mu\nu}$ given by (2.11).

As a last comment, let us mention that the four preceding formulations summarized by the equation (2.1) and the structure relations (2.2) *but* with specific representations of matrices and reduced numbers of components can be recovered from the following system

$$(\gamma^{\mu}_{(a)} p_{\mu} - m)\,\Psi(x) = 0,\quad P\Psi(x) = 0, \qquad a = 1, 2, \tag{2.28}$$

where, at the start, the γ-matrices are 16-dimensional ones satisfying Dirac anticommutation relations

$$\{\gamma^{\mu}_{(a)}, \gamma^{\nu}_{(a)}\} = 2g^{\mu\nu}\mathbf{1}_{16}. \tag{2.29}$$

The second equation (2.28) is expressed in terms of projection operators P $(P^2=P)$ evidently introduced for reducing the numbers of components according to the expected descriptions. With the possible choices (2.4) and the properties (2.5), we specify the following projection operators

$$P_1 = \frac{1}{16}[1_{16} - \gamma^\mu_{(1)}\gamma^{(2)}_\mu - \gamma^5_{(1)}\gamma^5_{(2)}]^2, \tag{2.30a}$$

$$P_2 = \frac{1}{4}\left[P_1 - \frac{1}{4}\gamma^\mu_{(1)}\gamma^{(2)}_\mu\right][1_{16} + \gamma^5_{(1)}\gamma^5_{(2)}], \tag{2.30b}$$

$$P_3 = \frac{1}{2}P_1[1_{16} + \gamma^5_{(1)}\gamma^5_{(2)}] \tag{2.30c}$$

and

$$P_4 = P_2 + \frac{1}{4}[1_{16} - \gamma^5_{(1)}\gamma^5_{(2)}]\left[1_{16} - \frac{1}{2}\gamma^\mu_{(1)}\gamma^{(2)}_\mu\right], \tag{2.30d}$$

where as usual

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.31}$$

These operators lead *respectively* to the spin 1-Kemmer equations, to the de Broglie ones, to the LOMONT-MOSES version [34] corresponding to the Hagen-Hurley description and, finally, to the Stueckelberg formulation. They are all such that

$$[P_a, S_{\mu\nu}] = 0, \quad a = 1, 2, 3, 4, \tag{2.32}$$

due to the required Lorentz invariance. Such an analysis could be developed in terms of projection operators analogous to those introduced by FUJIWARA [35] and exploited by FISCHBACH et al. [36] in order to recover the spin 1- and spin 0-Kemmer formulations. Here the spin 0-context is recovered through a fifth projector essentially given by $1_{16} - P_1 + P_2$ as it is easily verified.

2.B. On the spin one Hamiltonian formulations

Due to the specific properties (2.7) of the Kemmer matrices, the Hamiltonian form of the Kemmer equation has been obtained as constrained by an initial condition [37], which has the quality to eliminate the redundant components. By working here with the equation (2.1) and the β-matrices satisfying the properties (2.2), it is not difficult to get the corresponding Hamiltonian form given by

$$i\frac{\partial}{\partial t}\Psi(x) = H\Psi(x) = (\vec{B}\cdot\vec{p} + m\beta_0)\Psi(x) \tag{2.33}$$

where the matrices

$$B_j \equiv [\beta_0, \beta_j] + \beta_0^2\beta_j\beta_0 \quad (j = 1, 2, 3) \tag{2.34}$$

satisfy the generalized relations (see eq. (2.2)):

$$B_i B_j B_k + B_k B_j B_i + B_i B_k B_j + B_j B_k B_i + B_k B_i B_j + B_j B_i B_k =$$
$$= 2\delta_{ij} B_k + 2\delta_{jk} B_i + 2\delta_{ki} B_j. \tag{2.35}$$

The corresponding initial condition is once again given by

$$(H\beta_0 - m)\,\Psi(x) = 0 \tag{2.36}$$

and eliminates the components associated with the zero values of the energy as expected. By dealing with the matrix realizations (2.4), (2.15), (2.20) or (2.24), we thus get the so-defined new matrices (2.34) as well as the equations (2.33) and (2.36) in each case discussed in the preceding subsection.

Let us already point out (we plan to come back on this property in our conclusions) that the matrices B_j given by eqs. (2.34) reduce to the well known definitions

$$B_j^K \equiv [\beta_0, \beta_j], \quad B_j^D \equiv \beta_0 \beta_j = \alpha_j, \tag{2.37}$$

when we respectively admit the Kemmer relations (2.7) or the Dirac ones (2.29) showing that the corresponding Hamiltonian formulations are well adapted, not only for spin 1- and spin 0-particles but also for spin $\frac{1}{2}$-particles.

3. On Second Order Equations in the Free Context

In order to complete the description of *free* vector mesons, we want to add some information on the relativistic *second order* wave equations which have been proposed after the pioneering work of PROCA [39]. Among these proposals, the most famous one is certainly that of TAMM [40], SAKATA and TAKETANI [41] which has the advantage to be cast in a Hamiltonian form containing *only six* nonredundant components (see also [25, 42]). Besides the above two formulations, there are also many others which have some specificities [43–51]. Let us only remember here that the usual Kemmer equations (2.1) expressed in terms of the 10 by 10 matrices (2.20) comes out from the Proca equations by considering the 10-component wave function $\Psi(x)$ defined by

$$\Psi^T(x) = \{\Psi_{01}, \Psi_{02}, \Psi_{03}, \Psi_{23}, \Psi_{31}, \Psi_{12}, \Psi_1, \Psi_2, \Psi_3, \Psi_0\} \tag{3.1}$$

where the $\Psi_{\mu\nu}$'s appear as the skew-symmetric components of a second order tensor associated with the definitions of the "electric and magnetic fields" while the Ψ_μ's are the corresponding four components associated with the "fourpotential vector", i.e. through the definitions

$$\mathrm{im}\,\Psi_{\mu\nu} = p_\mu \Psi_\nu - p_\nu \Psi_\mu. \tag{3.2}$$

In that way, the first order Kemmer equation is equivalent to the six equations (3.2) supplemented by the following ones

$$\mathrm{im}\,\Psi_\nu = -p^\mu \Psi_{\mu\nu}, \tag{3.3}$$

so that, by introducing (3.3) into (3.2), we get back the second order Proca equations

$$(p_\mu p^\mu - m^2)\,\Psi_\nu - p_\nu p^\mu \Psi_\mu = 0. \tag{3.4}$$

The six components $\Psi_{\mu\nu}$ have been attractive for different authors. So, KYRIAKOPOULOS [43] has eliminated the fourvector dependence by introducing (3.3) into (3.4) in order to get the equations

$$p_\mu p^\sigma \Psi_{\sigma\nu} - p_\nu p^\sigma \Psi_{\sigma\mu} + m^2 \Psi_{\mu\nu} = 0. \tag{3.5}$$

In terms of the "dual wavefunction" Φ defined through

$$\Psi_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\varrho\sigma} \Phi^{\varrho\sigma}, \quad (\varepsilon_{0123} = 1), \tag{3.6}$$

others [44] have obtained, from (3.5), the following system

$$(p_\sigma p^\sigma - m^2) \Psi_{\mu\nu} - p_\mu p^\sigma \Psi_{\sigma\nu} + p_\nu p^\sigma \Psi_{\sigma\mu} = 0 \tag{3.7}$$

which offers its own interests. Let us moreover recall that KEMMER [45], himself, was also interested in writing his first order equation in a second order form summarized by the system

$$(p_\mu p^\mu - m^2) \Psi_{\nu\lambda} = 0, \tag{3.8a}$$

$$p_\mu \Psi_{\nu\lambda} + p_\nu \Psi_{\lambda\mu} + p_\lambda \Psi_{\mu\nu} = 0 \tag{3.8b}$$

where we evidently recognize the (free) Klein-Gordon equations in (3.8a) and the corresponding "Maxwell equations without sources" in (3.8b). In pointing out some Klein-Gordon equations, we can also stress the second order approach developed by STUECKELBERG [32] in complete agreement with the first order form (2.13). Supplemented by the definitions (3.2), we have the five equations

$$(p_\mu p^\mu - m^2) \Psi_\nu = 0, \quad (p_\mu p^\mu - m^2) \Psi_4 = 0, \quad p^\mu \Psi_\mu = \mathrm{im}\, \Psi_4, \tag{3.9}$$

where Ψ_4 enters in the eleven component wavefunction already given by (2.14). In covariant form, we finally want to point out the JOOS [46]-WEINBERG [47] formulation given by a 6 by 6 matrix description

$$(\gamma^{\mu\nu} p_\mu p_\nu + m^2) \Psi = 0 \tag{3.10}$$

where all the components of Ψ satisfy the Klein-Gordon equation. Here, the 6 by 6 matrices $\gamma^{\mu\nu}$ are the ones defined by BARUT, MUZINICH and WILLIAMS [48], i.e.

$$\begin{gathered}\gamma^{00} = \Sigma_1, \quad \gamma^{0j} = i\Sigma_2 S^j, \quad \gamma^{jk} = \Sigma_1 (S^j S^k + S^k S^j - \delta^{jk}), \\ \Sigma_1 = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix}, \quad \Sigma_2 = i \begin{pmatrix} 0 & -1_3 \\ 1_3 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1_3 & 0 \\ 0 & -1_3 \end{pmatrix}, \quad S^j = \begin{pmatrix} s^j & 0 \\ 0 & s^j \end{pmatrix},\end{gathered} \tag{3.11}$$

when s^j are the 3 by 3 spin matrices of the $su(2, \mathbb{C})$-representation $D^{(1)}$. Such a free description is equivalent to the particular TAMM-SAKATA-TAKETANI [40, 41] formulation which corresponds to the Hamiltonian

$$H_{\mathrm{TST}} = \Sigma_2 m + (\Sigma_2 + i\Sigma_1) \frac{\vec{p}^{\,2}}{2m} - \frac{i}{m} \Sigma_1 (\vec{S} \cdot \vec{p})^2, \tag{3.12}$$

a Hermitian operator with respect to the scalar product $(\Psi, \Sigma_3 \Psi)$. This last formulation can also be recovered from the (first order) Hamiltonian form (2.33)–(2.36) through a unitary transformation which takes the following form [17]

$$S = I + \frac{1}{m} \beta_j \beta_0^2 p_j, \quad S^{-1} = I - \frac{1}{m} \beta_j \beta_0^2 p_j. \tag{3.13}$$

As a last comment, let us add that other studies of Hamiltonian forms have also been developed [49–51] but we do not plan to discuss them in the present paper due to their already established relations with some of the above formulations.

4. On Parasupersymmetries in the Free Context

For completeness only, let us recall that we have already obtained all the first order Lie extended symmetries of the relativistic wave equations describing nonzero rest mass and spin $0, \frac{1}{2}$ or 1 particles [16]. In that context, the Kemmer matrices had to satisfy the relations (2.7) as usual but we can now also show that, if, actually, we require the more general relations (2.2), the *same* twenty six first order extended symmetries hold. Their explicit forms are given by the *ten* Poincaré well known symmetry operators

$$P_\mu \equiv p_\mu = i\partial_\mu,$$
$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad S_{\mu\nu} \equiv (2.11), \tag{4.1}$$

supplemented (besides the *unit* operator) by the *fifteen* operators belonging to the Poincaré enveloping algebra [16]:

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma} M^{\nu\varrho} P^\sigma = \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma} S^{\nu\varrho} P^\sigma, \tag{4.2a}$$

$$W_{\mu\nu} = \frac{1}{m}(P_\mu W_\nu - P_\nu W_\mu), \tag{4.2b}$$

$$A_\mu = \frac{1}{m}\left(\varepsilon_{\mu\nu\varrho\sigma} M^{\nu\varrho} W^\sigma - \frac{1}{2}P_\mu\right), \tag{4.2c}$$

$$B = \frac{1}{4}\varepsilon_{\mu\nu\varrho\sigma} M^{\mu\nu} M^{\varrho\sigma}. \tag{4.2d}$$

From this set (4.2) of extended Lie symmetries belonging to the enveloping Poincaré algebra, we can construct specific operators having all the typical properties of parasupercharges when, in the free case, we are asking for a parasupersymmetric Hamiltonian given by

$$H_{\mathrm{PSS}} = \vec{p}^{\,2} + m^2. \tag{4.3}$$

Such a motivation is justified through unexpected relativistic properties already obtained in the Dirac context [38] in connection with SSQM. Here, let us recall that PSSQM is characterized by the following double commutation relations if we adopt the Beckers-Debergh approach [11]:

$$[M_i, [M_j, M_k]] = 4(\delta_{ij} M_k - \delta_{ik} M_j) H_{\mathrm{PSS}}, \quad [M_i, H_{\mathrm{PSS}}] = 0. \tag{4.4}$$

Then it is easy (but tedious) to show that the following expressions are three ad-hoc parasupercharges

$$M_i = 2W_i + \varepsilon_{ijk} W_{jk}, \quad i = 1, 2, 3. \tag{4.5}$$

Such calculations are evidently based on the expressions (4.2) and the characteristic commutation relations of the Poincaré algebra given by

$$[P_\mu, P_\nu]=0, \quad [P_\mu, M_{\lambda\nu}]=i(g_{\mu\lambda}P_\nu-g_{\mu\nu}P_\lambda),$$
$$[M_{\mu\nu}, M_{\varrho\sigma}]=i(g_{\mu\sigma}M_{\nu\varrho}+g_{\nu\varrho}M_{\mu\sigma}-g_{\mu\varrho}M_{\nu\sigma}-g_{\nu\sigma}M_{\mu\varrho}). \tag{4.6}$$

As a complementary remark, let us mention that the three other combinations

$$M_i'=2W_i-\varepsilon_{ijk}W_{jk} \tag{4.7}$$

are also parasupercharges satisfying eqs. (4.4) with (4.3). The above sets $\{M_i\}$ and $\{M_i'\}$ have nevertheless to be considered as independent ones.

Such new properties are, once again, associated with the *symmetric* forms of the relativistic description of spin one particles developed in Section 2.A.

Let us also exploit the *Hamiltonian* forms developed in Section 2.B. Due to some recent properties obtained in SSQM in connection with the relativistic Dirac theory [6] as well as in PSSQM in connection with the relativistic Kemmer theory [15], we can stress that there are once again two other parasupercharges Q_1 or Q_2 leading, in this free context, to remarkable parasupersymmetries in *all* the above descriptions for spin 1-particles while we are dealing with the generalized relations (2.35) here essentially. Indeed we can construct the parasupercharges

$$Q_1=\frac{1}{\sqrt{2m}}\vec{B}\cdot\vec{p} \quad \text{and} \quad Q_2=\frac{1}{\sqrt{2m}}\vec{C}\cdot\vec{p} \tag{4.8}$$

where, besides the definitions of the B_j-matrices (2.34), we introduce the C_j's by

$$C_j=i\{\beta_0, \beta_j\}-i\beta_0\beta_j\beta_0 \tag{4.9}$$

satisfying once again the relations (2.35). With the parasupercharges (4.8), we get the typical relations

$$Q_1^3=Q_1H_{\text{PSS}}^{(0)}, \quad Q_2^3=Q_2H_{\text{PSS}}^{(0)}, \quad Q_1Q_2Q_1=-Q_2H_{\text{PSS}}^{(0)}, \dots, \tag{4.10}$$

so that we have an explicit Psqm(2)-algebra connected with these relativistic developments. Remember that, in this free context,

$$H_{\text{PSS}}^{(0)}=\frac{\vec{p}^{\,2}}{2m}, \tag{4.11}$$

while we are dealing with two (unitarily equivalent) relativistic Hamiltonians

$$H_1=\sqrt{2m}\,Q_1+m\beta_0, \quad H_2=\sqrt{2m}\,Q_2+m\beta_0. \tag{4.12}$$

The corresponding parasupersymmetries of both relativistic theories could evidently been cast in specific forms in terms of the above extended Lie symmetries.

Let us conclude by two remarks. First, the parasupersymmetries of the *second* order wave equations can evidently also be pointed out but we limit ourselves by recalling here that, through unitary transformations such as the one given by eqs. (3.13), it is easy to get them from the parasupersymmetries of the *first* order (equivalent) wave equations. Secondly, from this spin-one free context, we have to conclude that the already known results on supersymmetries of the (relativistic) theory of spin $\frac{1}{2}$-particles have a complete counterpart on parasupersymmetries of the (relativistic) theory of spin 1-particles: this is particularly illustrated through the conclusions issued from eqs. (2.34) and (2.37) as well as from the properties of the three parasupercharges (4.5) (in comparison with those pointed out in the Dirac context [16]) generating a parasuperstructure that we call Psqm (3) [12]. Such a parallelism in the results for spins $\frac{1}{2}$ and 1 is evidently due to the respective $p=1$- and $p=2$-parastatistical characteristics and to the recent results on Lie extended symmetries [16] valuable for arbitrary spin values.

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