J. Phys. A: Math. Gen. 26 (1993) L853–L857. Printed in the UK

LETTER TO THE EDITOR

On a hidden dynamical SU(3)-symmetry in parasupersymmetric quantum mechanics

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Received 13 April 1993

Abstract. The superposition of bosons and $p=2$-parafermions has led to the so-called parasupersymmetric quantum mechanics. We show that the structures subtending these developments contain a hidden dynamical SU(3)-symmetry effectively associated with the fundamental irreducible representation $\underline{3}$ (unitarily equivalent to the Gell–Mann one). The representation $\underline{3}^*$ is also visited. The generalization to arbitrary orders p of paraquantization is briefly discussed.

$N=2$-parasupersymmetric quantum mechanics dealing with bosons and $p=2$-parafermions [1, 2] has essentially been developed through two different approaches, respectively presented in the Rubakov–Spiridonov [3] and Beckers–Debergh [4] papers. Let us recall that N is the fixed number of charges leading to the corresponding Hamiltonian while p is the order of paraquantization.

Physically, these approaches are generalizations of the Witten supersymmetric model [5] concerned with bosons and $(p=1)$-fermions and they correspond to the *non-equivalent* Ξ- and Λ-types of three-level systems respectively [6].

Algebraically, the two (parasuper) structures subtending the above developments are fundamentally different: they lead to non-equivalent Hamiltonians included into sets of *trilinear* relations amongst the parasupercharges [7].

Here we want to point out that the two approaches are characterized by the same representation of the dynamical SU(3)-symmetry besides their typical non-equivalent parasuperhamiltonians.

In order to show the existence of such a hidden SU(3)-symmetry, let us restrict ourselves to oscillator-like interactions (for simplicity, but the arguments hold for arbitrary superpotentials $W_1(x)$ and $W_2(x)$). Moreover, let us decompose the two respective original parasupercharges Q and $Q^\dagger$ into the four simplest *odd* charges that we call $Q_k^\mp (k=1,2)$ or $q_k^\mp (k=1,2)$ in the Rubakov–Spiridonov or Beckers–Debergh developments respectively. These building charges are simple odd 3 by 3 matrices containing only one annihilation (a) or creation $(a^\dagger)$ bosonic operators. Moreover

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they are symmetries of (i.e. commute with) the corresponding Hamiltonian. They are given by the following explicit forms in the Rubakov–Spiridonov context [3]:

$$Q_1^+ = ae_{1,2} \qquad Q_1^- = a^\dagger e_{2,1} \qquad Q_2^+ = a^\dagger e_{3,2} \qquad Q_2^- = ae_{2,3} \tag{1}$$

and they lead to the parasuperhamiltonian

$$H_{RS} = (aa^\dagger + \tfrac{1}{2})e_{1,1} + (a^\dagger a + \tfrac{1}{2})e_{2,2} + (aa^\dagger - \tfrac{3}{2})e_{3,3} \tag{2}$$

where the $e_{j,k}$'s are evidently 3 by 3 matrices with all zero elements except those located at the intersection of the jth line and the kth column which are equal to unity. In the Beckers–Debergh context [4], we have the corresponding information in the forms

$$q_1^+ = ae_{1,2} \qquad q_1^- = a^\dagger e_{2,1} \qquad q_2^+ = ae_{3,2} \qquad q_2^- = a^\dagger e_{2,3} \tag{3}$$

and

$$h_{BD} = aa^\dagger(e_{1,1} + e_{3,3}) + a^\dagger a e_{2,2}. \tag{4}$$

The characteristics [(1), (2)] and [(3), (4)] can now be compared in a non-trivial way. As a starting point, let us search for information on the algebraic structure generated by the matrices (3) and (4) for example. By defining the four new (even) operators

$$[q_k^+, q_k^-] = Z_k \qquad (k=1,2) \qquad [q_1^+, q_2^-] = Z_3 \qquad [q_1^-, q_2^+] = Z_4 \tag{5}$$

it is easy to get the non-zero commutation relations (without summation on repeated indices):

$$\begin{aligned}
&[Z_k, q_k^\pm] = \pm 2h_{BD} q_k^\pm \qquad [Z_k, q_j^\pm] = \pm h_{BD} q_j^\pm \qquad (k \neq j)\\
&[Z_3, q_1^-] = -h_{BD} q_2^- \qquad [Z_3, q_2^+] = h_{BD} q_1^+\\
&[Z_4, q_1^+] = -h_{BD} q_2^+ \qquad [Z_4, q_2^-] = h_{BD} q_1^-\\
&[Z_1, Z_3] = [Z_3, Z_2] = h_{BD} Z_3 \qquad [Z_2, Z_4] = [Z_4, Z_1] = h_{BD} Z_4.
\end{aligned} \tag{6}$$

We are thus dealing, in (5) and (6), with eight operators $\{q_k^\pm, Z_\alpha(\alpha = 1,2,3,4)\}$ which commute with the parasuperhamiltonian h_{BD}. Such results immediately lead to the existence of dynamical symmetries [8], some of them leading to the explanation of specific degeneracies such as accidental ones. These symmetries generate a closed structure appearing as a simple Lie algebra if we restrict ourselves to subspaces of the original Hilbert space corresponding to eigenvalues E of the parasuperhamiltonian under study. By defining the new generators

$$q_k^{\pm\prime} = \frac{1}{\sqrt{E}} q_k^\pm \qquad Z'_\alpha = \frac{1}{E} Z_\alpha \tag{7}$$

the structure relations (5) and (6) fall into the following categories of commutators

$$[Z', Z'] \approx Z' \qquad [q', q'] \approx Z' \qquad [Z', q'] \approx q' \tag{8}$$

characterizing those of the simple Lie algebra $su(3, \mathbb{C})$. In order to convince ourselves of this result, let us act with these operators on a basis of oscillator-like vectors which are such that as usual

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{9}$$

By dealing with the following linear combinations

$$F_1=\frac{i}{2}(Z_3'+Z_4') \qquad F_2=\frac{1}{2}(Z_3'-Z_4') \qquad F_3=\frac{1}{2}(Z_1'-Z_2') \tag{10}$$

$$F_4=\frac{1}{2}(q_1^{+\prime}+q_1^{-\prime}) \qquad F_5=-\frac{i}{2}(q_1^{+\prime}-q_1^{-\prime}) \qquad F_6=-\frac{i}{2}(q_2^{+\prime}-q_2^{-\prime})$$

$$F_7=-\frac{1}{2}(q_2^{+\prime}+q_2^{-\prime}) \qquad F_8=-\frac{1}{2\sqrt{3}}(Z_1'+Z_2') \tag{11}$$

we immediately recover the structure relations of the F-spin [9]. Here we stress the two diagonal operators F_3 and F_8 as well as the operators (10) which generate a particular $su(2,\mathbb{C})$-subalgebra.

At this stage, it can be seen that we have in fact obtained a realization of the fundamental representation $\underline{3}$ of $su(3,\mathbb{C})$ which is unitarily equivalent through the following transformation

$$U=e_{1,1}+ie_{2,3}+e_{3,2} \qquad U^{\dagger}=U^{-1} \tag{12}$$

to the one given by Gell–Mann [9].

Completely parallel developments can also be realized from the Rubakov–Spiridonov characteristics [(1), (2)] but leading in correspondence with (7) to the (eight) generators

$$q_1^{\pm\prime}=\frac{1}{\sqrt{E-\frac{1}{2}}}Q_1^{\pm} \qquad q_2^{\pm\prime}=\frac{1}{\sqrt{E+\frac{1}{2}}}Q_2^{\pm} \qquad Z_1'=\frac{1}{E-\frac{1}{2}}[Q_1^+,Q_1^-]$$

$$Z_2'=\frac{1}{E+\frac{1}{2}}[Q_2^+,Q_2^-] \qquad Z_3'=\frac{1}{E^2-\frac{1}{4}}[Q_1^+,Q_2^-] \qquad Z_4'=\frac{1}{E^2-\frac{1}{4}}[Q_1^-,Q_2^+]. \tag{13}$$

Their commutation relations of the type (8) lead once again to a realization of the fundamental representation $\underline{3}$ of $su(3,\mathbb{C})$ as it can be verified.

Up to the specific forms of the parasuperhamiltonian H_{PSS} given by (2) or (4), we conclude that the same representation of the dynamical algebra $su(3,\mathbb{C})$ is present in *both* approaches of $N=2$-parasupersymmetric quantum mechanics describing bosons and $p=2$-parafermions. The complete structure generated by the nine operators is finally the direct sum

$$H_{\text{PSS}}\oplus su(3,\mathbb{C}) \tag{14}$$

in the fundamental (irreducible and unitary) representation $\underline{3}$.

Let us now end this letter with some comments.

First, we want to point out once again that the above $su(3,\mathbb{C})$-symmetry could play an analogous role to the $so(4)$-symmetry in the study of the hydrogen atom [8] where we remember that the time-independent Hamiltonian also commutes with all the other (six) operators, i.e. the orbital momentum and the Runge–Lenz vector. These examples are two typical applications requiring the determination of the symmetry Lie algebra for a Hamiltonian with accidental degeneracy [10]. Consequently, the four steps of the procedure proposed by Moshinsky *et al* [10] can be considered in the above parasupersymmetric context(s) in order to explain the (triple) degeneracies of the energy eigenvalues. In that way we can show that the above $su(3,\mathbb{C})$-symmetry is, in fact, too large for explaining these degeneracies: one of its $su(2,\mathbb{C})$-subalgebras [11] is already sufficient. The ladder operators are nothing else than the *original* para-supercharges Q and $Q^\dagger$ as previously understood [4], so that we get only one Z-operator *in the line of (5): it is then easy to see that* Q, $Q^\dagger$ *and* Z generate a $su(2,\mathbb{C})$-subalgebra which is effectively the symmetry algebra explaining the (triple) accidental degeneracies in the parasuperspectrum. Consequently, the above $su(3,\mathbb{C})$-symmetry also enhances additional properties with respect to the minimal $su(2,\mathbb{C})$-one. Let us end this first comment by noticing that the fourth step of the Moshinsky *et al* procedure [10] is always satisfied in our considerations: we effectively have that

$$H^2=\frac{1}{2}J^2 \qquad H^6=\frac{81}{9400}(C_1^3+C_2^2) \tag{15}$$

where J^2 is the usual Casimir operator of $su(2,\mathbb{C})$ while C_1 and C_2 are the Casimir operators of $su(3,\mathbb{C})$ given by

$$C_1=\sum_{i=1}^{8}F_i^2 \qquad C_2=\sum_{i,j,k=1}^{8}d_{ijk}F_iF_jF_k \tag{16}$$

the d_{ijk}'s being the symmetric structure constants [9] of $su(3,\mathbb{C})$.

Secondly, we have to understand that the way of constructing the operators (5) is simply related to the reduction of trilinear relations into bilinear ones [11] in order to get quadratic Sklyanin algebras [12] from Lie parasuperalgebras [13]. Indeed, we notice that the algebra quoted in eqs. (6) is nothing else than a quadratic algebra equivalent to the Lie parasuperalgebra subtending our approach [4] of parasupersymmetric quantum mechanics.

Thirdly, due to the fact that our results are enhancing the fundamental representation $\underline{3}$ of $su(3,\mathbb{C})$, we can ask if it is not possible to exploit the other (non-equivalent) fundamental representation $\underline{3}^*$ of $su(3,\mathbb{C})$, in order to develop a new form of parasupersymmetric quantum mechanics. The answer is negative: it can be shown that the $\underline{3}$- and $\underline{3}^*$-characteristics are, in our developments, simply related to each other by the only interchange of the so-called type-Q^- $(\equiv q_1^- + q_2^+)$ and type-P $(\equiv q_2^+ - q_1^-)$ parasupercharges defined elsewhere [13]. Such an interchange does not modify the physical context.

Fourthly, we recall that parasupersymmetry has to include supersymmetry, so that we can also ask for dynamical symmetries in the $p=1$-context. Here, the type P-supercharges do not exist in accordance with the fact that we get only two q's and only one Z (cf. (3) and (5)) besides the superhamiltonian H_{ss}. The Lie algebra subtending the dynamical supersymmetries is thus $su(2,\mathbb{C})$, so that, in correspondence with the structure (14), we have here

$$H_{SS}\oplus su(2,\mathbb{C}). \tag{17}$$

Let us notice that $su(2,\mathbb{C})$ admits only one fundamental representation ($\underline{2}\equiv\underline{2}^*$) and that the three symmetries are dynamical ones entering in the understanding of the degeneracies of the superspectrum [5]. This Lie algebra $su(2,\mathbb{C})$ is precisely the one which is also necessary in the $p=2$-context, where it appears as a subalgebra of $su(3,\mathbb{C})$ as discussed in the first comment.

Finally, let us conclude by mentioning that the above results can be extended to arbitrary orders p of paraquantization: the dynamical symmetry is then characterized by $[(p+1)^2-1]$ generators leading systematically to the simple Lie algebra

$su(p+1, \mathbb{C})$. In each p-context, the subalgebra $su(2, \mathbb{C}) \subset su(p+1, \mathbb{C})$ plays the main role as a part of the dynamical algebra explaining completely the degree $(p+1)$ of degeneracy of the energy eigenvalues contained in the associated parasuperspectrum. Such an argument is in complete agreement with the use of the $(p+1)$-dimensional representations $D^{(p/2)}$ of $su(2, \mathbb{C})$ required by the parafermionic variables [2].

One of us (AGN) wants to thank the Belgian Interuniversitary Institute for Nuclear Sciences (IISN) for financial support during his stay in Liège in December 1992.

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