On pararelativistic quantum oscillators

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(Received 15 January 1992; accepted for publication 5 June 1992)

Different choices of matrices characterizing $p=2$ parafermions are analyzed in connection with the description of relativistic spin-one particles through the Kemmer formulation. The free and interacting cases are considered and the relations between parasupersymmetry and Kemmer theory are enhanced as it is also the case between supersymmetry and Dirac theory. In that way the oscillatorlike context leads to the characterization of pararelativistic oscillators.

I. INTRODUCTION

As generalizations of ordinary ($p=1$) bosons and fermions, arbitrary (p-th order) parabosons and parafermions are described by parastatistics and associated parafields.[1–3] Their respective *superposition* led to supersymmetry[4] and parasupersymmetry[5] which appear now as fundamental tools characterizing respectively the so-called $N=2$ supersymmetric quantum mechanics (SSQM)[6] when bosons and fermions are concerned *and* the so-called $N=2$ parasupersymmetric quantum mechanics (PSSQM)[5] when bosons and $p=2$ parafermions are considered.

Essentially developed through *nonrelativistic* Schrödinger–Pauli type wave equations, these specific quantum contexts SSQM and PSSQM have different connections with *relativistic* developments. On the one hand, *supersymmetry* has indeed been related many times[7] with the *Dirac theory* and its unitarily equivalent Foldy–Wouthuysen representation.[8] On the other hand, *parasupersymmetry* has also to deal with a relativistic formulation if we recall that parafields have already been realized in terms of spin-one (Kemmer) matrices but only for $p=2$ parafermions.[9] We can thus relate *parasupersymmetry* with the Duffin–Kemmer–Petiau[10] equation hereafter called the *Kemmer equation*. These Dirac and Kemmer equations are the first simplest Bhabha formulations.[11–13]

If we have already concentrated our attention[8] on fermions (which are $p=1$ parafermions) and the Dirac theory in connection with SSQM and supersymmetry in general, let us focus our attention on $p=2$ parafermions (the *simplest nontrivial* parafermionic context) and the Kemmer theory in connection with PSSQM and $N=2$ parasupersymmetry in particular.

The contents of this paper are then distributed as follows: in Sec. II we will first consider *free* ($p=1$) particles (Sec. II A) in connection with Witten's supersymmetry[6] and, second, *free* ($p=2$) paraparticles (Sec. II B) in connection with a recent approach[14] of parasupersymmetry. Both subsections will be constructed in order to show that different supersymmetrization procedures are explicitly dependent of the choices of (para)fermionic matrices inside the (para)supercharges. This conclusion will lead us to new parasupersymmetrizations if we want (Sec. II C) to connect such developments with a relativistic *Kemmer* formulation. Section III will deal with the *interacting cases* mainly developed for spin-one (para) particles in harmonic oscillatorlike contexts (Sec. III A) *or* in interaction through arbitrary potentials (Sec. III B). In the (Dirac) spin one-half context, such developments led to relativistic oscillators (Ref. 8 and references therein) while we will get here typical informations on pararelativistic oscillators inside the Kemmer theory.

As far as relativistic considerations enter in this paper, we will adopt Bjorken and Drell's conventions[15] for the metric and the associated Lorentz covariance of the (Dirac as well as Kemmer) equations.

II. ON FREE $p=1$ PARTICLES AND $p=2$ PARAPARTICLES

Let us first distinguish in SSQM the possible procedures of supersymmetrization on *free* bosons and fermions, i.e., on $p=1$ (para)particles (Sec. II A) in order to exploit this way of reasoning in PSSQM for free $p=2$ paraparticles (Sec. II B) when bosons and parafermions are superposed along the Beckers–Debergh recent approach[14] in PSSQM. Then (Sec. II C) let us discuss the latter PSSQM characteristics in connection with the relativistic Kemmer theory.[10]

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A. Superposition of free bosons and fermions

After Witten,[6] the $N=2$ superalgebra sqm (2) expressed in terms of two *Hermitian* supercharges (Q_1,Q_2) and of the supersymmetric Hamiltonian H_{SS} is characterized by

$$\{Q_a,Q_b\}=\tfrac{1}{2}\delta_{ab}H_{SS},\quad [Q_a,H_{SS}]=0\quad (a,b=1,2), \tag{2.1}$$

containing, in particular, the three relations

$$Q_1^2=H_{SS},\quad Q_2^2=H_{SS},\quad \{Q_1,Q_2\}=0. \tag{2.2}$$

If the *free* context is visited in a three-dimensional space, we evidently ask for

$$H_{SS}^{(0)}=\frac{1}{2}\mathbf{p}^2,\quad Q_1=c_1\sum_{j=1}^{3}M_jp_j,\quad Q_2=c_2\sum_{k=1}^{3}N_kp_k, \tag{2.3}$$

where the superscript (0) refers to this free case, c_1 and c_2 being arbitrary coefficients $(\in\mathbb{R})$ and where the Hermitian matrices (M_1,M_2,M_3) and (N_1,N_2,N_3) have specific properties if the conditions (2.2) are required. It is not difficult to convince ourselves that Eqs. (2.2) lead to the respective conditions

$$\{M_j,M_k\}=\frac{1}{c_1^2}\delta_{jk},\quad \{N_j,N_k\}=\frac{1}{c_2^2}\delta_{jk},\quad \{M_j,N_k\}=c\Xi_{jk}, \tag{2.4}$$

where $\Xi_{jk}=-\Xi_{kj}$. The coefficient $c(\in\mathbb{R})$ determines that we are working within the original *standard* (Witten) procedure[6] of supersymmetrization iff $c=0$ *or* within the so-called *spin–orbit coupling* procedure[16] of supersymmetrization iff $c=1$ in this three-dimensional context (see also Refs. 17 and 18).

A refined discussion of such properties can be usefully displayed here by giving to the supercharges (2.3) more explicit expressions based on Green–Cusson Ansätze[1,19] that we have recently proposed.[20] Indeed, let us consider d-dimensional spaces with the two supercharges

$$Q_1=\frac{1}{\sqrt{2}}\sum_{j=1}^{n}\sum_{\mu=1}^{p}p_{\mu+(j-1)p}\,\xi_\mu\otimes\sigma_j \tag{2.5a}$$

and

$$Q_2=\frac{1}{\sqrt{2}}\sum_{j=1}^{n}\sum_{\mu=1}^{p}p_{\mu+(j-1)p}\,\xi_\mu\otimes\sigma_{j+n}, \tag{2.5b}$$

where evidently

$$d=n\cdot p, \tag{2.6a}$$

$$[p_j,p_k]=0,\quad j,k=1,\ldots,d, \tag{2.6b}$$

$$\{\xi_\mu,\xi_\nu\}=2\delta_{\mu\nu}, \tag{2.6c}$$

$$\{\sigma_j,\sigma_k\}=2\delta_{jk}=\{\sigma_{j+n},\sigma_{k+n}\}, \tag{2.6d}$$

$$\{\sigma_j,\sigma_{k+n}\}=2c\Sigma_{jk}=-2c\Sigma_{kj},\quad c=0\ \ or\ \ 1. \tag{2.6e}$$

The bosonic characteristics (2.6b) are the usual ones while the fermionic matrices ξ's and σ's generate Clifford algebras in direct $(c=0)$ or semidirect $(c=1)$ sums as already noticed.[21] These supercharges (2.5), directly connected with the ones given by (2.3), can now be discussed as follows. If $d=1$, we have the only possibility $n=1=p$ and we get

$$Q_1^{(1)}=\frac{1}{\sqrt{2}}p_1\xi_1\otimes\sigma_1=\frac{1}{\sqrt{2}}p_1M_1,\quad Q_2^{(1)}=\frac{1}{\sqrt{2}}p_1\xi_1\otimes\sigma_2=\frac{1}{\sqrt{2}}p_1N_1 \tag{2.7}$$

showing that

$$\{M_1,N_1\}=0 \tag{2.8}$$

and ensuring that we only get here the standard procedure $(c=0)$ as expected. If $d=2$, there are two possibilities which correspond to $n=1$, $p=2$ or $n=2$, $p=1$, so that we get two subcases. With $(n=1,\ p=2)$, we obtain

$$Q_a^{(2)}=\frac{1}{\sqrt{2}}\sum_{\mu=1}^{2}p_\mu\xi_\mu\otimes\sigma_a\quad (a=1,2), \tag{2.9a}$$

while, with $(n=2,\ p=1)$, we distinguish

$$Q_a^{(2)\prime}=\frac{1}{\sqrt{2}}\sum_{j=1}^{2}p_j\xi_a\otimes\sigma_j\quad (a=1,2). \tag{2.9b}$$

Both associated M and N matrices lead here to the two possibilities of standard or spin–orbit coupling procedures. If $d=3$, we have once again only two possibilities according to Eq. (2.6a): $(n=1,\ p=3)$ or $(n=3,\ p=1)$ and it is straightforward to show that the two procedures can appear. Finally, let us mention that, for $d\geqslant4$, Eq. (2.6a) gives more than two possibilities among which we evidently recover the two ones characterized by $n=1$ *or* $p=1$ leading once again to the above two possible procedures; all the *other* possibilities are not consistent with the free case (the free Hamiltonian does not result from the squares of the corresponding supercharges Q_a when $n\neq1$ and $p\neq1$) and ask for other procedures of supersymmetrization compatible with some interactions.[20,22]

Such considerations are useful for connecting SSQM results and the Dirac theory in Minkowski space-time. The latter includes the $d=3$ context, so that the above properties lead to the conclusion that the *two* procedures of supersymmetrization are admissible in accordance with previous results.[8] In fact, due to the four-dimensional matrices included in Dirac theory, we know that this strictly corresponds to $c=1$, so that the spin–orbit coupling scheme is here priviledged: it leads to a semidirect sum of two Clifford algebras $\mathscr{C}\ell_3$ issued from Eqs. (2.4) with $c_1=c_2=1/\sqrt{2}$. We are thus dealing with the four-dimensional representation of su(2|2) (Ref. 21) characterized in terms of Dirac matrices by the choices

$$M_j=\alpha_j,\quad N_j=i\beta\alpha_j,\quad j=1,2,3, \tag{2.10}$$

and satisfying the relations (2.4) on the forms

$$\{\alpha_j,\alpha_k\}=2\delta_{jk},\quad \{i\beta\alpha_j,i\beta\alpha_k\}=2\delta_{jk}, \\ \{\alpha_j,i\beta\alpha_k\}=i\beta[\alpha_k,\alpha_j]. \tag{2.11}$$

The above discussion enlightens the main role of the *spin–orbit coupling procedure* in connection with the Dirac equation: It also opens the way for the interacting context when oscillatorlike characteristics have to be included[8] leading in particular to the description of relativistic oscillators.

B. Superposition of free bosons and $p=2$ parafermions

Let us now consider PSSQM through two *Hermitian* parasupercharges Q_1 and Q_2 satisfying a Lie parasuperalgebra[23,24] characterized by the following nilpotent relations:

$$(Q_1-iQ_2)^3=0\quad (Q_1+iQ_2)^3=0 \tag{2.12}$$

and the *double* commutators

$$[Q_1,[Q_1,Q_2]]=Q_2H_{PSS},\quad [Q_2,[Q_2,Q_1]]=Q_1H_{PSS}, \tag{2.13}$$

besides the commutation relations ensuring the conserved character of Q_1 and Q_2, i.e.,

$$[H_{PSS},Q_1]=0,\quad [H_{PSS},Q_2]=0. \tag{2.14a}$$

Equations (2.12) and (2.13) can also be rewritten

$$Q_1^3-3Q_2Q_1Q_2=Q_1H_{PSS},\quad Q_2^3-3Q_1Q_2Q_1=Q_2H_{PSS}, \tag{2.14b}$$

$$2(Q_1Q_2^2+Q_2^2Q_1)-Q_2Q_1Q_2-Q_1^3=Q_1H_{PSS}, \\ 2(Q_2Q_1^2+Q_1^2Q_2)-Q_1Q_2Q_1-Q_2^3=Q_2H_{PSS}. \tag{2.14c}$$

In the $d=3$-dimensional space, H_{PSS} is the parasuper-Hamiltonian which reduces once again in the *free* case to

$$H_{PSS}^{(0)}=\tfrac{1}{2}\mathbf{p}^2 \tag{2.15}$$

and the parasupercharges can be defined as follows

$$Q_1=c_1\sum_{j=1}^{3}M_jp_j,\quad Q_2=c_2\sum_{j=1}^{3}N_jp_j, \tag{2.16}$$

where the matrices (M_1,M_2,M_3) and (N_1,N_2,N_3) have now specific properties issued from Eqs. (2.14), (2.15), and (2.16). Moreover, if we remember that $p=2$ parafermions have to deal with Kemmer matrices,[9] it is not difficult to show that Eqs. (2.14b) and (2.14c) imply the following properties:

$$M_jM_kM_l+M_lM_kM_j=\delta_{jk}M_l+\delta_{kl}M_j, \tag{2.17a}$$

$$N_jN_kN_l+N_lN_kN_j=\delta_{jk}N_l+\delta_{kl}N_j, \tag{2.17b}$$

$$N_jM_kN_l+N_lM_kN_j=-\frac{\frac{1}{2}-c_1^2}{3c_2^2}(\delta_{jk}M_l+\delta_{kl}M_j) \\ +d_1\Xi_{jk}^{(1)}\alpha_l^{(1)}+d_2\Xi_{kl}^{(2)}\alpha_j^{(2)}, \tag{2.17c}$$

$$M_jN_kM_l+M_lN_kM_j=-\frac{\frac{1}{2}-c_2^2}{3c_1^2}(\delta_{jk}N_l+\delta_{kl}N_j) \\ +d_3\Xi_{jk}^{(3)}\alpha_l^{(3)}+d_4\Xi_{kl}^{(4)}\alpha_j^{(4)}, \tag{2.17d}$$

$$M_jN_kN_l+N_lN_kM_j=\frac{1+2c_1^2+\frac{2}{3}c_2^2(\frac{1}{2}-c_1^2)}{4c_2^2}(a_1\delta_{jk}M_l \\ +b_1\delta_{kl}M_j)+d_5\Xi_{jk}^{(5)}\alpha_l^{(5)} \\ +d_6\Xi_{lj}^{(6)}\alpha_k^{(6)}+d_7\Xi_{kl}^{(7)}\alpha_j^{(7)} \\ +d_8\Xi_{jkl}^{(8)}, \tag{2.17e}$$

and

$$N_jM_kM_l+M_lM_kN_j=\frac{1+2c_2^2+\frac{2}{3}c_1^2(\frac{1}{2}-c_2^2)}{4c_1^2}(a_2\delta_{jk}N_l+b_2\delta_{kl}N_j)+d_9\Xi_{jk}^{(9)}\alpha_l^{(9)}+d_{10}\Xi_{lj}^{(10)}\alpha_k^{(10)}+d_{11}\Xi_{kl}^{(11)}\alpha_j^{(11)}+d_{12}\Xi_{jkl}^{(12)}. \tag{2.17f}$$

Here $d_1,d_2,...,d_{12}$ are arbitrary constant coefficients, the matrices $\Xi_{mn}^{(*)}$ and $\Xi_{mnq}^{(*)}$ are completely antisymmetric ones while the $\alpha_m^{(*)}$ are supplementary matrices with the same dimensions as the M_j and N_j.

The general results (2.17c)–(2.17f) do take into account possible extra terms which, contracted with the symmetric triple products $p_jp_kp_l$, give exactly null contributions in Eqs. (2.14b) and (2.14c). Such a system (2.17) can then be tested in connection with different matrix realizations essentially expressed as Kemmer matrices belonging to two Kemmer algebras $K(3)$ characterized by Eqs. (2.17a) and (2.17b).

By exploiting some recent results in three-dimensional spaces on $p=2$ parafermionic matrices typically introduced for the standard procedure[23] and for the spin–orbit coupling procedure,[24] it is *not* possible through the corresponding matrices M_j and N_j to satisfy *simultaneously* Eqs. (2.17) *and* the dimensional requirements for spin-one matrices in the Kemmer context. Let us only mention here their explicit forms according to the *standard* characteristics [see Eqs. (3.4a) and (3.4c) of Ref. 23 when $n=3$ therein]:

$$\begin{aligned}M_j^{\text{ST}}&=(1/\sqrt{2})(e_{7-j,7}+e_{7,7-j}+e_{7,4-j}+e_{4-j,7}),\\ N_j^{\text{ST}}&=(i/\sqrt{2})(e_{7,7-j}-e_{7-j,7}+e_{4-j,7}-e_{7,4-j}),\end{aligned} \tag{2.18}$$

where the notations $e_{k,l}$ refer to 7×7 matrices whose rows and columns are labeled from 1 to 7 containing zeros everywhere except units at the intersection of the kth row and lth column. For the *spin–orbit coupling* characteristics [see Eqs. (2.7) and (2.8a) of Ref. 24 when the summations are taken from 1 to 3 therein], we also get a realization of 6×6 matrices given by

$$\begin{aligned}M_j^{\text{S.O.}}&=(1/\sqrt{2})\sigma_j\otimes(-e_{1,3}+e_{2,3}-e_{3,1}+e_{3,2}),\\ N_j^{\text{S.O.}}&=-(i/\sqrt{2})\sigma_j\otimes(e_{1,3}+e_{2,3}-e_{3,1}-e_{3,2}),\end{aligned} \tag{2.19}$$

where, evidently, the matrices $e_{k,l}$ are here three-dimensional ones.

None of the above realizations is convenient, so that, at this stage, we cannot associate the Kemmer equation for spin one particles with the above two procedures of parasupersymmetrization in the free case. Consequently, every ten-dimensional (or six-dimensional) realization of the matrices M_j and N_j ($j=1,2,3$) satisfying Eqs. (2.17) will be associated with a *new* (para) supersymmetrization procedure.

C. The free case and the Kemmer formulation

Let us give an example through the Hamiltonian form[25] of the Kemmer theory.[10] With the Bjorken–Drell[15] conventions we are considering the covariant Kemmer equation expressed in terms of matrices β_μ ($\mu=0,1,2,3$) such that

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=g_{\mu\nu}\beta_\lambda+g_{\lambda\nu}\beta_\mu, \tag{2.20}$$

where $g_{00}=-g_{kk}=1$. The Kemmer Hamiltonian can then be written

$$H_K=\mathbf{B}\cdot\mathbf{p}+mB_0 \tag{2.21}$$

with

$$B_j\equiv\beta_0\beta_j-\beta_j\beta_0\quad(j=1,2,3),\quad B_0\equiv\beta_0. \tag{2.22}$$

Exploiting the Kemmer algebra,[12] we can propose the following choice

$$M_j=B_j,\quad N_j=i\eta_0B_j,\quad \eta_0=2\beta_0^2-I, \tag{2.23}$$

showing that Eqs. (2.17) are readily verified with

$$\begin{aligned}&c_1=c_2=\frac{1}{2\sqrt{2}},\quad d_1=d_2=\cdots=d_{12}=0,\\ &a_1=b_1=a_2=b_2=\frac{16}{41}.\end{aligned} \tag{2.24}$$

The Eqs. (2.17) expressed in this context (2.23)–(2.24) lead to a semidirect sum of two $K(3)$–Kemmer algebras corresponding to a procedure which is neither a standard one, nor a spin–orbit coupling one. In a parallel way with the Dirac theory,[8] we thus can construct two Kemmer Hamiltonians as follows:

$$H_K^{(1)}=2\sqrt{2}Q_1+m\beta_0,\quad H_K^{(2)}=2\sqrt{2}Q_2+m\beta_0, \tag{2.25}$$

which are unitarily equivalent through the operator

$$U=(1/\sqrt{2})(1+i\eta_0). \tag{2.26}$$

Let us end this free case by two remarks. First, let us point out that the above ten-dimensional Kemmer realization has to be completed by an initial condition eliminating the (4) redundant components in the wave function. The well-known Sakata–Taketani formulation[26] corresponds to this point of view and deals with 6×6 matrices. We just want to mention here that, if we consider the respective substitutions in M_j and N_j,

$$B_j \leftrightarrow \sigma_1 \otimes S_j, \quad i\eta_0 B_j \leftrightarrow \sigma_2 \otimes S_j, \tag{2.27}$$

where $\{S_j\ (j=1,2,3)\}$ correspond to the $D^{(1)}$ representation of the Lie algebra su(2,C), we immediately satisfy the relations (2.17) with (2.24), so that we get an isomorphic semidirect sum as expected. This corresponds to the well-known reduction leading to the $2(2s+1)$ description for $s=1$ relativistic particles. As a second remark, let us mention that the above study could be developed through the Rubakov–Spiridonov approach[5] of PSSQM which differs from ours essentially through (2.13) and consequently through (2.14b) and (2.14c). From the start,[27] we know that these developments do not permit the spin–orbit coupling procedure, so that we have decided to discuss the more general context we have at our disposal.

III. ON INTERACTING BOSONS AND $p=2$ PARAFERMIONS

As already discussed in SSQM,[8] the oscillator interacting context is related to the Dirac equation and there we speak about *relativistic oscillators.* We now propose to examine the corresponding case in PSSQM (Sec. III A) by dealing evidently with $p=2$ parafermions (instead of $p=1$ fermions) and by exploiting the results of Sec. II on the Kemmer formulation: we construct in that way what we call here *pararelativistic oscillators.* This first step then leads easily to the case of *arbitrary* interactions in the spin-one developments (Sec. III B).

A. On pararelativistic oscillators

The interacting context with oscillatorlike characteristics corresponds to a first generalization of Eqs. (2.15) and (2.16) but by maintaining the validity of Eqs. (2.14). Indeed, we can define the new parasupercharges

$$Q_1=\frac{1}{2\sqrt{2}}\left(\sum_{j=1}^{3} B_j p_j + i\omega\eta_0 \sum_{j=1}^{3} B_j x_j\right) \tag{3.1a}$$

and

$$Q_2=\frac{1}{2\sqrt{2}}\left(i\eta_0 \sum_{j=1}^{3} B_j p_j - \omega \sum_{j=1}^{3} B_j x_j\right), \tag{3.1b}$$

where ω is, as usual, the harmonic angular frequency and the matrices B_j, η_0 are ten-dimensional ones already introduced in Sec. II C. We evidently have included in such parasupercharges the information (2.23) and (2.24) but also extra terms analogous to those introduced in the spin one-half case.[8] Relatively tedious but straightforward calculations show that the parasupersymmetric Hamiltonian issued from such developments is given by

$$H_{PSS}^{\text{H.O.}}=\tfrac{1}{2}\mathbf{p}^2+\tfrac{1}{2}\omega^2\mathbf{x}^2+\omega\eta_0(\tfrac{3}{2}-\mathbf{B}\cdot\mathbf{B}), \tag{3.2}$$

so that Eqs. (2.14) are satisfied ensuring, in particular, that the parasupercharges are conserved ones. We recognize in (3.2) the expected bosonic part $(\frac{1}{2}\mathbf{p}^2+\frac{1}{2}\omega^2\mathbf{x}^2)$ supplemented by a typical spin-one term which is not of a spin–orbit coupling type (also expected from our information issued from the free case in Sec. II). Through a 10×10 realization of the matrices $\beta_\mu \equiv (2.20)$ given for example by

$$\begin{aligned}
&\beta_0=e_{1,7}+e_{2,8}+e_{3,9}+e_{7,1}+e_{8,2}+e_{9,3},\\
&\beta_1=-i(e_{1,10}-e_{5,9}+e_{6,8}+e_{8,6}-e_{9,5}+e_{10,1}),\\
&\beta_2=-i(e_{2,10}+e_{4,9}-e_{6,7}-e_{7,6}+e_{9,4}+e_{10,2}),\\
&\beta_3=-i(e_{3,10}-e_{4,8}+e_{5,7}+e_{7,5}-e_{8,4}+e_{10,3}),
\end{aligned} \tag{3.3}$$

it is straightforward to see the Hamiltonian (3.2) as the diagonal matrix

$$\begin{aligned}
H_{PSS}^{\text{H.O.}}=\omega\ \text{diag}[&N+1,N+1,N+1,N+2,N+2,N\\
&+2,N+2,N+2,N+2,N+3],
\end{aligned} \tag{3.4}$$

where N refers here to the usual bosonic degeneracy

$$N=2n+l, \quad \sigma=2l+1, \quad l=0,1,\ldots. \tag{3.5}$$

Let us now consider the two Kemmer Hamiltonians (2.25) with the parasupercharges (3.1). For example, we evidently have

$$H_K^{(1)\text{H.O.}}=\mathbf{B}\cdot(\mathbf{p}-i\omega\eta_0\mathbf{x})+m\beta_0 \tag{3.6}$$

leading to a Kemmer formulation characterized by a six-component wave function due to the fact that the Hamiltonian equation has to be completed by an initial condition[25] saying that

$$(H_K\beta_0-m)\Psi=0. \tag{3.7}$$

Within the representation (3.3), the condition (3.7) is equivalent to the four equations

$$\begin{aligned}
&m\Psi_4=i(p_3-i\omega x_3)\Psi_8-i(p_2-i\omega x_2)\Psi_9,\\
&m\Psi_5=i(p_1-i\omega x_1)\Psi_9-i(p_3-i\omega x_3)\Psi_7,\\
&m\Psi_6=i(p_2-i\omega x_2)\Psi_7-i(p_1-i\omega x_1)\Psi_8,\\
&im\Psi_{10}=(p_1-i\omega x_1)\Psi_1+(p_2-i\omega x_2)\Psi_2+(p_3-i\omega x_3)\Psi_3,
\end{aligned} \tag{3.8}$$

when we have denoted the transposed wave function $\Psi^T = (\Psi_1,\Psi_2,\Psi_3,...,\Psi_{10})$. These equations express the expected redundancy in terms of the four components Ψ_4, Ψ_5, Ψ_6, and Ψ_{10}.

Now, if we take account of this redundancy in the *nonrelativistic* context in order to get a self-consistent physical theory simply related to a (physical) pararelativistic oscillator, we suppress in (3.4) the fourth, fifth, sixth, and tenth lines and columns and get the six-dimensional nonrelativistic Hamiltonian

$$H_{PSS}^{\mathrm{H.O.}1} = \omega \,\mathrm{diag}[N+1,N+1,N+1,N+2,N+2,N+2]. \tag{3.9}$$

It gives ω as the minimal energy corresponding to the fundamental ground state. The eigenvalues $k\omega$ ($k=1,2,...$) are equidistant and each of them is characterized by a total degeneracy equal to $3k^2$.

Complementary discussions on Foldy–Wouthuysen transformations[8] in this spin-one context, on equivalent Hamiltonians, etc., can evidently be developed at this stage. Let us only end this paper by considering the extension of our oscillatorlike results to the case of *arbitrary* interactions.

B. Toward arbitrary (parasuper) potentials

In the harmonic case, the construction of the parasupercharges (3.1) corresponds to a potential $W(\mathbf{x})$ such that

$$\nabla W(\mathbf{x}) = \omega \mathbf{x}. \tag{3.10}$$

So we can immediately propose the *general* parasupercharges

$$Q_1 = (1/2\sqrt{2})(\mathbf{B}\cdot\mathbf{p} + i\eta_0 \mathbf{B}\cdot\nabla W(\mathbf{x})) \tag{3.11a}$$

and

$$Q_2 = (1/2\sqrt{2})(i\eta_0 \mathbf{B}\cdot\mathbf{p} - \mathbf{B}\cdot\nabla W(\mathbf{x})). \tag{3.11b}$$

According to Eqs. (2.14), we get here the general parasuper-Hamiltonian

$$H_{PSS} = \tfrac{1}{2}\mathbf{p}^2 + \tfrac{1}{2}(\nabla W)^2 + \tfrac{1}{2}\eta_0[\Delta W - 2(\mathbf{B}\cdot\nabla)^2 W], \tag{3.12}$$

which has to deal with the two unitarily equivalent Kemmer Hamiltonians

$$H_K^{(1)} = \mathbf{B}\cdot\mathbf{p} + i\eta_0 \mathbf{B}\cdot\nabla W(\mathbf{x}) + m\beta_0 \tag{3.13a}$$

or(and)

$$H_K^{(2)} = i\eta_0 \mathbf{B}\cdot\mathbf{p} - \mathbf{B}\cdot\nabla W(\mathbf{x}) + m\beta_0, \tag{3.13b}$$

subject to supplementary conditions for avoiding redundancy as discussed in the previous harmonic context. In principle, physical applications (other than oscillatorlike ones) such as electromagnetic interactions can evidently be developed through these characteristics.

ACKNOWLEDGMENT

One of us (A.G.N.) thanks the Belgian Interuniversitary Institute for Nuclear Sciences (I.I.S.N.) for financial support during his stay in Liège.

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