Systems of Reaction Diffusion Equations and their Symmetry Properties

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Abstract

A constructive algorithm is proposed for the investigation of symmetries of partial differential equations. The algorithm is used to present classical Lie symmetries of systems of two non-linear reaction diffusion equations.

1 Introduction

Mathematical models which ultimately involve the analysis of coupled systems of nonlinear diffusion equations are often discussed in the literature. The Schrödinger equation in m-dimensional space is an obvious example from mathematical physics whilst activator-inhibitor reactions [1], predator-prey systems, $\lambda - \omega$ reaction systems [2] are common examples from mathematical biology. These systems are very complex in nature and admit fundamental particular solutions (for example, travelling waves and, spiral waves) which have a clear group-theoretical interpretation and which can be obtained using the classical Lie approach. The existence of such solutions suggests an important role for both the classical and non-classical symmetry analysis of systems of reaction diffusion equations. However, to the best of our knowledge, a comprehensive group analysis has not been undertaken previously although analyses of some special cases do exist.

In this paper we discuss symmetries of equations in the general form

$$\frac{\partial u}{\partial t} - A \sum_i \frac{\partial^2 u}{\partial x_i^2} - f(u) = 0 \tag{1}$$

where $u \in \mathbb{R}^n$, $f \in \mathbb{R}^n$, $(x, t) \in \mathbb{R}^m \times \mathbb{R}$ and A is a $n \times n$ constant non-singular matrix. Actually we restrict ourselves to the case $n = 2$ but all results present in Section 2 and the main part of results of Section 3 are valid for arbitrary n.

We note that equation (1) with $m = n = 1$, $f \equiv 0$ was the subject of a group analysis by Sophus Lie [3]. In addition, classical Lie symmetries of equation (1) with, $n = m = 1$, were investigated by Ovsiannikov [4] whose results were completed by Dorodnitsyn [5] and then generalized to the case $m = 2, 3$ by Dorodnitsyn, Kniazeva and Svirishchevski [6]. The related conditional (nonclassical) symmetries were described by Fushchych and Serov [7] and Clarkson and Mansfield [8]. The classical symmetries are summarized in Table 1.

Table 1: Symmetries of the scalar diffusion equation with source

$f(u)$	Infinitesimals Generator, X
arbitrary	X_0
0	$X_0 + \nu G_0 + \mu \widehat{D}_0 + \lambda \hat{A}_0 + \sigma u \frac{\partial}{\partial u} + \psi_0 \frac{\partial}{\partial u}$
1	$X_0 + \nu G_1 + \mu \widehat{D}_1 + \lambda \hat{A}_1 + \sigma (u - t) \frac{\partial}{\partial u} + \psi_0 \frac{\partial}{\partial u}$
u	$X_0 + \nu G_0 + \mu \widehat{D}_2 + \alpha \hat{A}_2 + \sigma u \frac{\partial}{\partial u} + \psi_1 \frac{\partial}{\partial u}$
$u^n,\ n \neq 1$	$X_0 + \mu \widehat{D}_3$
e^u	$X_0 + \mu \widehat{D}_4$
$f = u \ln u$	$X_0 + \nu e^t \left(\frac{\partial}{\partial x} - \frac{1}{2} x u \frac{\partial}{\partial u} \right) + \mu e^t u \frac{\partial}{\partial u}$

Here Greek letters denote arbitrary parameters and are the translation, X_0; Galilei, G_α ($\alpha = 0, 1$); scale, $\widehat{D}_\mu$ ($\mu = 0, 1, ..., 4$) and conformal, $\hat{A}_s$ ($\sigma = 0, 1, 2$) generators respectively, where :

$$X_0 = \alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial x}, \quad G_\alpha = t \frac{\partial}{\partial x} - \frac{1}{2} x u \frac{\partial}{\partial u} + \frac{1}{2} \delta_{\alpha 1} t x \frac{\partial}{\partial u}, \quad \widehat{D}_0 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x},$$
$$\widehat{D}_1 = \widehat{D}_0 + 2t \frac{\partial}{\partial u}, \quad \widehat{D}_2 = \widehat{D}_0 + 2tu \frac{\partial}{\partial u}, \quad \widehat{D}_3 = \widehat{D}_0 - \frac{2}{n-1} u \frac{\partial}{\partial u}, \tag{2}$$
$$\widehat{D}_4 = D_0 - 2 \frac{\partial}{\partial u}, \quad \hat{A}_\mu = \frac{1}{2} t \widehat{D}_\mu - \frac{1}{4} (x^2 + 2t)(1 - \delta_{\mu 1} t) u \frac{\partial}{\partial u}.$$

In addition, $\delta_{\mu\nu}$ is the Kronecker symbol and $\psi_n(n = 0, 1)$ is an arbitrary solution of the linear diffusion equation $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\psi_n = n\psi_n$.

A systematic investigation of the symmetries of the general equation (1) with $m > 1$ and $n > 1$ has so far not been considered in detail although partial results are available in [10], [11], [9], [2]. In recent paper [12] Lie symmetries of a subclass of systems (1) for $n = 2$ and arbitrary m were investigated. In this subclass the matrix A was diagonal but not a multiple of the unit matrix. We shall demonstrate that the results [12] are incomplete.

It is the aim here to undertake such an investigation for the most general case of an *arbitrary* 2×2 matrix A. In this way we present a complete description of Lie symmetries of all possible systems (1) including the cases of the unit matrix A. This latter case corresponds to the most rich of symmetries with many interesting applications. In particular, we find all nonlinear Schrödinger equations admitting nontrivial Lie symmetries since these equations can also be represented in the form (1) with an *antidiagonal* matrix A, as is demonstrated in Section 7. Moreover, we find new symmetries of (1) also in the subclass studied in [12].

The additional main aim of this paper is to present the novel use of a rather conventional algorithm for investigation of symmetries of a special class of partial differential equations which includes (1) as a particular case. More precisely we will show that the classical Lie approach (refer, for example, [13],[11]) when applied to systems (1) admits a rather simple formulation in terms of commutator algebra which may also be applied to extended classes of partial differential equations. Furthermore the algorithm may be used to determine non-classical (or, conditional) symmetries of (1). It will be shown that for $n > 1$ there is a proliferation of symmetries, including for the case when $f(u)$ is linear which do not have origins in symmetries of Table 1 for the one dimensional scalar diffusion equation with source.

2 An Algorithm for the Determining equations of symmetries for the system (1)

We require form-invariance of the system of reaction diffusion equations (1) with respect to the one-parameter group of transformations:

$$t \to t'(t, x, \varepsilon), \qquad x \to x'(t, x, \varepsilon), \qquad u \to u'(t', x', \varepsilon), \tag{3}$$

where ε is a group parameter. In other words, we require that $u'(t', x', \varepsilon)$ satisfies the same equation as $u(t, x)$:

$$L'u' = f(u'), \qquad L' = \frac{\partial}{\partial t'} - A\sum_i \frac{\partial_i^2}{\partial x_i'^2}. \tag{4}$$

From the infinitesimal transformations:

$$t \to t' = t + \Delta t = t + \varepsilon\eta, \qquad x_a \to x_a' = x_a + \Delta x_a = x_a + \varepsilon\xi^a,$$
$$u_a \to u_a' = u_a + \Delta u_a = u_a + \varepsilon\pi_a \tag{5}$$

we obtain the following representation for the operator L':

$$L' = \left[1 + \varepsilon \left(\eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a}\right)\right] L \left[1 - \varepsilon \left(\eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a}\right)\right] + O(\varepsilon^2). \tag{6}$$

Using the classical Lie algorithm it is possible to find the determining equations for the functions η, ξ_a and π_a which specify the generator X of the symmetry group:

$$X = \eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a} - \pi^b \frac{\partial}{\partial u_b} \tag{7}$$

where a summation from 1 to m and from 1 to n is assumed over repeated indices a and b respectively. This system will not be reproduced here but we note that three of the equations are:

$$\frac{\partial \eta}{\partial u_a} = 0, \quad \frac{\partial \xi^a}{\partial u_b} = 0, \quad \frac{\partial^2 \pi^a}{\partial u_c \partial u_b} = 0. \tag{8}$$

So from (8) η and ξ^a are functions of t and x_a and, π^a is linear in u_a. Thus:

$$\pi^a = -\pi^{ab} u_b - \omega^a \tag{9}$$

where π^{ab} and ω^a are functions of t and $x = (x_1, x_2, \ldots x_m)$.

From (4) it is possible to deduce all the remaining determining equations. Indeed, substituting (5), (9) into (6), using (1) and neglecting the terms of order ε^2 we find that:

$$[Q, L] u - L\omega = \pi f + \frac{\partial f}{\partial u_a} \left(-\pi^{ab} u_b - \omega^a\right), \qquad Q = \eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a} + \pi \tag{10}$$

and π is a matrix whose elements π^{ab} are defined by the relation (9).

To guarantee that equation (10) is compatible with (1) and does not impose new nontrivial conditions for u in addition to (1) it is necessary to suppose that the commutator $[Q, L]$ admits the representation:

$$[Q, L] = \Lambda L + \varphi(t, x) \tag{11}$$

where Λ and φ are $n \times n$ matrices dependent on (t, x_a).

Substituting (11) into (10) the following determining equations for f are obtained:

$$\left(\Lambda^{kb} - \pi^{kb}\right) f^b + \varphi^{kb} u^b - (L\omega)^k = -(\omega^a + \pi^{ab} u_b) \frac{\partial f^k}{\partial u^a}. \tag{12}$$

Thus, to find all non-linearities f^k generating Lie symmetries for equation (1) it is necessary to solve the operator equation (11) for L, Q given in (4), (10) and determine the corresponding matrices Λ, π, φ and functions η and ξ. In the second step the non-linearities f^a may be found by solving the system of first order equations (12) with their known coefficients.

Equation (11) is a straightforward generalization of the invariance condition for the *linear* system of diffusion equations (1) with $f(u) = 0$, so that $[Q, L] = \Lambda L$, which may readily be solved. By means of this "linearization" the problem of investigating

symmetries of systems of nonlinear diffusion equations is reduced to the rather simple application of elements of matrix calculus in order to classify non-equivalent solutions of the determining equations.

We notice that this approach is valid for the extended class of equations $Lu = f(u)$ where L is a linear differential operator with constant coefficients and where $u \in \mathbb{R}^n$. We note also that calculations of the non-classical (conditional) symmetries for the system (1) may be reduced to the solution of the determining equations (12) where now Λ, π, φ, η and ξ are defined as solutions of the following relationship:

$$[Q, L] = \Lambda L + \varphi(t, x) + \mu(t, x)Q, \tag{13}$$

and where $\mu(t, x)$ is an unknown function of the independent variables.

3 The Symmetry Operators and their Simplification

We now determine the general solutions for matrices Λ, φ, π and also the functions ξ, η, π which satisfy (12), (11).

Evaluating the commutator in (11) and equating the coefficients for linearly independent differential operators we obtain the five determining equations:

$$A(\xi_b^a + \xi_a^b) = -\delta_{ab}(\Lambda A + [A, \pi]), \qquad \dot{\eta}_a = 0, \qquad \dot{\eta} = \Lambda, \tag{14}$$

$$\dot{\xi}^a - 2A\pi_a - A\xi_{nn}^a = 0, \qquad \varphi = A\pi_{nn} - \dot{\pi}. \tag{15}$$

Here the dots denotes derivatives with respect to t and subscripts denote derivatives with respect to the spatial variables, so for example, $\eta_a = \frac{\partial \eta}{\partial x_a}$.

From (14) Λ is proportional to the unit matrix, $\Lambda = \lambda I$ and from (14) $[A, \pi] = 0$. Indeed, choosing $a = b$ we obtain

$$\pi - A^{-1}\pi A = (2\xi_a^a - \lambda)I. \tag{16}$$

The trace of the left hand side of (16) is equal to zero, and so $2\xi_a^a - \lambda \equiv 0$ and $A\pi - \pi A = 0$.

Equations (14)–(15) contain matrices which commute, and so they may easily be integrated using, for example, the method of characteristics. The general solution of (14)–(15) is:

$$\begin{aligned}
\xi^a &= C^{[ab]}x_b + \dot{d}x^a + g^a, & \eta &= -2d, \\
\pi &= \tfrac{1}{2}A^{-1}(\tfrac{\ddot{d}}{2}x^2 + \dot{g}^a x^a) + C, & \Lambda &= -2\dot{d}I, \\
\varphi &= \tfrac{m}{2}\ddot{d} - \dot{C} - \tfrac{1}{2}A^{-1}(\tfrac{\dddot{d}}{2}x^2 + \ddot{g}^a x^a)
\end{aligned} \tag{17}$$

where d, g^a are arbitrary functions of t and C is a t-dependent matrix commuting with A.

By considering the x-dependence of functions (17) it is convenient to represent still unknown functions ω_a, occurring in (12), as:

$$\omega_a = \omega_2^a x^2 + \omega_1^{ab}x_b + \omega_0^a + \mu^a \tag{18}$$

where $\omega_2^a, \omega_1^{ab}, \omega_0^a$ are functions of t, and μ^a is a function of t and x. Without loss of generality we suppose that all terms in the right hand side of (18) are linearly independent. Then comparing with (12), (17) the functions μ^k have to satisfy:

$$(L\mu)^k = \lambda^{kb}\mu^b + \xi_0^k + \xi_1^{kb}x_b + \xi_2^k x^2 \tag{19}$$

where λ^{kb} are constants and $\xi_0^k, \xi_1^{kb}, \xi_2^k$ are functions of t.

The final step is to substitute (17), (19) into (12) and equate coefficients for all different powers of x_a. As a result we obtain the system of equations:

$$\ddot{d}(A^{-1})^{kb}f^b + \dddot{d}(A^{-1})^{kb}u^b - \ddot{d}(A^{-1})^{ab}u^b\frac{\partial f^k}{\partial u^a} = 4\left(\dot{\omega}_2^k + \xi_2^k - \omega_2^b\frac{\partial f^k}{\partial u_b}\right), \tag{20}$$

$$\dot{g}^a(A^{-1})^{kb}f^b + \ddot{g}^a(A^{-1})^{kb}u^b - \dot{g}^a(A^{-1})^{kb}u^b\frac{\partial f^k}{\partial u^a} = 2\left(\dot{\omega}_1^{ka} + \xi_1^{ka} - \omega_1^{ba}\frac{\partial f^k}{\partial u_a}\right), \tag{21}$$

$$2\dot{d}f^k - C^{kb}f^b - (\dot{C}^{kb} + \frac{m}{2}\ddot{d}\delta^{kb})u^b + (\omega_0^a + C^{ab}u^b)\frac{\partial f^k}{\partial u_a} = \dot{\omega}_0^k - 2mA^{kb}\omega_2^b + \xi_0^k, \tag{22}$$

$$\frac{\partial f^k}{\partial u_b}\mu^b = \lambda^{kb}\mu^b. \tag{23}$$

Thus, the general form of symmetry group generators for equation (1) is given by relations (7), (17),(18) where $d, g^a, C^{ab}, \omega_0^k, \omega_1^{kb}, \omega_2^a, \mu^a$ are functions of t to be specified using equations (20)-(23). These results are valid for equation (1) with arbitrary $n \times n$ constant nonsingular matrix A.

In the following we restrict ourselves to the case of 2×2 matrix A. The related equation (1) is reduced to the form:

$$\frac{\partial}{\partial t}u_1 - \sum_i\frac{\partial^2}{\partial x_i^2}\left(A^{11}u_1 + A^{12}u_2\right) = f^1,$$

$$\frac{\partial}{\partial t}u_2 - \sum_i\frac{\partial^2}{\partial x_i^2}\left(A^{21}u_1 + A^{22}u_2\right) = f^2 \tag{24}$$

where A^{11}, A^{12}, A^{21} and A^{22} are elements of matrix A , f^1 and f^2 are functions of u_1, u_2 to be specified.

First we present all non-equivalent matrices A which have to be considered in the analysis. The *ad hoc* non-equivalent versions of equation (24) correspond to the following matrices A:

$$Ia. A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad Ib. A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}; \quad II. A = \begin{pmatrix} b & -c \\ c & b \end{pmatrix}; \quad III. A = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \tag{25}$$

where a, b, c and e are arbitrary parameters, $a \neq 0, 1$. All 2×2 matrices can be reduced to one of the forms (25) using linear and scale transformations of the dependent variables. Moreover, without loss of generality it is possible to set $c = 1$ and $e = 1$.

The analysis of symmetries of systems of nonlinear diffusion equations present in paper [12] was restricted to the case when matrix A had the form Ib. We consider here the most general case, that is, all possible forms of matrix the A given by relations (25).We present now an outline of the approach used to solve the system (20)-(23)and begin by noticing that:

- Except the functions μ_a depending on t, x and the constant matrix A^{-1} all quantities in the determining equations belong to one of two classes. Either the quantities depend of u but are independent of t, or, alternatively, the quantities depend of t but are independent of u. This enables us to separate variable and so decouple the equations.

- To simplify calculations it is convenient to use the equivalence transformations

$$u^a \to (u')^a = \lambda^{ab} u^b + \sigma^a, \qquad f^a \to (f')^a = \lambda^{ab} f^b \tag{26}$$

where λ^{ab} is an invertable constant matrix and σ^a are constants. Such transformations leave equation (1) form invariant and so make it possible to reduce A^{-1} (and other matrices used in the analysis) to canonical forms (25).

- Many solutions of (20)-(23) are such that f is linear in u. Such possibilities are considered separately.

An important *a priori* simplification of the determining equations can be obtained by considering the compatibility of (20)-(23).

Lemma 1.*Let equations (20), (22)be compatible and f^k are nonlinear in u. Then, up to equivalence transformations (26), the functions d and ω_2^k have to satisfy one of the following relations*

$$\dddot{d} = 0, \quad \omega_2^1 = 0, \quad \dot{\omega}_2^k = \mu \omega_2^k - \xi_2^k + \mu \ddot{d}, \quad or \quad \dddot{d} = \mu \ddot{d}, \quad \omega_2^1 = \omega_2^2 = 0 \tag{27}$$

where $\mu \neq 0, \nu_1$ and ν_2 are constants, at least one of constants ν_α is nonzero and nontrivial ω_2^k are linearly independent of $\ddot{d}$.

The proof of Lemma 1 depends on the analysis of the consequences of differentiating equation (20) first, with respect to u_c and then with respect to u_a and further considering the conditions for consistency of the resulting system. Details of this analysis are straightforward and are omitted here.

An analogous result, with a similar proof is valid for equation (21) which generates the following restrictions:

$$\ddot{g}_a = 0, \quad \omega_1^{1b} = 0, \quad \dot{\omega}_1^{2b} = \nu \omega_1^{2b} - \xi_1^{2b}, \quad or \quad \ddot{g}_a = \nu \dot{g}_a, \quad \omega_1^{1b} = \omega_1^{2b} = 0. \tag{28}$$

These conditions are compatible with (20), (21) only when:

$$\dot{d} = d_3 t + d_4, \qquad g^a = g_3^a t + g_4^a, \qquad \omega_1^{2b} = \nu^b \omega_2^2, \tag{29}$$

$$\dot{d} = d_1 \exp(\nu t) + d_2, \qquad g^a = g_1^a \exp(\nu t) + g_2^a, \qquad \omega_1^{2b} = \nu^b \omega_2^2 \tag{30}$$

where $\nu, \nu^a, g_k^a, d_k, k = 1 \ldots 4$ are constant.

Let μ^α are trivial then substituting these into (20)–(22) we find that

$$(A^{-1})^{kb} f_b = (A^{-1})^{ab} u_b \frac{\partial f^k}{\partial u^a}, \qquad d_3 \neq 0 \quad \text{or/and} \quad g_3^a \neq 0, \tag{31}$$

$$[-2(d_3 t + d_4)\delta^{kb} + C^{kb}] f^b + (\tfrac{m}{2} d_3 \delta_{kb} + \dot{C}^{kb}) u^b = \\ = \dot{\omega}_0^k - 2m A^{k2} \omega_2^2 - (\omega_0^a - C^{ab} u^b) \frac{\partial f^k}{\partial u^a}, \tag{32}$$

$$\omega_2^2 \left(\mu\delta_{k2} - \frac{\partial f^k}{\partial u_2} \right) = 0, \tag{33}$$

or alternatively, the system which includes (33) and the two following equations

$$(A^{-1})^{kb}(f^b + \mu u^b) = (A^{-1})^{ab}u_b \frac{\partial f^k}{\partial u^a}, \qquad \mu \not\models 0, \quad d_1 \not\models 0 \quad \text{or} \quad g_1 \not\models 0, \tag{34}$$

$$[(2d_1 \exp(\mu t) + d_2)\delta^{kb} + C^{kb}]f^b + (\tfrac{1}{2}\mu d_1 \exp(\nu t)\delta^{kb} + \dot{C}^{kb})u^b = \\ \dot{\omega}_0^k - 2A^{k2}\omega_2^2 - (\omega_0^a - C^{ab}u_b)\frac{\partial f^k}{\partial u^a}. \tag{35}$$

Thus, the investigation of symmetries for systems of diffusion equations depends on solving the determining equations (31)-(33) and (34), (35), (33). These include arbitrary parameters d_k, μ, arbitrary functions of t, i.e., ω_0^a, ω_2^a, and an arbitrary matrix C^{ab} which commutes with A and also depend on t. The form of C is given as follows.

Lemma 2. *A necessary condition of compatibility of system* (31)-(33) *is that the matrix function C^{ab} has the form*

$$C^{ab} = \phi_0 F^{ab} + \phi_1 B^{ab} + \nu(\delta^{ab} - F^{ab}) \tag{36}$$

where F^{ab}, B^{ab} are constants and ϕ_0, ϕ_1 are functions of t satisfying the conditions

$$F^{ak}B^{kb} - B^{ak}F^{kb} = \alpha B^{ab} + \beta F^{ab}, \\ \dot{\phi}_0 = k_0\phi_0 + k_1\phi_1 + k_2, \quad \dot{\phi}_1 = n_0\phi_0 + n_1\phi_1 + n_2. \tag{37}$$

Here α, β, ν, k_0, k_1, k_2, n_0, n_1 and n_2 are arbitrary constants.

Proof is straightforward but rather cumbersome, so we present its sketch only.

An arbitrary 2×2 matrix C whose elements are C^{ab} can be expanded as

$$C = C_0 I + C_1\sigma_1 + C_2\sigma_2 + C_3\sigma_3 \tag{38}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{39}$$

C_0, C_1, C_2 and C_3 are functions of t. Let k $(1 \leq k \leq 4)$ of these functions be linearly independent, then equating in (32) the coefficients for these functions we obtain k systems of equations for f^a.

Considering the case $k = 4$ it is possible to convince ourselves that the related overdetermined systems are incompatible. For $k = 3$ the compatibility condition for (32) reduces to the form (36). Then equating coefficients for independent functions ϕ_0 and ϕ_1 in (32) we come to systems of equations for f^a, which are compatible provided relations (36) and (37) are satisfied.

Substituting (27), (28) into (32) we obtain

$$\phi_0 \left[F^{kb}(f^b + k_0 u^b) - F^{ab}u_b\frac{\partial f^k}{\partial u_a} + n_0 B^{kb}u^b \right] + \phi_1 \left[B^{kb}(f^b + n_1 u^b) - \\ -B^{ab}u_b\frac{\partial f^k}{\partial u^a} + k_1 F^{kb}u^b \right] + \nu \left[(\delta^{kb} - F^{kb})F^b - (\delta^{ab} - F^{ab})u_b\frac{\partial f^k}{\partial u_a} \right] + (n_2 F^{kb} \\ + k_2 B^{ab})u_b - 2(d_3 t + d_4)f^k + \tfrac{1}{2}d_3 u^k = -\dot{\omega}_0^k + 2m\delta_{k2}A^{k2}\omega_2^2 + 2\omega_0^a \frac{\partial f^k}{\partial u^a}. \tag{40}$$

This equation has to be imposed together with (31) provided $g_3^a \neq 0$ or $d_3 \neq 0$. Moreover, different combinations of values of these parameters correspond to different systems of determining equations. We specify the following five cases:

$$d_3 = 0, \quad g_3^a = 0, \quad \phi_0 \neq 0, \quad \omega_2^2 = \omega_1^{2b} = 0, \quad \omega_0^a = const, \quad n_2 = k_2 = 0,$$
$$d_3 = 0, \quad g_3^a = 0, \quad \phi_0 \equiv 0, \quad \omega_2^2 = \omega_1^{2b} = 0, \quad \omega_0^a = const, n_2 = k_2 = 0,$$
$$d_3 = 0, \quad g_3^a \neq 0, \quad \omega_2^2 = \omega_1^{2b} = 0, \quad \omega_0^a = const,$$
$$d_3 \neq 0, \quad \omega_2^2 = \omega_1^{2b} = 0, \quad \omega_0^a = const,$$
$$\omega_2^2 \neq 0, \quad \omega_1^{2b} \neq 0.$$

In this way the system of equations (31)-(33) may be solved explicitly using the method of characteristics to determine f^k and their corresponding symmetries. A similar approach can be used for the alternative system (34) and (35) and for the case when μ^α are not trivial. In the last case very strong restriction are imposed on f^k by relation (23) which has only few solutions.

4   Non-linearities and symmetries

We will not give the detailed calculations but present the general solution of relations (20)-(23). In the following tables we present the results of the symmetry analysis for the case where f^k is non-linear in u.

First we present the list of non-linearities of the most general form which are defined up to arbitrary functions. To make this we specify matrices B which commute with A (25) according to the following categories:

$$I. \ B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ if } d \neq 1; A^{-1} \text{ is arbitrary if } d = 1;$$
$$II. \ B = \begin{pmatrix} d & -1 \\ 1 & d \end{pmatrix}, \quad A^{-1} = \frac{1}{b^2+c^2} \begin{pmatrix} b & c \\ -c & b \end{pmatrix},$$
$$IIIa. \ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix},$$
$$IIIb. \ B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}. \tag{41}$$

Table 2. Non-linearities with arbitrary functions

No	Nonlinear terms	Type of matrix B (41)	Arguments of φ_1, φ_2	Conditions for parameters	Symmetries, Z_N for $A^{-1} \neq \kappa B$ & Z_E for $A^{-1} = \kappa B$
1.	$f^1 = u_1^{k+1}\varphi_1,$ $f^2 = u_1^{k+d}\varphi_2$	I	$\frac{u_2}{u_1^d}$	$k \neq 0$	$Z_N = Z_E =$ $= X_0 + \nu D_1$
				$k = 0,$ $d \neq 0$	$Z_N = X_0 + \lambda\hat{B},$ $Z_E = Z_N + \sigma_a G_a$
				$k = d = 0,$ $\varphi_1 \neq$const	$Z_N = X_0 + \alpha u_1\frac{\partial}{\partial u_1}$
				$k = d = 0,$ $\varphi_1 = n$	$Z_N = X_0$ $+\alpha u_1\frac{\partial}{\partial u_1} + \tilde{\psi}_n\frac{\partial}{\partial u_1}$
2.	$f^1 = e^{k\theta}\left(\varphi_1 u_2 + \varphi_2 u_1\right),$ $f^2 = e^{k\theta}\left(\varphi_2 u_{2-}\varphi_1 u_1\right)$	II	$R\,e^{-d\theta}$	$k \neq 0$	$Z_N = Z_E =$ $= X_0 + \nu D_1$
3.	$f^1 = \varphi_1 u_1^{k+1},$ $f^2 = (\varphi_1 \ln u_1 + \varphi_2)\, u_1^{k+1}$	$IIIb$	$u_1 e^{\frac{-u_2}{u_1}}$	$k \neq 0$	$Z_N = Z_E = X_0 + \nu D_1$
				$k = 0$ $\varphi_1 \neq$ const ,	$Z_N = X_0 + \nu u_1\frac{\partial}{\partial u_2}$
				$k = 0,$ $\varphi_1 = n$	$Z_N = X_0 + \nu u_1\frac{\partial}{\partial u_2}$ $+\mu\psi_n\frac{\partial}{\partial u_2}$
4.	$f^1 = e^{k\frac{u_2}{u_1}}\varphi_1 u_1,$ $f^2 = e^{k\frac{u_2}{u_1}}\left(\varphi_1 u_2 + \varphi_2\right)$	$IIIa$	u_1	$k \neq 0$	$Z_N = X_0 + \nu D_1$
5.	$f^1 = u_1(n \ln u_1 + \varphi_1),$ $f^2 = u_2(n \ln u_2 + \varphi_2)$	I	$\frac{u_2}{u_1^d}$	$n \neq 0$	$Z_N = X_0 + \mu e^{nt}\hat{B},$ $Z_E = Z_N + \nu_a\hat{G}_a$
6.	$f^1 = \varphi_1 u_2 + \varphi_2 u_1$ $+\frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_1 - u_2),$ $f^2 = \varphi_2 u_{2-}\varphi_1 u_1$ $+\frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_2 + u_1)$	$II,$ $d \neq 0$	$R\,e^{-d\theta}$	$n \neq 0$	$Z_N = X_0 + \mu e^{nt}\hat{B},$ $Z_E = Z_N + \nu_a\hat{G}_a$
				$n = 0$	$Z_N = X_0 + \mu\hat{B},$ $Z_E = Z_N + \lambda_a G_a$
7.	$f^1 = (\varphi_1 - n\theta)u_2 + \varphi_2 u_1,$ $f^2 = \varphi_2 u_2 - \varphi_1 u_1$	$II,$ $d = 0$	R	$n \neq 0$	$Z_N = X_0 + \mu e^{nt}\hat{B},$ $Z_E = Z_N + \nu_a\hat{G}_a$
				$n = 0$	$Z_N = X_0 + \mu\hat{B},$ $Z_E = Z_N + \lambda_a G_a$
8.	$f^1 = \varphi_1 u_1 + nu_2,$ $f^2 = \varphi_1 u_2 + u_1\varphi_2$ $+nu_2\left(1 + \frac{u_2}{u_1}\right)$	$IIIb$	$\frac{u_2}{u_1} - \ln u_1$	$n \neq 0$	$Z_N = X_0 + \mu e^{nt}\hat{B},$ $Z_E = Z_N + \nu_a\hat{G}_a$
				$n = 0$	$Z_N = X_0 + \mu\hat{B},$ $Z_E = Z_N + \lambda_a G_a$
9.	$f^1 = \varphi u_1 - su_1,$ $f^2 = \varphi u_2 - nu_1$	$IIIa$	u_1	$s = 0, n \neq 0$	$Z_N = X_0 + \mu u_1\frac{\partial}{\partial u_2}$ $+\nu Y_1$
				$s \neq 0, n = 0$	$Z_N = X_0 + \nu u_2\frac{\partial}{\partial u_2}$ $+\mu e^{st}u_1\frac{\partial}{\partial u_2}$
				$n = s = 0$	$Z_N = X_0 + \nu u_2\frac{\partial}{\partial u_2}$ $+\mu u_1\frac{\partial}{\partial u_2}$

Table 2. Continued

10.	$f^1 = \varphi_1 u_1,$ $f^2 = \varphi_1 u_2 + \varphi_2 u_1 + nu_2$	$IIIa$	u_1	$\varphi_1 \neq 0$ $\varphi_1 = 0$	$Z_N = X_0 + \mu e^{nt}\hat{B},$ $Z_N = X_0 + \mu\hat{B} + \psi_n\frac{\partial}{\partial u_2}$
11.	$f^1 = \varphi_1 u_1^{1-k},$ $f^2 = \varphi_2 u_1^{-k}$	$I,$ $d = 0$	$\ln u_1 + nu_2$	$k \neq 0$	$Z_N = X_0 + \nu D_2$
12.	$f^1 = \varphi_1 u_1 + \frac{s}{n}u_1 u_2,$ $f^2 = \varphi_2 + su_2$	$I,$ $d = 0$	$u_2 - n\ln u_1$	$n \neq 0, s = 0$ $s \neq 0, n \neq 0$	$Z_N = X_0 + \lambda Y_2$ $Z_N = X_0 + \lambda Y_3$
13.	$f^1 = e^{ku_1}\varphi_1,$ $f^2 = e^{ku_1}(\varphi_2 - n\varphi_1 u_1)$	$IIIa$	$nu_1^2 + 2u_2$	$k \neq 0$ $k = 0, n \neq 0$	$Z_N = X_0 + \lambda D_3$ $Z_N = X_0 + \lambda Y_4$
14.	$f^1 = nu_1 + \varphi_1,$ $f_2 = ku_2 + \varphi_1 u_1 + \varphi_2$	$IIIb$	$nu_1^2 + 2u_2$	$k \neq 0$	$Z_N = X_0+$ $+\nu e^{nt}\left(u_1\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_1}\right)$
15.	$f^1 = n,$ $f^2 = ku_2 + \varphi$	$IIIa$	u_1	$\varphi \neq$ const	$Z_N = X_0 + \lambda Y_5$ $+\psi_k\frac{\partial}{\partial u_2}$
16	$f^1 = (n-r)u_1 + \varphi_1,$ $f_2 = -ru_2 + \varphi_2$	$IIIb$	u_2		$Z_N = X_0+$ $+\nu e^{nt}\psi(x)\frac{\partial}{\partial u_1}$
17	$f^1 = u_1\varphi_1 + nu_1\ln u_1,$ $f^2 = \varphi_2$	$I,$ $d = 0$	u_2	$n \neq 0$	$Z_N = X_0 + \lambda Y_6$
18.	$f^\alpha = e^{ku_1}\varphi^\alpha, \alpha = 1,2$	any	$\omega_1 u_2 - \omega_2 u_1$	$k \neq 0$	$Z_N = Z_E = X_0 + \nu D_4$
19.	$f^1 = \varphi_1 + su_1,$ $f^2 = \varphi_2 + su_2$	any	$u_2 - ku_1$	$k \neq 0,$ $A \neq \varkappa I$	$Z_N = Z_E = X_0$ $+e^{st}\psi(x)\left(\frac{\partial}{\partial u_1} + k\frac{\partial}{\partial u_2}\right)$
		$A = \varkappa I$	u_1	$s = 0, A = \varkappa I$	$Z_N = X_0 + \tilde{\psi}_0\frac{\partial}{\partial u_2}$
20	$f^1 = \varphi_1, f^2 = \varphi_2$	any	u_1, u_2		$Z_N = Z_E = X_0$

Here $R = \sqrt{u_1^2 + u_2^2}, \theta = \arctan(\frac{u_2}{u_1})$, the Greek letters in the right column denote arbitrary coefficients whilst D_μ, G_a^i and $\bar{G}_a^i, X_A, Y_a, \hat{B}$ are various types of dilatation, Galilei and special transformation generators as follows:

$$D_0 = 2t\frac{\partial}{\partial t} + x_a\frac{\partial}{\partial x_a}, \quad D_1 = D_0 - \frac{2}{k}\hat{B}, \quad D_2 = D_0 + \frac{2}{k}\left(\frac{\partial}{\partial u_2} + nu_1\frac{\partial}{\partial u_1}\right),$$
$$D_3 = D_0 - \frac{2}{k}\left(\frac{\partial}{\partial u_1} - nu_1\frac{\partial}{\partial u_2}\right), \quad D_4 = D_0 - \frac{2}{k}\omega_a\frac{\partial}{\partial u_a},$$
$$G_a = t\frac{\partial}{\partial x_a} - \frac{1}{2}x_a(A^{-1})^{nb}u_b\frac{\partial}{\partial u_n}, \quad \hat{G}_a = e^{nt}\left(\frac{\partial}{\partial x_a} - \frac{1}{2}nx_a(A^{-1})^{nb}u_b\frac{\partial}{\partial u_n}\right),$$
$$X_0 = \alpha\frac{\partial}{\partial t} + \beta_a\frac{\partial}{\partial x_a} + \nu^{[a,b]}x_a\frac{\partial}{\partial x_b}, \quad \nu^{[a,b]} = -\nu^{[b,a]}, \tag{42}$$
$$Y_1 = ntu_1\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial u_2}, \quad Y_2 = u_1\frac{\partial}{\partial u_1} + n\frac{\partial}{\partial u_2}, \quad Y_3 = e^{st}\left(u_1\frac{\partial}{\partial u_1} + n\frac{\partial}{\partial u_2}\right),$$
$$Y_4 = nu_1\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}, Y_5 = e^{kt}\left(u_1\frac{\partial}{\partial u_2} + \frac{nx^2}{2m}\frac{\partial}{\partial u_2}\right), Y_6 = e^{nt}u_1\frac{\partial}{\partial u_1},$$
$$\hat{B} = B^{ab}u_b\frac{\partial}{\partial u_a}$$

where B^{ab} are elements of the corresponding matrices (41). In addition, $\psi(x)$ and $\widetilde{\psi}_0$ are arbitrary solutions of equations $\Delta\psi(x) = 0$ and $(\frac{\partial}{\partial t} - \Delta)\widetilde{\psi}_0 = 0$ respectively were Δ is the Laplace operator.

In the following Tables 3 and 4 we use triplets of matrices (F, B, A^{-1}) with F and B forming two-dimensional Lie algebra and commuting with A (25). We classify such

triplets according to the categories:

$$
\begin{aligned}
I. \quad & F = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}; \\
IIa. \quad & F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^{-1} = \tfrac{1}{b^2+c^2}\begin{pmatrix} b & c \\ -c & b \end{pmatrix}; \\
IIb. \quad & F = \begin{pmatrix} d & -1 \\ 1 & d \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{-1} = \tfrac{1}{b^2+c^2}\begin{pmatrix} b & c \\ -c & b \end{pmatrix}; \quad . \\
IIIa. \quad & F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \\
IIIb. \quad & F = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}
\end{aligned}
\tag{43}
$$

where a, b, c and d are real parameters $(a \neq 0, b^2 + c^2 \neq 0)$.

Table 3. Non-linearities which generate dilatation symmetry

No	Non-linear terms	Conditions for parameters	Symmetries, Z_N for, $A^{-1} \neq \kappa F$ & Z_E for $A^{-1} = \kappa F$	Matrices (43) and generator parameters
1	$f^1 = (gu_1^q u_2^r + n)\, u_1,$	$q \neq 0, -1,$ $p \neq 0, n = 0$ $q + r = \frac{4}{m}$	$Z_N = X_0 + \mu\hat{F} + \nu D_5,$ $Z_E = Z_N + \sigma_a G_a + \lambda\hat{A}$	$I, k = \frac{4}{m},$ $d = -\frac{q}{r}$
	$f^2 = \left(pu_1^q u_2^r - \frac{qn}{r}\right) u_2$	$q = -1,$ $p = n = 0,$ $r = \frac{4+m}{m}$	$Z_N = X_0 + \mu\hat{F} + \nu D_5$ $+\widetilde{\psi}_0\frac{\partial}{\partial u_1},$ $Z_E = Z_N + \sigma_a G_a + \lambda\hat{A}$	$I, k = \frac{4}{m},$ $d = \frac{1}{r}$
		$q \neq 0, -1, r \neq 0$ $q + r \neq 0, \frac{4}{m}$	$Z_N = X_0 + \nu\hat{F} + \mu D_5,$ $Z_E = Z_N + \sigma_a G_a$	$I, k = r + q,$ $d = -\frac{q}{r}$
		$p = n = 0, g \neq 0$ $q = -1, r \neq 0, -1$	$Z_N = X_0 + \nu\hat{F}$ $+\mu D_6 + \tilde{\psi}_0\frac{\partial}{\partial u_1},$ $Z_E = Z_N + \sigma_a G_a$	$I, d = \frac{1}{r},$ $k = r$
		$q = g = 0,$ $r \neq 0, -1; p \neq 0$	$Z_N = X_0 + \nu\hat{F}$ $+\mu D_6 + \tilde{\psi}_n\frac{\partial}{\partial u_1}$	$I, d = 0,$ $k = r$
2	$f^1 = e^{q\theta} R^r (gu_1$ $-pu_2) + su_2 - lu_1,$ $f^2 = e^{q\theta} R^r (gu_2$ $+pu_1) - su_1 - lu_2,$	$s = l = 0,$ $r = \frac{4}{m}$	$Z_N = X_0 + \nu\hat{F} + \mu D_5,$ $Z_E = Z_N + \sigma_a G_a + \lambda\hat{A}$	$IIb, k = \frac{4}{m},$ $d = -\frac{q}{r}, n = 0$
		$r \neq \frac{4}{m}, r \neq 0,$ $s = l = 0$	$Z_N = X_0 + \nu\hat{F} + \mu D_5,$ $Z_E = Z_N + \sigma_a G_a$	$IIb, k = r,$ $d = -\frac{q}{r}, n = 0$
	$R^2 = u_1^2 + u_2^2$	$l = -\frac{sq}{r},$ $s \neq 0, r \neq 0$	$Z_N = X_0 + \nu\hat{F} + \mu D_6,$ $Z_E = Z_N + \sigma_a G_a$	$IIb, k = r,$ $n = -s,$ $d = -\frac{q}{r}$
	$\theta = \arctan\left(\frac{u_2}{u_1}\right)$	$s = 0, l \neq 0,$ $q \neq 0, r = 0$	$Z_N = X_0 + \nu\hat{F} + \mu D_6,$ $Z_E = Z_N + \sigma_a G_a$	$IIa, k = q,$ $n = -l$

Table 3. Continued 1

3	$f^1 = pu_1^{r+1}e^{q\frac{u_2}{u_1}}$ $-su^1,$	$r = -q = \frac{4}{m},$ $s = 0$	$Z_N = X_0 + \nu\hat{F} + \mu D_5,$ $Z_E = Z_N + \alpha_a G_a + \lambda\hat{A}$	$IIIb, d = 1,$ $k = \frac{4}{m}$
	$f^2 = e^{q\frac{u_2}{u_1}}(pu_2$ $+gu_1)u_1^r$	$-q = r \neq \frac{4}{m},$ $s = 0, r \neq 0$	$Z_N = X_0 + \nu\hat{F} + \mu D_5,$ $Z_E = Z_N + \alpha_a G_a$	$IIIb, d = 1,$ $k = r$
	$-s\left(u_2 - \frac{r}{q}u_1\right)$	$q \neq 0, s \neq 0$	$Z_N = X_0 + \nu\hat{F} + \mu D_6,$ $Z_E = Z_N + \alpha_a G_a$	$IIIb, k = q,$ $n = -sq,$ $d = -\frac{r}{q}$
4	$f^1 = pu_1^{k+1},$	$d + k \neq 1,$ $k \neq 0, q \neq 0,$ $s = n \neq 0, p \neq 0$	$Z_N = X_0 + \nu\hat{F}$ $+\mu D_6$	$IIIa$
	$f^2 = u_1^k\left(pu_2 + qu_1^d\right)$ $+\frac{s}{d+k-1}u_1$	$k \neq 0, s = 0,$ $q = 0, p \neq 0$	$Z_N = X_0 + \nu\hat{F}$ $+\mu D_5 + \lambda u_2\frac{\partial}{\partial u_2}$	$IIIa, d = 0$
		$pk \neq 0, q = 0,$ $d = 0,$ $s = n(k-1) \neq 0$	$Z_N = X_0 + \nu\hat{F}$ $+\mu D_5 + \lambda Y_1$	$IIIa, d = 0$
		$d + k \neq 0, 1,$ $k \neq 0, q \neq 0,$ $s = n \neq 0, p = 0$	$Z_N = X_0 + \nu\hat{F}$ $+\mu D_6 + \psi_0\frac{\partial}{\partial u_a}$	$IIIa$
5	$f^1 = pu_1^{k+1},$ $f^2 = pu_1^k u_2$ $-knu_1\ln u_1$	$k \neq 0, n \neq 0,$ $p \neq 0$	$Z_N = X_0 + \nu\hat{F} + \mu D_6$	$IIIa,$ $d = 1 - k$
		$k \neq 0, n \neq 0,$ $p = 0$	$Z_N = X_0 + \nu\hat{F}$ $+\mu D_6 + \psi_0\frac{\partial}{\partial u_2}$	$IIIa,$ $d = 1 - k$
6	$f^1 = qu_1^{r+1}e^{ku_2} + su_1,$ $f^2 = pu_1^r e^{ku_2} - \frac{sr}{k}$	$r \neq 0, -1; k \neq 0,$ $p \neq 0, q \neq 0$	$Z_n = X_0 + \nu D_7$ $+\mu Y_2$	$I, d = 0,$ $n = -\frac{r}{k}$
		$p = 0, r = -1,$ $q \neq 0, s = 0,$ $k \neq 0$	$Z_N = X_0 + \nu D_7$ $+\mu Y_2 + \tilde{\psi}_0\frac{\partial}{\partial u_1}$	$I, d = 0,$ $r = -1,$ $n = \frac{1}{k}$
		$k \neq 0,$ $s = r = 0$	$Z_N = X_0 + \nu u_1\frac{\partial}{\partial u_1}$ $+\mu\left(D_0 - \frac{2}{k}\frac{\partial}{\partial u_1}\right)$	I
7	$f^1 = pe^{(n-k)u_2} + hu_2,$ $f^2 = ge^{-ku_2} + q$	$q = h = 0,$ $k \neq 0, n \neq 0$	$Z_N = X_0 + \nu D_2$ $+\tilde{\psi}_0\frac{\partial}{\partial u_1}$	$I, d = 0$
		$q = 0,$ $k \neq 0, h \neq 0$	$Z_N = X_0 + \nu D_8 + \psi_0\frac{\partial}{\partial u_1}$	$I, d = 0$
		$p = q = 0,$ $h = 0, k \neq 0$	$Z_N = X_0 + \nu D_2$ $+\sigma u_1\frac{\partial}{\partial u_1} + \tilde{\psi}_0\frac{\partial}{\partial u_1}$	$I, d = 0,$ $n = 0$
		$n = q = g = 0,$ $p \neq 0$	$Z_N = X_0 + \nu D_9$ $+\mu u_2\frac{\partial}{\partial u_1} + \tilde{\psi}_0\frac{\partial}{\partial u_1}$	$IIIa$

Table 3. Continued 2

8	$f^1 = ge^{u_2+ku_1},$ $f^1 = qe^{u_2+ku_1} + p$	$k \neq 0$	$Z_N = Z_E$ $= X_0 + \nu D_{10}$ $+\psi(x)\left(\frac{\partial}{\partial u_1} - k\frac{\partial}{\partial u_2}\right)$	$I, a \neq 1$
			$Z_N = Z_E$ $= X_0 + \nu D_{11}$ $+\psi(x)\left(\frac{\partial}{\partial u_1} - k\frac{\partial}{\partial u_2}\right)$	$IIa, IIb,$ $c \neq 0$
		$k = 0$	$Z_N = Z_E = X_0$ $+\nu D_{12} + \psi(x)\frac{\partial}{\partial u_1}$	$IIIb,$ $e \neq 0$
9	$f^1 = p\left(u_2 + nu_1^2\right)^{s+\frac{1}{2}}$ $+\frac{1}{2n(2s+1)},$ $f^2 = -\frac{1}{2s+1}u_1$ $+ (q - 2npu_1)$ $\times(u_2 + nu_1^2)^{s+\frac{1}{2}}$	$s \neq 0, -\frac{1}{2},$ $p \neq 0, n \neq 0$	$Z_N = X_0 + \nu D_{13}$	$IIIa$
10	$f_1 = gu_1^{k+1},$ $f_2 = (p \ln u_1 + q)\, u_1^{k+1}$	$k = -1, q = 0,$ $p \neq 0, g \neq 0$	$Z_N = Z_E = X_0$ $+\psi_0\frac{\partial}{\partial u_2}$ $+\nu\left(D_5 + 2pt\frac{\partial}{\partial u_2}\right)$	$I, k = -1$ $d = 0$
		$k = -1, p \neq 0,$ $q = g = 0$	$Z_N = Z_E = X_0$ $+\nu\left(D_5 + 2pt\frac{\partial}{\partial u_2}\right)$ $+\lambda\left(B\widehat{F} + pt\frac{\partial}{\partial u_2}\right)$ $+\psi_0\frac{\partial}{\partial u_2}$	$I, k = -1$ $d = 0$
		$k \neq 0, -1,$ $g = p$	$Z_N = Z_E = X_0$ $+\nu D_1 + \psi_0\frac{\partial}{\partial u_2}$	$IIIb, d = 1$
11	$f^1 = g\left(u_2 + su_1\right)^{k+1},$ $f^2 = q\left(u_2 + su_1\right)^{k+1}$ $+p$	$k \neq 0, -1;$ $s \neq 0$	$Z_N = Z_E$ $= X_0 + \nu D_{14}$ $+\psi(x)\left(\frac{\partial}{\partial u_1} - s\frac{\partial}{\partial u_2}\right)$	$I, a \neq 1$
		$k \neq 0, -1;$ $s = 0$	$Z_N = Z_E = X_0$ $+\nu D_{15} + \psi(x)\frac{\partial}{\partial u_1}$	$IIa, IIb,$ $c \neq 0$ $IIIb, b = 1$ $c \neq 0$
12	$f^1 = p \ln(u_2 + su_1),$ $f^2 = q \ln(u_2 + su_1)$	$s \neq 0$	$Z_N = Z_E$ $= X_0 + \nu D_{16}$ $+\psi(x)\left(\frac{\partial}{\partial u_1} - s\frac{\partial}{\partial u_2}\right)$	$I, a \neq 1,$ $k = -1$
		$s = 0$	$Z_N = Z_E = X_0$ $+\nu D_{17} + \psi(x)\frac{\partial}{\partial u_1}$	$IIa, IIb,$ $k = -1,$ $c \neq 0;$ $IIIb, k = -1,$ $b = 1,$ $c \neq 0$
13	$f^1 = gu_1^{k+1},$ $f^2 = qu_1^{k+d}$	$g \neq 0, k \neq 0$	$Z_N = Z_E = X_0$ $+\nu(D_0 - \frac{2}{k}\widehat{F}) + \psi_0\frac{\partial}{\partial u_2}$	I

Table 4. Further non-linearities with arbitrary parameters

No	Non-linear terms	Matrix Class (43)	Conditions for parameters and matrices $\mathcal{F}_a$	Symmetries, Z_N for $A^{-1} \neq \kappa\mathcal{F}_a$ $\&Z_E$ for $A^{-1} = \kappa\mathcal{F}_1$ $\&\tilde{Z}_E$ for$A^{-1} = \kappa\mathcal{F}_2$
1	$f^1 =$ $(k_0 \ln u_1$ $+k_1 \ln u_2$ $+q)u_1,$ $f^2 =$ $(n_0 \ln u_1$ $+n_1 \ln u_2$ $+p)u_2$	$I,$ $d = 0$	$\delta > 0, \Delta \neq 0, k_1 \neq 0,$ $\mathcal{F}_1 = k_1 F + (n_+ - k_0)B,$ $\mathcal{F}_2 = k_1 F + (n_- - k_0)B,$ $n_\pm = \frac{k_0 + n_1}{2} \pm \delta$	$Z_N = X_0 + \lambda e^{n_+ t}\hat{\mathcal{F}}_1 + \nu e^{n_- t}\hat{\mathcal{F}}_2$ $Z_E = Z_N + \sigma_a \hat{G}_a \; (n = n_+),$ $\tilde{Z}_E = Z_N + \sigma_a \hat{G}_a \; (n = n_-)$
			$k_1 = n_0 = 0, k_0 n_1 \neq 0$	$Z_N = X_0 + \nu e^{k_0 t} u_1 \frac{\partial}{\partial u_1}$ $+\mu e^{n_1 t} u_2 \frac{\partial}{\partial u_2}$
			$k_1 = n_0 = n_1 = 0$	$Z_N = X_0 + \psi_p \frac{\partial}{\partial u_2} + \nu e^{k_0 t} u_1 \frac{\partial}{\partial u_1}$
			$\Delta = 0, k_1 \neq 0,$ $k_0 + n_1 = n \neq 0$ $\mathcal{F}_1 = k_1 F + n_1 B,$ $\mathcal{F}_2 = k_0 B - k_1 F;$	$Z_N = X_0$ $+\lambda e^{(n_1 + k_0)t}\hat{\mathcal{F}}_1 + \nu\hat{\mathcal{F}}_2,$ $Z_E = Z_N + \sigma_a \hat{G}_a \; (n_1 \neq 0)$ $\hat{Z}_E = Z_N + \sigma_a G_a \; (k_0 \neq 0)$
			$k_0 = n_1 = n,$ $k_1 = n_0 = 0,$ $\mathcal{F}_1 = \mu F + \nu B, \mu\nu \neq 0$	$Z_N = X_0 + \alpha e^{nt} u_1 \frac{\partial}{\partial u_1}$ $+\lambda e^{nt} u_2 \frac{\partial}{\partial u_2},$ $\tilde{Z}_E = Z_N + \sigma_a \hat{G}_a$
			$k_0 = n_1 = n,$ $n_0 = 0, k_1 \neq 0$	$Z_N = X_0 + \lambda e^{nt} u_1 \frac{\partial}{\partial u_1}$ $+\nu e^{nt}(k_1 t u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2})$
			$\delta = 0, \Delta \neq 0, k_0 + n_1 = 2n$ $\mathcal{F}_1 = 2k_1 F + (n_1 - k_0)B,$ $\mathcal{F}_2 = t\mathcal{F}_1 + 2B$	$Z_N = X_0 + \nu X_1 + \mu X_2,$ $Z_E = Z_N + \sigma_a \hat{G}_a$
			$\delta = \Delta = 0,$ $n_0 k_1 \neq 0,$ $\mathcal{F}_1 = k_1 F - k_0 B$	$Z_N = X_0 + \nu[k_1 t u_1 \frac{\partial}{\partial u_1}$ $+(1 - k_0 t)u_2 \frac{\partial}{\partial u_2}]$ $+\mu(k_1 u_1 \frac{\partial}{\partial u_1} - k_0 u_2 \frac{\partial}{\partial u_2}),$ $Z_E = Z_N + \sigma_a G_a$
			$k_1 = n_1 = 0, q = p,$ $k_0 = n_0 = n, a = 1$ $\mathcal{F}_1 = F + B$	$Z_N = X_0 + \nu u_2 \frac{\partial}{\partial u_2} + \mu u_1 \frac{\partial}{\partial u_2}$ $+\lambda e^{nt}\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\right),$ $Z_E = Z_N + \sigma_a \tilde{G}_a$
			$k_1 = n_1 = 0, p = 0,$ $k_0 = n_0 = n, q \neq 0,$ $a = 1, \mathcal{F}_1 = F + B$	$Z_N = X_0 + \nu u_2 \frac{\partial}{\partial u_2} + \mu e^{pt} u_1 \frac{\partial}{\partial u_2}$ $+\lambda e^{nt}\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\right),$ $Z_E = Z_N + \sigma_a \tilde{G}_a$
			$k_0 = k_1 = n_1 = 0,$ $n_0 \neq 0$	$Z_N = X_0 + \nu u_2 \frac{\partial}{\partial u_2}$ $+\mu(u_1 \frac{\partial}{\partial u_1} + n_0 t u_2 \frac{\partial}{\partial u_2})$
			$\delta = -\omega^2, 2n = k_0 + n_1$	$Z_N = X_0 + \nu X_3 + \mu X_4$

Table 4. Continued 1

2	$f^1 = qu_1,$ $f^2 = nu_2 + pu_1^d$ $+ su_1 + k$	$IIIa$	$d \neq 0, 1, 2; k = 0,$ $s = 0, n \neq q, \mathcal{F}_1 = B$	$Z_N = X_0 + \nu\hat{B} + \psi_n\frac{\partial}{\partial u_2},$ $Z_E = Z_N + \sigma_a G_a$
			$d \neq 0, 2; n = 0,$ $s = 0,$ $\mathcal{F}_1 = B$	$Z_N = X_0 + \psi_0\frac{\partial}{\partial u_2}$ $+\nu\left(\hat{B} - dkt\frac{\partial}{\partial u_2}\right),$ $Z_E = Z_N + \sigma_a G_a$
			$d \neq 0, 1, 2; k = 0,$ $s = \frac{1}{1-d}, \mathcal{F}_1 = B$	$Z_N = X_0 + \nu\hat{F}$ $+\mu\hat{B} + \psi_q\frac{\partial}{\partial u_2},$ $Z_E = Z_N + \sigma_a G_a$
			$d = 2, k = 0, s \neq 0$	$Z_N = X_0 + \mu Y_7 + \psi_n\frac{\partial}{\partial u_2}$
			$n = 2(q+p)$ $d = 2, k = s = 0,$ $\mathcal{F}_1 = B$	$Z_N = X_0 + \nu Y_8$ $+\mu\hat{B} + \psi_0\frac{\partial}{\partial u_2},$ $Z_E = Z_N + \sigma_a G_a$
			$d = 2, p = -q,$ $n = s = 0$	$Z_N = X_0 + \nu Y_8 + \psi_0\frac{\partial}{\partial u_2}$ $+\mu\left(\hat{B} - 2kt\frac{\partial}{\partial u_2}\right)$
		$IIIb$	$k = s = 0,$ $d = 2, \mathcal{F}_1 = B$	$Z_N = X_0 + \nu\hat{B}$ $+\mu e^{qt}u_1\frac{\partial}{\partial u_2} + \psi_0\frac{\partial}{\partial u_2},$ $Z_E = Z_N + \sigma_a G_a$
3	$f^1 = ku_1\ln u_1 + pu_1,$ $f^2 = bu_2 + n\ln u_1$	I	$b = k = q = 0,$ $p \neq 0$	$Z_N = X_0 + \nu Y_9$ $+\psi_0\frac{\partial}{\partial u_2}$
			$q = p = 0, b = k$	$Z_N = X_0 + \nu Y_{10} + \psi_b\frac{\partial}{\partial u_2}$
			$q = p = 0,$ $b \neq k, b \neq 0$	$Z_N = X_0 + \nu Y_{11}$ $+\psi_b\frac{\partial}{\partial u_2}$
			$k = 0, nb \neq 0$	$Z_N = X_0 + \nu e^{bt}\frac{\partial}{\partial u_2}$ $+\mu(bu_1\frac{\partial}{\partial u_1} - n\frac{\partial}{\partial u_2})$
			$b = p = 0,$ $k \neq o, n \neq 0$	$Z_N = X_0 + \psi_0\frac{\partial}{\partial u_2}$ $+\mu e^{kt}(ku_1\frac{\partial}{\partial u_1} + n\frac{\partial}{\partial u_2})$
4	$f^1 = k_0 u_1\ln u_1 + k_1 u_1 u_2,$ $f^2 = n_0\ln u_1 + n_1 u_2$	$I,$ $d = 0$	$k_1 \neq 0, \delta = 0$	$Z_N = X_0 + \nu Y_{12}$ $+\mu\left(tY_{12} + e^{\frac{k_0+n_1}{2}t}\frac{\partial}{\partial u_2}\right)$
			$k_1 \neq 0, \delta > 0$	$Z_N = X_0 + \nu Y_{13}^+ + \mu Y_{13}^-$
			$k_1 \neq 0, \delta = -\omega^2 < 0$	$Z_N = X_0 + \nu Y_{14} + \mu Y_{15}$
5	$f^1 = pu_2^k + qu_1,$ $f^2 = gu_2$	$I,$ $d = \frac{1}{k}$	$p \neq 0, kg \neq q,$ $k \neq 0, 1; \mathcal{F}_1 = F$	$Z_N = X_0 + \nu\hat{F} + \tilde{\psi}_q\frac{\partial}{\partial u_1},$ $Z_E = Z_N + \sigma_a G_a$
6	$f^1 = nu_1\ln u_1,$ $f^2 = nu_2\ln u_1 + pu_1$	$I,$ $d = 0$ $a = 1$	$n \neq 0, p \neq 0,$ $\mathcal{F}_1 = B + F$	$Z_N = X_0 + \mu u_1\frac{\partial}{\partial u_2}$ $+\nu\left(\hat{B} - ptu_1\frac{\partial}{\partial u_2}\right)$ $+\lambda e^{nt}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right),$ $Z_E = Z_N + \sigma_a\tilde{G}_a$
7	$f^1 = pu_1^{k+1},$ $f^2 = pu_1^k u_2 + su_2$	$IIIa,$ $d = 0$	$k \neq 0, s \neq 0$	$Z_N = X_0 + \nu u_2\frac{\partial}{\partial u_2}$ $+\mu e^{st}u_1\frac{\partial}{\partial u_2}$

Table 4. Continued 2

8	$\begin{array}{l} f^1 = \\ (k_0 u_1 - n_0 u_2) \ln R \\ +\theta (k_1 u_1 - n_1 u_2) \\ +p u_1 - q u_2, \end{array}$	IIa	$\begin{array}{l} \delta > 0, \Delta \neq 0, k_1 \neq 0, \\ \mathcal{F}_1 = k_1 F + (n_+ - k_0) B, \\ \mathcal{F}_2 = k_1 F + (n_- - k_0) B, \\ n_\pm = \frac{k_0 + n_1}{2} \pm \delta \end{array}$	$\begin{array}{l} Z_N = X_0 \\ +\lambda e^{n_+ t} \hat{\mathcal{F}}_1 + \nu e^{n_- t} \hat{\mathcal{F}}_2, \\ Z_E = Z_N + \sigma_a \hat{G}_a, \\ \tilde{Z}_E = Z_E \ (k_1 \neq 0^*) \end{array}$
	$\begin{array}{l} f^2 = \\ (k_0 u_2 + n_0 u_1) \ln R \\ +\theta (n_1 u_1 + k_1 u_2) \\ +q u_1 + p u_2 \end{array}$		$\begin{array}{l} \Delta = 0, k_1 \neq 0 \\ k_0 + n_1 = n \neq 0, : \\ \mathcal{F}_1 = k_1 F + n_1 B, \\ \mathcal{F}_2 = k_0 B - k_1 F; \\ k_1 = 0, n_0 \neq 0 : \\ \mathcal{F}_1 = n_0 B + k_0 F, \\ (k_0 \neq 0^*) \\ \mathcal{F}_2 = n_1 F - n_0 B; \\ (n_1 \neq 0^*) \end{array}$	$\begin{array}{l} Z_N = X_0 + \mu e^{nt} \hat{\mathcal{F}}_1 \\ +\nu \hat{\mathcal{F}}_2, \\ Z_E = Z_N + \sigma_a \hat{G}_a, \\ \tilde{Z}_E = Z_N + \sigma_a G_a, \end{array}$
9	$\begin{array}{l} f^1 = (k_0 \ln u_1 + \\ q) u_1 + k_1 u_2, \end{array}$	$\begin{array}{l} IIIb, \\ d = 0 \end{array}$	$\begin{array}{l} k_1 = n_0 = 0, \\ k_0 = n_1 = n, \\ \mathcal{F}_1 = \nu F + \mu B \\ (\nu \neq 0^*) \end{array}$	$\begin{array}{l} Z_N = X_0 + \alpha e^{nt} \hat{\mathcal{F}}_1, \\ Z_E = Z_N + \sigma_a \hat{G}_a \end{array}$
	$\begin{array}{l} f^2 = \\ (n_0 u_1 + k_0 u_2) \ln u_1 \\ +k_1 \frac{u_2^2}{u_1} + p u_1 \\ + (n_1 + q) u_2 \end{array}$		$\begin{array}{l} \delta = n_0 = 0, \Delta \neq 0, \\ k_1 \neq 0, \mathcal{F}_1 = F, \\ n = \frac{1}{2}(k_0 + n_1) \end{array}$	$\begin{array}{l} Z_N = X_0 + \mu e^{nt}(k_1 t \hat{F} \\ +\hat{B}) + \nu e^{nt} \hat{F}, \\ Z_E = Z_N + \sigma_a \hat{G}_a \end{array}$
			$\begin{array}{l} \delta = k_1 = 0, \\ k_0 = n_1, \\ n_0 \neq 0, \mathcal{F}_1 = B \end{array}$	$\begin{array}{l} Z_N = X_0 + \mu e^{n_1 t}(\hat{F} \\ +n_0 t \hat{B}) + \nu e^{n_1 t} \hat{B}, \\ Z_E^{**} = Z_N + \sigma_a \hat{G}_a \end{array}$
			$\begin{array}{l} \delta = 0, \Delta \neq 0, \\ k_0 + n_1 = 2n \\ \mathcal{F}_1 = 2 k_1 F \\ +(n_1 - k_0) B, \\ \mathcal{F}_2 = t \mathcal{F}_1 + 2B \end{array}$	$\begin{array}{l} Z_N = X_0 + \nu X_1 + \mu X_2, \\ Z_E = Z_N + \sigma_a \hat{G}_a \end{array}$
			$\begin{array}{l} k_0 = k_1 = n_1 = 0, \\ n_0 \neq 0, \mathcal{F}_1 = B \end{array}$	$\begin{array}{l} Z_N = X_0 + \nu \hat{B} \\ +\mu \left(\hat{F} + n_0 t \hat{B} \right), \\ Z_E^{**} = Z_N + \sigma_a G_a \end{array}$
			$\begin{array}{l} k_0 = n_0 = n_1 = 0, \\ k_1 \neq 0, \mathcal{F}_1 = F \end{array}$	$\begin{array}{l} Z_N = X_0 + \nu \hat{F} \\ +\mu \left(\hat{B} + k_1 t \hat{F} \right) \\ Z_E = Z_N + \sigma_a G_a \end{array}$
			$\begin{array}{l} \delta = \Delta = 0, \\ n_0 k_1 < 0, \\ \mathcal{F}_1 = k_1 F - k_0 B \end{array}$	$\begin{array}{l} Z_N = X_0 + \mu(k_1 \hat{F} - k_0 \hat{B}) \\ +\nu[k_1 t \hat{F} + (1 - k_0 t) \hat{B}], \\ Z_E = Z_N + \sigma_a G_a \end{array}$
			$\delta = -\omega^2, k_0 + n_1 = 2n$	$Z_N = X_0 + \nu X_3 + \mu X_4$

In Table 4 columns 4 and 5 refer to the both non-linearities specified as Cases 8 and 9. However, the conditions marked by * are related to Case 8 only and the symmetries marked by ** correspond only to Case 9. In addition, $\kappa =$ constant, $\Delta = k_0 n_1 - n_0 k_1$, $\delta = \frac{1}{4}(k_0 - n_1)^2 + k_1 n_0$, $\hat{\mathcal{F}}_\alpha = \mathcal{F}_\alpha^{ab} u_b \frac{\partial}{\partial u_a}$, $\hat{F} = F^{ab} u_b \frac{\partial}{\partial u_a}$ and:

$$\left(\frac{\partial}{\partial t} - A \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}\right) \begin{pmatrix} 0 \\ \psi_n \end{pmatrix} = n \begin{pmatrix} 0 \\ \psi_n \end{pmatrix}, \quad \left(\frac{\partial}{\partial t} - A \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}\right) \begin{pmatrix} \tilde{\psi}_n \\ 0 \end{pmatrix} = n \begin{pmatrix} \tilde{\psi}_n \\ 0 \end{pmatrix}$$

and n is an arbitrary parameter (including the case $n = 0$). The generators $D_\mu, \hat{A}, G_a, \hat{G}_a, X_\nu, Y_s$ when not specified in (42) are given by:

$$\hat{A} = t^2 \frac{\partial}{\partial t} + tx_a \frac{\partial}{\partial x_a} - \frac{1}{4}x^2 (A^{-1})^{ab} u_b \frac{\partial}{\partial u_a} - \frac{m}{2}t\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\right) + nt^2 \hat{F},$$

$$D_5 = D_0 - \frac{2}{k}\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\right), \quad D_6 = D_0 + 2tn\hat{F} - \frac{2}{k}\hat{B},$$

$$D_7 = D_0 + 2\left(st - \frac{1}{r}\right)u_1 \frac{\partial}{\partial u_1} - \frac{2srt}{k}\frac{\partial}{\partial u_2}, \quad D_8 = D_0 + \frac{2n}{k}u_1 \frac{\partial}{\partial u_1} + \frac{2ht}{k}\frac{\partial}{\partial u_1} + \frac{2}{k}\frac{\partial}{\partial u_2},$$

$$D_9 = D_0 - 2htu_2 \frac{\partial}{\partial u_1} - \frac{2}{k}\frac{\partial}{\partial u_2},$$

$$D_{10} = D_0 - 2\frac{\partial}{\partial u_2} - \frac{p}{k(1-a)}\left(\frac{x^2}{m} + 2t\right)\left(\frac{\partial}{\partial u_1} - k\frac{\partial}{\partial u_2}\right),$$

$$D_{11} = D_0 - 2\frac{\partial}{\partial u_2} - \frac{p}{c(1+k^2)}\left(\frac{x^2}{m} + 2t(ck + b)\right)\left(\frac{\partial}{\partial u_1} - k\frac{\partial}{\partial u_2}\right),$$

$$D_{12} = D_0 - 2\frac{\partial}{\partial u_2} - \frac{p}{e}\left(\frac{x^2}{m} + 2t\right)\frac{\partial}{\partial u_1}, \tag{44}$$

$$D_{13} = D_0 - \frac{1}{s}\left(u_1 \frac{\partial}{\partial u_1} + 2u_2 \frac{\partial}{\partial u_2}\right) + \frac{t}{2sn}\frac{\partial}{\partial u_1} - \frac{t}{s}u_1 \frac{\partial}{\partial u_2},$$

$$D_{14} = D_5 + \frac{p(k+1)}{ks(a-1)}\left(2t + \frac{x^2}{m}\right)\left(\frac{\partial}{\partial u_1} - s\frac{\partial}{\partial u_2}\right),$$

$$D_{15} = D_5 - \frac{p(k+1)}{kc}\left(2tb + \frac{x^2}{m}\right)\frac{\partial}{\partial u_1},$$

$$D_{16} = D_5 + \frac{1}{s(a-1)}\left[2t\left(q + ps(a-1)\right) + q\frac{x^2}{m}\right]\left(\frac{\partial}{\partial u_1} - s\frac{\partial}{\partial u_2}\right),$$

$$D_{17} = D_0 + 2\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\right) - \frac{1}{c}\left(2t(bq - pc) + q\frac{x^2}{m}\right)\frac{\partial}{\partial u_1},$$

$$X_1 = \exp(nt)\left(2k_1\hat{F} + (n_1 - k_0)\hat{B}\right), \quad X_2 = tX_1 + 2\exp(nt)\hat{B},$$

$$X_3 = \exp(nt)\left[k_1 \cos(\omega t)\hat{F} + ((\lambda_0 - k_0)\cos(\omega t) - \omega \sin(\omega t))\hat{B}\right],$$

$$X_4 = \exp(nt)\left[k_1 \sin(\omega t)\hat{F} + ((\lambda_0 - k_0)\sin(\omega t) + \omega \cos(\omega t))\hat{B}\right],$$

$$Y_7 = \exp(nt)\left(u_1 \frac{\partial}{\partial u_2} - \frac{q}{2p}\left(\frac{sx^2}{2m}\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right)\right), \quad Y_8 = \exp(nt)(u_1 \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}),$$

$$Y_9 = u_1 \frac{\partial}{\partial u_1} + nt\frac{\partial}{\partial u_2}, \quad Y_{10} = \exp(kt)\left(u_1 \frac{\partial}{\partial u_1} + nt\frac{\partial}{\partial u_2}\right),$$

$$Y_{11} = \exp(kt)u_1 \frac{\partial}{\partial u_1} + \frac{n}{k-b}\frac{\partial}{\partial u_2}, \quad Y_{12} = \exp(\lambda_0 t)\left(k_1 u_1 \frac{\partial}{\partial u_1} + \frac{n_1 - k_0}{2}\frac{\partial}{\partial u_2}\right),$$

$$Y_{13}^\pm = \exp(\lambda_\pm t)\left(k_1 u_1 \frac{\partial}{\partial u_1} + (\lambda_\pm - k_0)\frac{\partial}{\partial u_2}\right),$$

$$\lambda_\pm = \frac{1}{2}(k_0 + n_1) \pm \sqrt{\delta}, \lambda_0 = \frac{1}{2}(k_0 + n_1),$$

$$Y_{14} = \exp(\lambda_0 t)\left[k_1 \cos(\omega t)u_1 \frac{\partial}{\partial u_1} + ((\lambda_0 - k_0)\cos(\omega t) - \omega \sin(\omega t))\frac{\partial}{\partial u_2}\right],$$

$$Y_{15} = \exp(\lambda_0 t)\left[k_1 \sin(\omega t)u_1 \frac{\partial}{\partial u_1} + ((\lambda_0 - k_0)\sin(\omega t) + \omega \cos(\omega t))\frac{\partial}{\partial u_2}\right].$$

5   Linear Systems

Consider now the linear case when f^1 and f^2 have the form, $f^a = \Lambda^{ab} u_b + \lambda_a$. In contrast to the one-dimensional cases we find non trivial possibilities corresponding to the non-

commutation of the matrices A and Λ where we specify A in formulae (25).

Table 5: Symmetries for linear systems

No	Form of f^1 and f^2	Type of A (25)	Conditions	Symmetries
1	$f^1 = f^2 = 0$	Ia		$X_5 + \lambda u_1 \frac{\partial}{\partial u_2}$ $+\nu u_2 \frac{\partial}{\partial u_2} + \mu u_2 \frac{\partial}{\partial u_1}$
		Ib		$X_5 + \mu u_2 \frac{\partial}{\partial u_2}$
		II		$X_5 + \lambda \left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1} \right)$
		III		$X_5 + \lambda u_1 \frac{\partial}{\partial u_2}$
2	$f^1 = u_2,$ $f^2 = 0$	Ib		$X_0 + \psi_\Lambda^a \frac{\partial}{\partial u_a}$ $+\alpha(D_0 + 2u_1 \frac{\partial}{\partial u_1})$ $+\lambda \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right)$
3	$f^1 = au_1 + bu_2,$ $f^2 = cu_1 + du_2$	Ib	$\delta > 0$	$X_0 + \psi_\Lambda^a \frac{\partial}{\partial u_a}$ $+\nu \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right)$
4	$f^1 = au_1 + bu_2,$ $f^2 = cu_1 + du_2$	II	$(a - d)^2 + (b + c)^2 \neq 0$	$X_0 + \psi_\Lambda^a \frac{\partial}{\partial u_a}$ $+\nu \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right)$
5	$f^1 = au_1 + bu_2,$ $f^2 = cu_1 + du_2$	III	$b^2 + (a - d)^2 \neq 0$	$X_0 + \psi_\Lambda^a \frac{\partial}{\partial u_a}$ $+\nu \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right)$

In this table the following notation has been employed:

$\delta = bc - \frac{1}{4} (a - d)^2 ,$

$X_5 = X_0 + \nu \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right) + \lambda D_0 + \psi_\Lambda^a \frac{\partial}{\partial u_a}$

$+\mu \left(t^2 \frac{\partial}{\partial t} + tx_a \frac{\partial}{\partial x_a} - \frac{1}{4}x^2 (A^{-1})^{ab} u_b \frac{\partial}{\partial u_a} - \frac{m}{2} t \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right) + t^2 \lambda^{ab} u_b \frac{\partial}{\partial u_a} \right) .$

Note that symbols X_0 and D_0 has been defined in (42) and (25) and further ψ_Λ is an arbitrary solution of the homogeneous equation:

$$\left(\frac{\partial}{\partial t} - \sum_{i=1}^{m} \frac{\partial^2}{\partial x_1^2} \right) \psi_\Lambda^j = \Lambda^{jk} \psi_\Lambda^k.$$

6   Equivalence transformations

The solutions of the determining equations presented in Tables 2-5 are defined up to the equivalence transformations (26) which do not change the shape of equations (1) for *arbitrary* f^k. However for some *particular* f^k it is possible to indicate more extended groups of equivalence transformations which include (26) as a subgroup. Here we discuss such transformations.

The most extended equivalence groups appear for the case of linear f^k presented in Table 5. Let the related solutions have the following general form

$$f^k = \Lambda^{kb} u^b \tag{45}$$

where Λ^{kb} is a matrix which commutes with $(A^{-1})^{kb}$. So there exist an additional equivalence transformation

$$u_a \longrightarrow \exp\left(-t\hat{\Lambda}^{cb}u_b\frac{\partial}{\partial u_c}u_a\right) = \exp\left(-t\hat{\Lambda}^{ab}\right)u_b \tag{46}$$

where $\hat{\Lambda}^{cb}$ is a matrix which commutes with A^{ab} and Λ^{ab}. The transformed equation has the form (45) where $f'^k = \left(\Lambda^{kb} - \hat{\Lambda}^{kb}\right)u_b$. In particular we can choose $\hat{\Lambda}^{kb} = \Lambda^{kb}$ and reduce the related f^k to zero.

For A diagonal there exist also the following equivalence transformation

$$u_1 \to u_1\exp(-kt), \qquad u_2 \to u_2 + nt, \tag{47}$$

$$u_1 \to u_1 - tu_2 + \frac{t^2}{2}p, \quad u_2 \to u_2 - tp, \tag{48}$$

$$u_1 \to u_1 + \frac{pt^2}{2}, \quad u_2 \to u_2 + pt \tag{49}$$

moreover, (48) is valid only for the case when A is proportional to the unit matrix.

Solutions present in Table 5 are defined up to equivalence transformations (26), (46) - (49).

Using the transformation

$$u_1 \to \exp(nt)u_1, \quad u_2 \to \exp\left(-\frac{qn}{r}\right)u_2$$

it is possible to reduce to zero the parameter n in Table 3, Item 1 (we will refer to this case as [T3.1]). The transformations $u_1 \to u_1, u_2 \to u_2 + \nu t + \mu x^2$ enable to make zero parameters C_1 and C_2 in the cases when f^1 and f^2 have the form $f^1 = \varphi_1 + C_1, f^2 = \varphi_2 + C_2$ where φ_1 and φ_2 are given functions of u_1. Such transformations can be applied in the cases [T3.10] and [T3.8] (in the last case it is necessary to change roles of u_1 and u_2).

Transformations $u_2 \to \exp(-nt)u_2$ and $u_2 \to \exp(-bt)u_2$ give rise to new n and b in solutions 2 and 3 from Table 4 respectively. The transformation

$$u_1 \to u_1\exp(-st), \qquad u_2 \to u_2 + \frac{sr}{k}t$$

reduces to zero parameter s in non-linearities [T3.6].

Consider further the scale transformation

$$u_1 \to \mu u_1, \qquad u_2 \to \nu u_2, \qquad t \to \lambda t \qquad x \to \sqrt{\lambda}x. \tag{50}$$

Under obvious conditions for the parameters defining the functions f_1 and f_2 transformations (50) enable the reduction of nonzero coefficients p and q to zero coefficients for solutions 1, 8, and 9 from Table 4.

We see that using equivalence transformations it is possible to make "cosmetic" improvements to the solutions found for f^k. These transformations were not used systematically because they do not change the principal classes of solutions. Also in some cases their use would complicate the presentation of the results in standard form and make them less convenient for applications.

7 Discussion

In this paper we have found all possible versions of systems of diffusion equations that admit a nontrivial Lie symmetry. These results can be used to construct mathematical models with required symmetry properties in for example, physics, biology, chemistry.

In the case when the matrix A is proportional to the unit matrix I equation (24) admits nontrivial symmetries for *all* f^1, f^2 given in Tables 2-4. In other words in this case we have obtained the richest spectrum of possible symmetries. If A is not proportional to I the number of symmetries is sufficiently reduced by the requirement of commutativity of A^{-1} with the chosen matrices B and $F(43)$.

In the particular case when matrix A has the form Ib from (25) our results can be compared with those of [12]. Our results are quite similar. However, a number of our solutions, namely, seven of those presented in Table 4, Case 1 (which correspond to symmetries Z_N), solutions [T4.3] for $b = k = q = 0$, [T5.2] , [T3.11], [T3.12] and all solutions [T4.4], [T4.5], [T4.7] are missing in [12]. In addition, [T2.17] are presented in [12] incorrectly (see Table 5, item 9 here).

Consider the examples of reaction diffusion equations mentioned above in Section 1.

- The activator-inhibitor reaction equations [1] are given by

$$\dot{u}_1 - \frac{\partial^2 u_1}{\partial x^2} = \frac{u_1^2}{u_2} - bu_1, \qquad \dot{u}_2 - a\frac{\partial^2 u_2}{\partial x^2} = u_1^2 - u_2$$

  and these are a particular case of equation (24) with the non-linearities given in [T 2.5] with $d = 2, k = 0, \varphi_1 = \frac{u_1^2}{u_2} - b, \varphi_2 = \frac{u_1^2}{u_2} - 1$, and so admits the symmetry:

$$X = X_0 + \alpha\left(u_1\frac{\partial}{\partial u_1} + 2u_2\frac{\partial}{\partial u_2}\right).$$

- The primitive predator-prey system can be defined by [1]

$$\dot{u}_1 - D\frac{\partial^2 u_1}{\partial x_2} = -u_1 u_2, \qquad \dot{u}_2 - \lambda D\frac{\partial^2 u_2}{\partial x^2} = u_1 u_2.$$

  This is a particular case of (24) with non-linearities [T2.1] where $d = 1, k = 1, \varphi_2 = -\varphi_1 = \frac{u_2}{u_1}$, and so it admits the symmetry:

$$X = X_0 + \alpha\left(D_0 - 2u_1\frac{\partial}{\partial u_1} - 2u_2\frac{\partial}{\partial u_2}\right).$$

- The $\lambda - \omega$ reaction-diffusion system

$$\dot{u}_1 = D\Delta u_1 + \lambda(R)u_1 - \omega(R)u_2, \qquad \dot{u}_2 = D\Delta u_2 + \omega(R)u_1 + \lambda(R)u_2, \qquad (51)$$

  where $R^2 = u_1^2 + u_2^2$, and Δ is the Laplacian operator has symmetries that were analyzed in paper [2]. Again we recognize that this system is a particular case of

(24) with non-linearities [T2.6] with $n = 0$. Hence it admits the five dimensional Lie algebra generated by:

$$X = X_0 + \alpha \left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1} \right) \tag{52}$$

which is in accordance with results of paper [2] for the case when functions λ and ω are arbitrary. Moreover, using Table 3, Case [T3.2] we find that for the cases when

$$\lambda = \kappa_1 R^r, \quad \omega = \kappa_2 R^r \tag{53}$$

equation (51) admits additional symmetry with respect to scaling transformations given by the operator:

$$X = X_0 + \alpha \left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1} \right) + \nu D_5. \tag{54}$$

- The nonlinear Schrödinger equation in $m-$dimensional space is given by:

$$\left(i \frac{\partial}{\partial t} - \Sigma_{i=1}^m \frac{\partial^2}{\partial x_i^2} \right) \psi = F(\psi, \psi^*) \psi \tag{55}$$

also is a particular case of (24). If we denote

$$u_1 = \frac{1}{2}(\psi + \psi^*), \quad u_2 = \frac{1}{2i}(\psi - \psi^*) \tag{56}$$

then (55) reduces to the form (24) with $A = -\sigma_2$ and

$$f^1 = \frac{1}{2}(F^* + F)u_2 + \frac{1}{2i}(F - F^*)u_1, \quad f^2 = \frac{1}{2i}(F - F^*)u_2 - \frac{1}{2}(F + F^*)u_1. \tag{57}$$

In other words, any solution given in Table 2 with matrices belonging to Classes I ($d = 0$) and II, and solutions given in Tables 3-5 with matrices belonging to Class II give rise to the non-linearity

$$F = \frac{1}{R^2} \left(u_2 f^1 - u_1 f^2 + i(u_2 f^2 + f^1 u^1) \right) \tag{58}$$

for the nonlinear Schrödinger equation (55) that admits a nontrivial Lie symmetry.In the cases [T3.7] with $n = 0$, [T3.2] for $s = l = q = 0$, $r \neq \frac{4}{m}$ and $r = \frac{4}{m}$, [T4.8] for $k_1 = n_1 = k_0 = 0$, $p = q$ we recognize the well-known non-linearities [11]

$$F = F(\psi^* \psi), \quad F = (\psi^* \psi)^k, \quad F = (\psi^* \psi)^{\frac{2}{m}}, \quad F = \ln(\psi^* \psi) \tag{59}$$

which correspond to extended symmetries. Our analysis makes it possible to describe all other possible versions of the nonlinear Schrödinger equation with a nontrivial symmetry. We plan to discuss these elsewhere.

Higher symmetries of the linear and nonlinear Schrödinger equations where investigated in [14], extended supersymmetries where studied in [15] The nonlinear Schrödinger equations and equations (24) for diagonal A with *ad hoc* required symmetry with respect to the (extended) Galilei group were analyzed in [11] and [16]. We notice that the algorithm used in the present paper reduce such an analyzis to routine and simple calculations.For example, to find all systems (1) *with arbitrary n* which are invariant with respect to the Galilei group it is sufficient to solve the system of homogeneous linear equations (31) which is easy integrated for any given invertable matrix A .

In the present paper we have restricted ourselves to a complete description of all possible non-linearities which generate Lie symmetry of equation (1). We have not analyzed non-classical symmetries that may be found with using condition (13) nor have any symmetry reductions been presented. These problems will be a subject of further investigations. Finally we remark that some of the results of this paper have been presented in [17].

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