Symmetry analysis of the nonlinear Schrödinger equation

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Abstract

A complete description of nonlinear Schr̈dinger equations with $t-$ and x-independent nonlineariries is present which admit a nontrivial (i.e., distinct from displacements and rotations) Lie symmetry.

1 Introduction

Complete investigation of symmetries of the *linear* Schrödinger equation was carried out by Niederer and Boyer [1] who described all potentials which generate nontrivial Lie symmetries. Recently higher symmetries of this equation were described also [2].

In the present paper the complete description of symmetries of the *nonlinear* Schrödinger equation (NSE)

$$i\frac{\partial}{\partial t}\psi = -\frac{\partial^2}{\partial x^2}\psi + F(\psi, \psi^*)\psi \tag{1}$$

is presented, where ψ is a function of $t, x_1, x_2, ..., x_m,$ F is *an arbitrary* function of ψ and ψ^*.

Considering real and imaginary part of ψ as new variables, we can reduce (1) to the system of nonlinear diffusion equations

$$Lu - f(u) \equiv \frac{\partial u}{\partial t} - \Lambda \sum_i \frac{\partial^2 u}{\partial x_i^2} - f(u) = 0, \tag{2}$$

where $u, f(u)$ are related to ψ, ϕ as follows

$$u = \begin{pmatrix} u_1, \\ u_2 \end{pmatrix}, f = \begin{pmatrix} f_1, \\ f_2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_1 = \frac{1}{2}(\psi + \psi^*), \quad u_2 = \frac{1}{2i}(\psi - \psi^*) \quad .$$
$$f^1 = \frac{1}{2}(F^* + F)u_2 + \frac{1}{2i}(F - F^*)u_1 \quad f^2 = \frac{1}{2i}(F - F^*)u_2 - \frac{1}{2}(F + F^*)u_1.$$

If A is an arbitrary matrix then relations (2) define the general system of nonlinear diffusion equations which has many important applications in mathematical physics, chemistry and biology.

2 Determining equations

Investigation of symmetries of general equation (2) can be carried out in frames of the classical Lie algorithm [3]. In application to systems (2) this algorithm admits rather simple formulation which is valid for an extended class of partial differential equations. We present it here as two successive steps:

1. Solve the operator equation

$$[Q, L] = \Lambda L + \varphi(t, x), \tag{3}$$

where

$$\hat{X} = \eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a} - \pi^a \frac{\partial}{\partial u_a}, \tag{4}$$

L is operator (2), Λ and φ are $n \times n$ matrices dependent on (t, x_a), and find the corresponding matrices Λ, π, φ and functions η and ξ.

2. Solve the following system of equations for unknown nonlinearities $f(u)$:

$$\left(\Lambda^{kb} - \pi^{kb}\right) f^b + \varphi^{kb} u^b + L\omega^k = -(\omega^a + \pi^{ab} u_b) \frac{\partial f^k}{\partial u^a}. \tag{5}$$

The corresponding Lie generators X will have the form $X = \eta \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x_a} - \pi^a \frac{\partial}{\partial u_a}$.

We shall not present here detailed calculations. The list of nontrivial symmetries is rather extended and present in the following section.

3 Nonlinearities and symmetries

The results of our analysis are present in the following tables.

Table 1. **Nonlinearities with arbitrary functions**

No	Nonlinear terms	Type of matrix B	Arguments of φ_1, φ_2	Conditions for parameters	Symmetries
1.	$F = \frac{1}{R^2} u_1^k (u_1 u_2 \varphi_1 - u_1^2$, $f^2 = u_1^{k+d} \varphi_2$	$B = I$	$\frac{u_2}{u_1^d}$	$k \neq 0$	$X_0 + \nu D_1$

2.	$f^1 = \exp(k\theta)[\varphi_1 u_2 + \varphi_2 u_1]$ $f^2 = \exp(k\theta)[\varphi_1 u_1 - \varphi_2 u_2]$	$B = A$	$R\exp(-d\theta)$	$k \neq 0$	$X_0 + \nu D_1$
3.	$f^1 = u_1(n\ln u_1 + \varphi_1)$ $f^2 = u_2(n\ln u_2 + \varphi_2)$	$B = I$	$\frac{u_2}{u_1}$	$n \neq 0$	$X_0 + \nu_a \tilde{G}_a$ $+\mu\exp(nt)\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}+$ $\nu_a G_a$
4.	$f^1 = \varphi_1 u_2 + \varphi_2 u_1 +$ $+\frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_1 - u_2)$ $f^2 = \varphi_1 u_1 - \varphi_2 u_2 +$ $\frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_2 + u_1)$ $R^2 = u_1^2 + u_2^2,$ $\theta = \arctan\left(\frac{u_2}{u_1}\right)$	$B = A + dI$ $d \neq 0$	$R\exp(-d\theta)$	$n \neq 0$	$X_0 + \nu_a \hat{G}_a$ $+\mu\exp(nt)\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}+$ $\lambda_a G_a$
5.	$f^1 = (\varphi_1 - n\theta)u_2 + \varphi_2 u_1,$ $f^2 = \varphi_1 u_1 - \varphi_2 u_2$	$B = A$ $d = 0$	R	$n \neq 0$	$X_0 + \nu_a \hat{G}_a$ $+\mu\exp(nt)\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}+$ $\lambda_a G_a$
6.	$f^1 = u_1^{\left(k + \frac{k^2}{\sqrt{k^2+s^2}}\right)}$ $\times\exp\left[\left(s + \frac{sk}{\sqrt{k^2+s^2}}u_2\right)\right]\varphi_1$ $f^2 = u_1^k\exp(su_2)\varphi_2$	$B = I$	$s\ln u_1 + ku_2$		$X_0 + \nu D_2$
7.	$f^1 = \varphi_1\exp\left(\frac{nu_2}{n^2+1}\right)u_1^{\frac{1}{n^2+1}}$ $+\frac{s}{n^2+1}u_1(nu_2 + \ln u_1)$ $f^2 = \varphi_2 + \frac{ns}{n^2+1}(nu_2 + \ln u_1)$	$I,$	$u_2 - n\ln u_1$		$X_0 + Y_2$
8.	$f^\alpha = \exp[\frac{k}{\omega}(\omega_1 u_1 + \omega_2 u_2)]\varphi^\alpha$ $\alpha = 1, 2,\ \omega^2 = \omega_1^2 + \omega_2^2$	*any*	$\omega_1 u_2 -$ $-\omega_2 u_1$	$k \neq 0$	$X_0 + \nu D_3$
9.	$f^1 = \varphi_1,\ f^2 = \varphi_2$	*any*	(u_1, u_2)	-	X_0

Here the Greek letters denote arbitrary coefficients, D_μ, G_a^i and $\bar{G}_a^i, X_A, Y_a, \hat{B}$ are different kinds of dilatation, Galilei and special transformation generators, specified in the following formulae

$$
\begin{aligned}
&D_0 = 2t\frac{\partial}{\partial t} + x_a\frac{\partial}{\partial x_a},\ D_1 = D_0 - \frac{2}{k}F,\ D_2 = D_0 - \frac{2s}{\sqrt{k^2+s^2}}u_1\frac{\partial}{\partial u_1} - \frac{2s}{\sqrt{k^2+s^2}}\frac{\partial}{\partial u_2},\\
&D_3 = D_0 - \frac{2}{k}\omega_a\frac{\partial}{\partial u_a},\ G_a = t\frac{\partial}{\partial x_a} - \frac{1}{2}x_a(A_1^{-1})^{nb}u_b\frac{\partial}{\partial u_n},\\
&\hat{G}_a = \exp(nt)\left(\frac{\partial}{\partial x_a} - \frac{1}{2}nx_a(A_1^{-1})^{nb}u_b\frac{\partial}{\partial u_n}\right),\\
&X_0 = \alpha\frac{\partial}{\partial t} + \beta_a\frac{\partial}{\partial x_a} + \nu^{[ab]}x_a\frac{\partial}{\partial x_b}, \quad \nu^{[ab]} = -\nu^{[ba]},\\
&Y_1 = nt\hat{F} - \hat{B},\ Y_2 = \exp(st)\left(u_1\frac{\partial}{\partial u_1} + n\frac{\partial}{\partial u_2}\right), \hat{B} = B^{ab}u_b\frac{\partial}{\partial a_a}.
\end{aligned}
\tag{6}
$$

In the following table we present nonlinearities which depend on arbitrary parameters and generate symmetry w.r.t. dilatation

Table 2

No	Nonlinear terms	Conditions for parameters	Symmetries	Class of matrices and parameters of symmetry generators
1.	$f^1 = (gu_1^q u_2^r - s)\, u_1$ $f^2 = \left(pu_1^q u_2^r - \frac{rs}{q}\right) u_2$	$s = 0, q \neq 0,$ $q + r = \frac{4}{m}$	$X_0 + \mu\hat{F} +$ $+\nu D_5$	$I,$ $d = -q/r$
		$s = 0, q \neq 0$ $q + r \neq \frac{4}{m}, r \neq 0$ $r + q \neq 0$	$X_0 + \nu\hat{F} +$ $+\mu D_4$	$I, k = r + q,$ $d = -q/r$
		$s \neq 0, q \neq 0$	$X_0 + \nu\hat{F} +$ $+\mu D_6$	$I, k = r,$ $d = -q/s$
2.	$f^1 = \exp(q\theta)R^r(gu_1 -$ $pu_2) + su_2 - lu_1$ $f^2 = \exp(q\theta)R^r(gu_2 +$ $pu_1) - su_1 - lu_2$ $R^2 = u_1^2 + u_2^2,$ $\theta = \arctan(\frac{u_2}{u_1})$	$s = l = 0,$ $r = 4/m$	$X_0 + \nu\hat{F}$ $+\mu D_4$	$IIb, k = \frac{4}{m}$ $d = -q/r, g \neq 0$
			$X_0 + \nu\hat{F} + \sigma_a G_a$ $\mu D_4 + \lambda A_0$	$IIb, k = \frac{4}{m}$ $d = 0$
		$r \neq \frac{4}{m}, r \neq 0$ $s = l = 0$	$X_0 + \nu\hat{F}$ $+\mu D_4$	$IIb, k = r$ $d = -q/r$
			$X_0 + \nu\hat{F}$ $\mu D_4 + \sigma_a G_a$	$IIb, k = r$ $d = 0$
		$l = sq(1 + \frac{1}{r})$ $s \neq 0, r \neq 0$	$X_0 + \nu\hat{F}$ $+\mu D_5 + \sigma_a G_a$	$IIb, k = q$ $n = sq,$ $d = 0$
			$X_0 + \nu\hat{F} + \mu D_5$	$IIB, k = q,$ $n = sq, d \neq 0$ $d = q(1 + \frac{1}{r}) \neq 0$
		$s = 0, l \neq 0$ $q \neq 0, r = 0$	$X_0 + \nu\hat{F}$ $+\mu D_5$	$IIa, k = q,$ $n = lq$
3.	$f^1 = pu_1^{k+1}$ $f^2 = \left(pu_2 + qu_1^d\right) u_1^k -$ $-\frac{n}{d+k-1}u_1$	$d + k \neq 1,$ $k \neq 0, q \neq 0$	$X_0 + \nu\hat{F}$ $+\mu D_6$	$IIIa$
		$k \neq 0, n = 0,$ $q = 0$	$X_0 + \nu\hat{F} +$ $+\mu D_5 + \lambda\hat{B}$	$IIIa, d = 0$
		$k \neq 0, n = 0,$ $q \neq 0$	$X_0 + \nu\hat{F} +$ $\mu D_5 + \lambda Y_1$	$IIIa, d = 0$
4.	$f^1 = qu_1^{r+1}\exp(ku_2) +$ $+\frac{ks}{r^2}u_1$ $f^2 = pu_1^r \exp(ku_2) + \frac{s}{r}$	$r \neq 0, -1$ $p \neq 0$	$X_0 + \nu D_7 +$ $+\mu Y_7$	$I,$

5.	$f^1=\exp u_2+su_1+q$ $f^2=n$	$q=0$	$X_0+\nu D_7+$ $+\mu Y_7+\psi_n\frac{\partial}{\partial u_1}$	$I,$ $k=-r=1$
		$n=0,$ $q\neq 0$	$X_0+\nu D_1+$ $\mu\left(Y_7-qt\frac{\partial}{\partial u_1}\right)+$ $\psi_0\frac{\partial}{\partial u_1}$	$I, d=0$ $k=-r=1$
6.	$f^1=k_1\exp u_2-pA^{21}$ $f^2=k_2\exp u_2-pA^{11}+q$		X_0+D_8+ $+\psi_0\frac{\partial}{\partial u_1}$	*any*
7.	$f^1=p\left(u_2+nu_1^2\right)^{s+\frac{1}{2}}$ $+\frac{1}{2n(2s+1)}$ $f^2=q\left(u_2+nu_1^2\right)^{s+1}$ $-2npu_1\left(u_2+nu_1^2\right)^{s+\frac{1}{2}}$ $-\frac{1}{2s+1}u_1$	$s\neq 0, p\neq 0$ $s\neq -\frac{1}{2},\ n\neq 0$	$X_0+\nu D_9$	I
8.	$f^1=pu_1^{2k+1}-2msA^{11}$ $f^2=qu_1^{2k+1}-2msA^{21}$	$k\neq 0$	$X_0+\nu D_{10}$ $+\psi_0\frac{\partial}{\partial u_2}$	any

Here (F,B) is one of following pairs matrices:

$$I.\quad F=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\0&1\end{pmatrix},\ IIa.\quad F=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$
$$IIb.\quad F=\begin{pmatrix}d&-1\\1&d\end{pmatrix}\quad B=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

and gerators $D_\mu, \hat{A}_\alpha, G_a, \hat{G}_a, X_\nu, Y_\sigma$ (if no specified in (6)) have the following form

$$\begin{aligned}
&\hat{A}_0=t^2\frac{\partial}{\partial t}+tx_a\frac{\partial}{\partial x_a}-\frac{1}{4}x^2(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a}-\frac{m}{2}+\left(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}\right),\\
&\hat{A}_1=\hat{A}_0+nt^2\hat{F}-\frac{m}{2}\hat{B},\ D_5=D_0-\frac{m}{2}(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}),\ D_6=D_0-2tn\hat{F}-\frac{2}{k}\hat{B},\\
&D_7=D_0+\frac{2}{r}(\frac{kst}{r}-1)u_1\frac{\partial}{\partial u_1}-\frac{2st}{r}\frac{\partial}{\partial u_2},\ D_8=D_0-(\frac{p}{m}x^2+2qt)\frac{\partial}{\partial u_1}-2\frac{\partial}{\partial u_2},\\
&D_9=D_0-\frac{1}{n}\left(u_1\frac{\partial}{\partial u_1}+2u_2\frac{\partial}{\partial u_2}\right)+\frac{t}{2sn}\frac{\partial}{\partial u_1}-\frac{t}{s}u_1\frac{\partial}{\partial u_2},\\
&D_{10}=D_0-\frac{1}{k}\left(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}-s(2k+1)x^2\frac{\partial}{\partial u_1}\right).
\end{aligned}$$

Thus we have found all possible versions of nonlinear Schrödinger equations which admit a nontrivial Lie symmetry. Part of them is given in the tables. These results can be used to construct mathematical models with required symmetry properties and to search for their exact solutions.

References

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